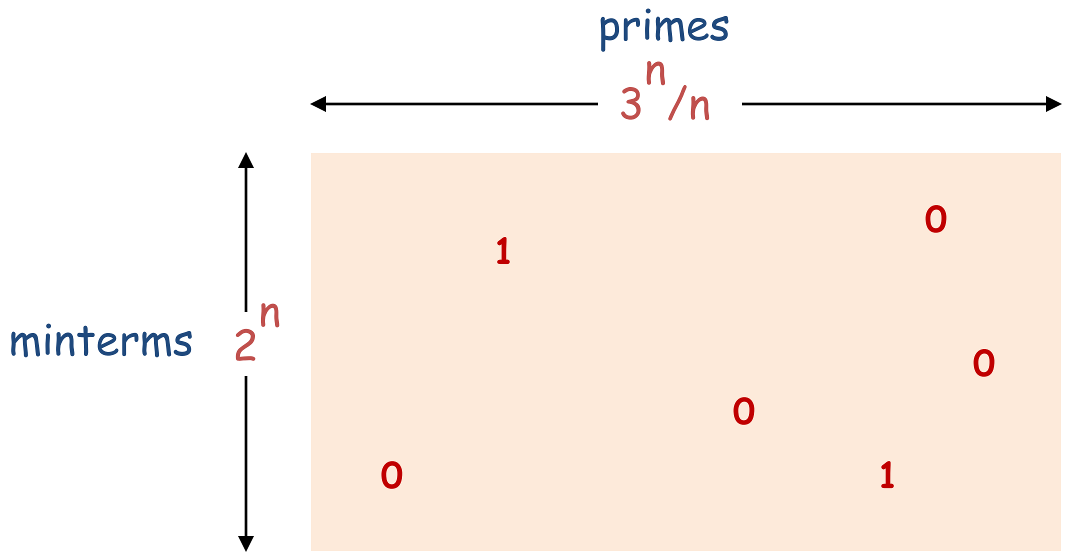
Quine-McCluskey (Tabular) Minimization

- ✓ Two step process utilizing tabular listings to:
  - Identify prime implicants (implicant tables)
  - Identify minimal PI set (cover tables)
- ✓ All work is done in tabular form
  - Number of variables is not a limitation
  - Basis for many computer implementations
  - Don't cares are easily handled
- ✓ Proper organization and term identification are key factors for correct results

Difficulty

Note:

- ✓ 2^n minterms
- ✓ ~ $3^n/n$ primes



- ✓ Thus $O(2^n)$ rows and $O(3^n/n)$ columns and minimum covering problem is NP-complete.

Example

$F = \bar{w}\bar{x}\bar{y}\bar{z} + \bar{w}x\bar{y}z + \bar{w}xy\bar{z} + wx\bar{y}z$

$d = x\bar{y}\bar{z} + w\bar{y}\bar{z} + xyz + \bar{x}y\bar{z} + wy\bar{z} + \bar{w}\bar{x}\bar{y}z$

Karnaugh map

wx \ yz ($\bar{w}\bar{y}$, $\bar{z}$, x marked as groupings)

1	d	0	d
d	1	d	1
d	1	d	d
d	0	0	d

	$\bar{z}$	x	$\bar{w}\bar{y}$
$\bar{w}\bar{x}\bar{y}\bar{z}$	1	0	1
$\bar{w}x\bar{y}z$	0	1	1
$\bar{w}xy\bar{z}$	1	1	0
$wx\bar{y}z$	0	1	0

Primes: $\bar{z} + x + \bar{w}\bar{y}$

Covering Table

Solution: {1,2} $\Rightarrow$ $\bar{z} + x$ is minimum prime cover.

(also $x + \bar{w}\bar{y}$)

Covering Table

Primes of f+d

Minterms of f

	$\bar{z}$	x	$\bar{w}\bar{y}$
$\bar{w}\bar{x}\bar{y}\bar{z}$	1	0	1
$\bar{w}x\bar{y}z$	0	1	1
$\bar{w}xy\bar{z}$	1	1	0
$\bar{w}\bar{x}\bar{y}\bar{z}$	0	1	0

Row singleton (essential minterm) ← (last row)

Essential prime ↑ (column x)

- ✓ Definition: An essential prime is any prime that uniquely covers a minterm of f.

Quine-McCluskey Minimization (cont.)

- ✓ Terms are initially listed one per line in groups
  - Each group contains terms with the same number of true and complemented variables
  - Terms are listed in numerical order within group
- ✓ Terms and implicants are identified using one of three common notations
  - full variable form
  - cellular form
  - 1,0,- form

Example of Different Notations

$F(A, B, C, D) = \sum_m (4,5,6,8,10,13)$

	Full variable	Cellular	1,0,-
1	$\bar{A}B\bar{C}\bar{D}$	4	0100
	$A\bar{B}\bar{C}\bar{D}$	8	1000
2	$\bar{A}B\bar{C}D$	5	0101
	$\bar{A}BC\bar{D}$	6	0110
	$A\bar{B}C\bar{D}$	10	1010
3	$AB\bar{C}D$	13	1101

Notation Forms

- ✓ Full variable form: variables and complements in algebraic form
  - hard to identify when adjacency applies
  - very easy to make mistakes
- ✓ Cellular form: terms are identified by their decimal index value
  - easy to tell when adjacency applies; indexes must differ by power of two (one bit)
- ✓ 1,0,- form: terms are identified by their binary index value
  - easier to translate to/from full variable form
  - easy to identify when adjacency applies, one bit is different
  - shows variable(s) dropped when adjacency is used
- ✓ Different forms may be mixed during the minimization

Implication Table (1,0,-)

✓ Quine-McCluskey Method

- Tabular method to systematically find all prime implicants

  $f(A,B,C,D) = \Sigma_m(1,2.5,6,7,9,10) + \Sigma_d(0,13,15)$

- Part 1: Find all prime implicants
- Step 1: Fill Column 1 with onset and DC-set minterm indices. Group by number of true variables (# of 1's).

NOTE THAT DCs ARE INCLUDED IN THIS STEP!

Implication Table		
Column I 0000 0001 0010 0101 0110 1001 1010 0111 1101 1111		

Minimization - First Pass (1,0,-)

✓ Quine-McCluskey Method

- Tabular method to systematically find all prime implicants
- $f(A,B,C,D) = \Sigma_m(1,2.5,6,7,9,10) + \Sigma_d(0,13,15)$
- Part 1: Find all prime implicants
- Step 2: Apply Adjacency - Compare elements of group with N 1's against those with N+1 1's. One bit difference implies adjacent. Eliminate variable and place in next column.

  E.g., 0000 vs. 0100 yields 0-00

  0000 vs. 1000 yields -000

  When used in a combination, mark with a check. If cannot be combined, mark with a star. These are the prime implicants.

  Repeat until nothing left.

Implication Table

Column I		Column II		
0000	0	000-	0,1	
		00-0	0,2	
0001	1			
0010	2	0-01	1,5	
		-001	1,9	
0101	5	0-10	2,6	
0110	6	-010	2,10	
1001	9			
1010	10	01-1	5,7	
		-101	5,13	
0111	7	011-	6,7	
1101	13	1-01	9,13	
1111	15	-111	7,15	
		11-1	13,15	

Minimization - Second Pass (1,0,-)

- Quine-McCluskey Method
  - Step 2 cont.: Apply Adjacency - Compare elements of group with N 1's against those with N+1 1's. One bit difference implies adjacent. Eliminate variable and place in next column.

    E.g., 0000 vs. 0100 yields 0-00

    00-0 vs. 10-0 yields -0-0

    When used in a combination, mark with a check. If cannot be combined, mark with a star. **THESE ARE THE PRIME IMPLICANTS.**

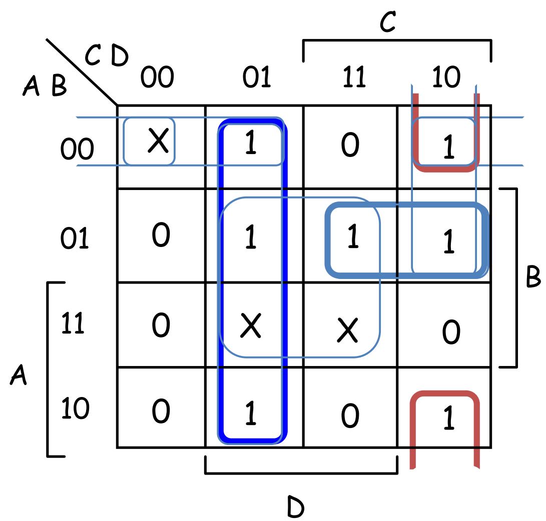
    Repeat until nothing left.
- The set of * constitutes the Complete Sum Σ_c

Implication Table

Column I		Column II		Column III	
0000 ✓	0	000-	0,1	--01	1,5,9,13
		00-0	0,2	- 1-1	5,7,13,15
0001 ✓	1				
0010 ✓	2	0-01	1,5		
		-001	1,9		
0101 ✓	5	0-10	2,6		
0110 ✓	6	-010	2,10		
1001 ✓	9				
1010 ✓	10	01-1	5,7		
		-101	5,13		
0111 ✓	7	011-	6.7		
1101 ✓	13	1-01	9,13		
1111 ✓	15	-111	7,15		
		11-1	13,15		

Prime Implicants

$f(A,B,C,D) = \Sigma_m (1,2.5,6,7,9,10) + \Sigma_d (0,13,15)$



Stage 2: find smallest set of prime implicants that cover the active-set

Note that essential prime implicants must be in the final expression

Coverage Table

rows = prime implicants
columns = ON-set elements (minterms)

place an "X" if ON-set element is covered by the prime implicant

NOTE: DON'T INCLUDE DCs IN COVERAGE TABLE; THEY DON'T HAVE TO BE MANDATORY COVERED

Coverage Chart

		1	2	5	6	7	9	10
0,1	000-	X						
0,2	00-0		X					
2,6	0-10		X		X			
2,10	-010		X					X
6,7	011-				X	X		
1,5,9,13	--01	X		X			X	
5,7,13,15	-1-1			X		X		

Row and Column Dominance

- ✓ Definition: Given two rows i_1 and i_2, a row i_1 is said to dominate i_2 if it has checks in all columns in which i_2 has checks, i.e. it is a superset of i_2

Example:

i_1	×	×	×	×	×	×
i_2	×	×		×	×	

i_1 dominates i_2

- ✓ We can remove row i_2, because we would never choose i_2 in a minimum cover since it can always be replaced by i_1 (i_2 is anymore a prime implicant).

DOMINATED ROWS CAN BE ELIMINATED

Row and Column Dominance

- Definition: Given two colums j_1 and j_2, if the set of primes of column j_2 is contained in the set of primes of column j_1

Example:

j1	j2
×	
×	×
×	
×	×

j_2 dominates j_1

- We can remove column j_1 since we have to choose a prime to cover j_2, any such prime also covers j_1, that would result covered as well.

DOMINATED COLUMNS CAN BE ELIMINATED

Pruning the Covering Table

1. Remove all rows covered by essential primes (columns in row singletons). Put these primes in the cover G.
2. Group identical rows together and remove dominated rows.
3. Remove dominating columns. For equal columns, keep just one to represent them.
4. Newly formed row singletons define n-ary essential primes.
5. Go to 1 if covering table decreased.

- ✓ The algorithm may terminate successfully with a set of primes and an emty table.
- ✓ In case it terminate with a non empty table, the resulting reduced covering table is called the cyclic core. This has to be solved. A minimum solution for the cyclic core must be added to the resulting G.

Coverage Table (cont.)

Coverage Chart

		1	2	5	6	7	9	10
0,1	000-	X						
0,2	00-0		X					
2,6	0-10		X		X			
2,10	-010		X					X
6,7	011-				X	X		
1,5,9,13	--01	X		X			X	
5,7,13,15	-1-1			X		X		

		1	2	5	6	7	9	10
0,1	000-	X						
0,2	00-0		X					
2,6	0-10		X		X			
2,10	-010		X					X
6,7	011-				X	X		
1,5,9,13	--01	X		X			X	
5,7,13,15	-1-1			X		X		

If column has a single x, than the implicant associated with the row is essential.
It must appear in the minimum cover

Coverage Table (cont.)

		1	2	5	6	7	9	10
0,1	000-	X						
0,2	00-0		X					
2,6	0-10		X		X			
2,10	-010		X					X
6,7	011-				X	X		
1,5,9,13	--01	X		X			X	
5,7,13,15	-1-1			X		X		

Eliminate all columns covered by essential primes

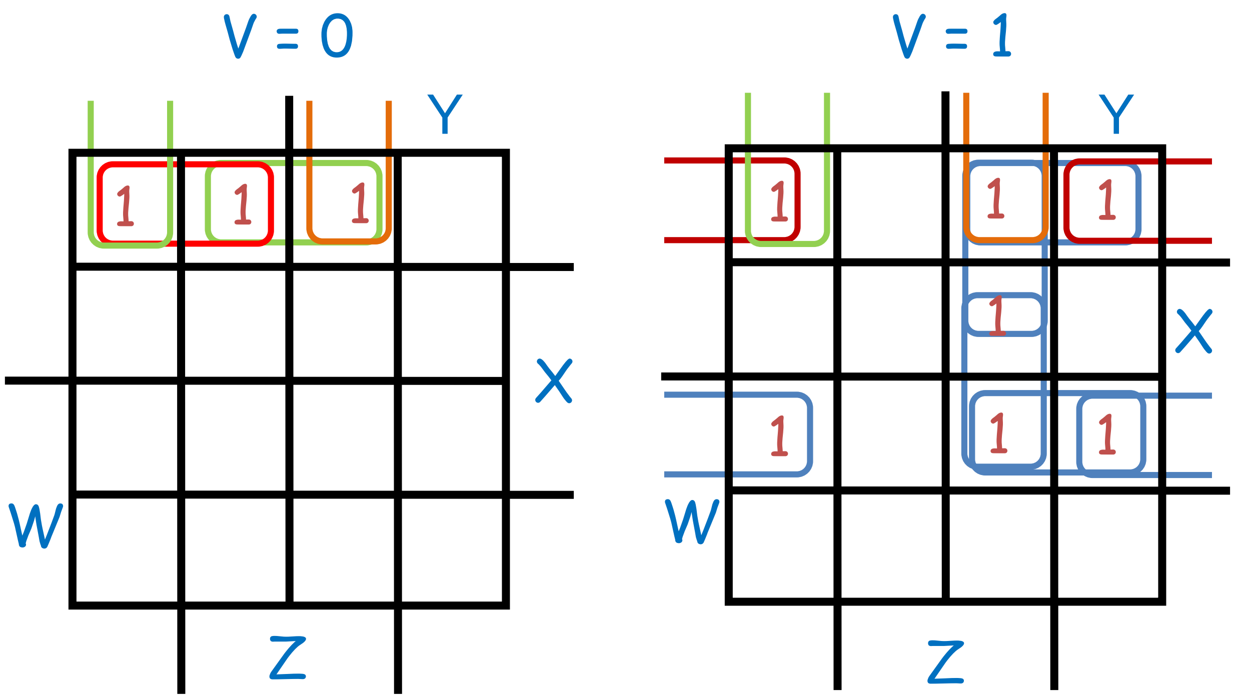
		1	2	5	6	7	9	10
0,1	000-	X						
0,2	00-0		X					
2,6	0-10		X		X			
2,10	-010		X					X
6,7	011-				X	X		
1,5,9,13	--01	X		X			X	
5,7,13,15	-1-1			X		X		

Find minimum set of rows that cover the remaining columns

$$F = \bar{B}C\bar{D} + \bar{A}BC + \bar{C}D$$

Quine Mc Clunskey: Cyclic Core example

$$F=\Sigma_m(0,1,3,16,18,19,23,28,30,31)$$



F=v'w'y'x'+v'w'x'z+w'x'y'z'+w'x'yz+vw'x'z'+vw'x'y+vwxz'+vwxy+vw'yz+vxyz

A B C D E F G H I J

Implication Table (1,0,-)

✓ Quine-McCluskey Method

- Tabular method to systematically find all prime implicants
- $f(v,w,x,y,z) = \Sigma m(0,1,3,16,18,19,23,28,30,31)$
- Part 1: Find all prime implicants
- Step 1: Fill Column 1 with active-set and DC-set minterm indices. Group by number of true variables (# of 1's).

Implication Table

	Column I	Column II	
0	00000	0000-	A: 0 1
1	00001	-0000	C: 0 16
16	10000	000-1	B: 1 3
3	00011	100-0	E: 16 18
18	10010	-0011	D: 3 19
19	10011	1001-	F: 18 19
28	11100	10-11	I: 19 23
23	10111	111-0	G: 28 30
30	11110	1-111	J: 23 31
31	11111	1111-	H: 30 31

F=v′w′y′x′+v′w′x′z+w′x′y′z′+w′x′yz+vw′x′z′+vw′x′y+vwxz′+vwxy+vw′yz+vxyz

A B C D E F G H I J

Quine Mc Clunskey

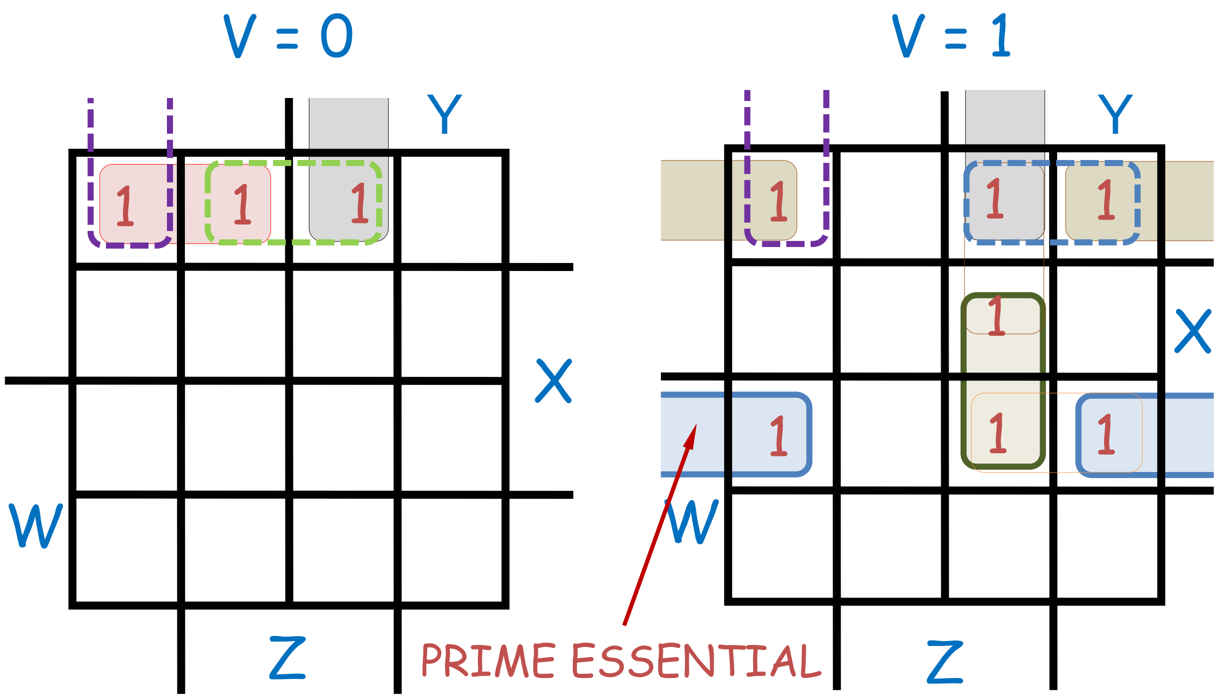
$F=\sum_5(0,1,3,16,18,19,23,28,30,31)$

	0	1	3	16	18	19	23	28	30	31
A	X	X								
B		X	X							
C	X			X						
D			X			X				
E				X	X					
F					X	X				
G								X	X	
H									X	X
I						X	X			
J							X			X

G+J+A+D+E;

Quine Mc Clunskey: Cyclic Core example

$F=\Sigma_m(0,1,3,16,18,19,23,28,30,31)$



G:28 30
J:23 31

A: 0 1
D: 3 19
E:16 18

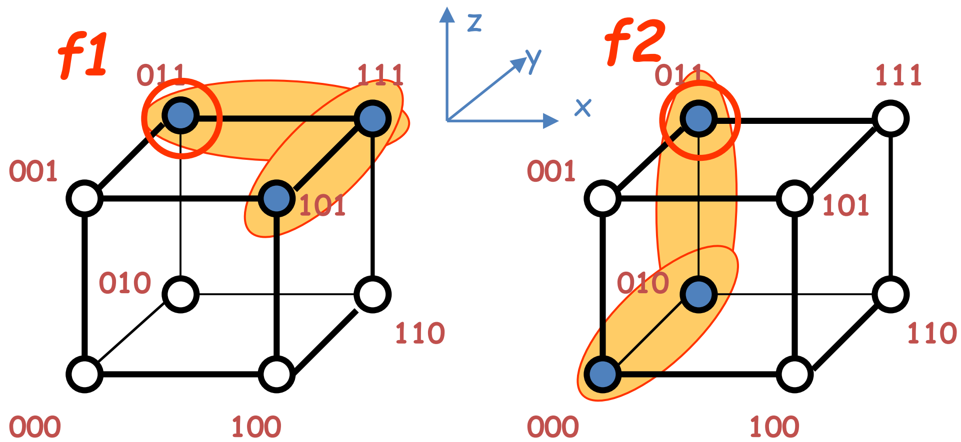
F=v'w'y'x'+v'w'x'z+w'x'y'z'+w'x'yz+vw'x'z'+vw'x'y+**vwxz'**+vwxy+vw'yz+**vxyz**

A B C D E F G H I J

G+J+A+D+E;

Generating Primes - multiple outputs

Example: $f1(x, y, z) = \sum m(3,5,7)$, $f2(x, y, z) = \sum m(0,2,3)$

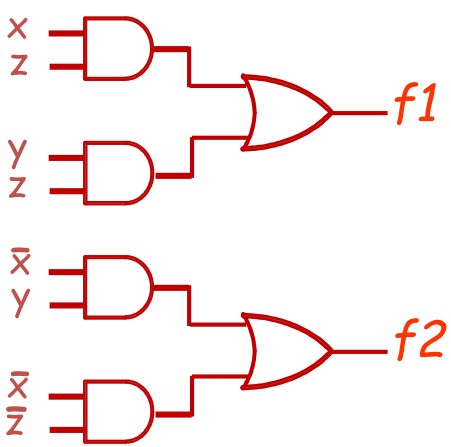


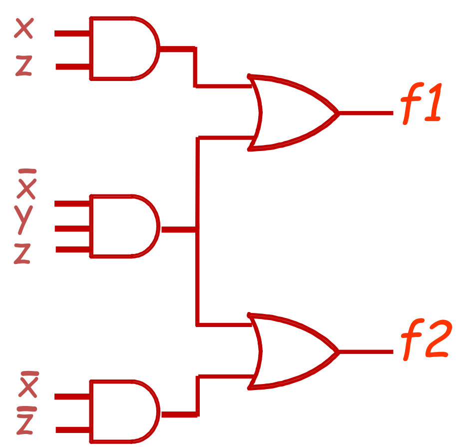
x y z	
0 1 1	- 1 1
1 0 1	1 - 1
1 1 1	

$f1 = yz + zx$

x y z	
0 0 0	0 - 0
0 1 0	0 1 -
0 1 1	

$f2 = \bar{x}\bar{z} + \bar{x}y$





The idea is that we can share terms:

- using separate optimizations 6 gates and G=12
- sharing a term 5 gates and G=11.

$f1\ f2 = (yz+zx)(\bar{x}\bar{z}+\bar{x}y)$

$f1\ f2 = \bar{x}yz$

Generating Primes - multiple outputs

- ✓ Theorem: if p_1 is a prime implicant for f_1, and p_2 is a prime implicant for f_2, then if $p_1.p_2 \neq 0$, $p_1.p_2$ is a prime implicant of $f_1.f_2$
- ✓ Theorem: if p_3 is a prime implicant for $f_1.f_2$, then there exist p_1 for f_1, and p_2 for f_2, such that $p_3=p_1.p_2$
- ✓ We can conclude that all prime implicants of $f_1.f_2$ are minimal sharable products for f_1 and f_2: and that all prime implicants for $f_1.f_2$ are created by products of prime implicants for f_1 and f_2
- ✓ The way to use this is to make the prime implicants of $f_1.f_2$ available to the minimizations of f_1 and f_2 by extending the table concept

Generating Primes - multiple outputs

- ✓ Procedure similar to single-output function, except: include also the primes of the products of individual functions

	f_1 minterms	f_2 minterms
Rows for f_1 prime implicants: mark only f_1 columns		
Rows for f_2 prime implicants: mark only f_2 columns		
Rows for $f_1 f_2$ prime implicants: mark both f_1 and f_2 columns		

Minimize multiple-output cover

✓ Example, cont.

f_1: $m_3 = 011$, $m_5 = 101$, $m_7 = 111$; $p_1 = y\,z$, $p_2 = x\,z$

f_2: $m_0 = 000$, $m_2 = 010$, $m_3 = 011$; $p_3 = \bar{x}\,y$, $p_4 = \bar{x}\,\bar{z}$

$f_1\,f_2$: $m_3 = 011$; $p_5 = \bar{x}\,y\,z$

Min cover has 3 primes:

$F = \{\, p_2, p_4, p_5 \,\}$

	m_3	m_5	m_7	m_0	m_2	m_3
p_1	✓		✓			
p_2		✓	✓			
p_3					✓	✓
p_4				✓	✓	
p_5	✓					✓

	m_3	m_3
p_1	✓	
p_3		✓
p_5	✓	✓

Note that selecting p_5 and removing all columns the marks coverage is complete