Example 2: Boolean Algebraic Proofs

✓ $AB + \overline{A}C + BC = AB + \overline{A}C$ Consensus Theorem

Proof Steps

$$AB + \overline{A}C + BC$$
$$= AB + \overline{A}C + 1 \cdot BC$$
$$= AB + \overline{A}C + (A + \overline{A}) \cdot BC$$
$$= AB + \overline{A}C + ABC + \overline{A}BC$$
$$= AB \cdot (1 + C) + \overline{A}C \cdot (1 + B)$$
$$= AB + \overline{A}C$$

✓ $(A+B) \cdot (\overline{A}+C) \cdot (B+C) = (A+B) \cdot (\overline{A}+C)$ Dual identity

Tison's method of iterated consensus

- ✓ The prime implicants are determined by generalization of the consensus operation which is performed systematically
- ✓ A property of these generalized consensus relations is that the consensus of two implicants of a formula gives another implicant
- ✓ A biform variable is a variable which occurs both positively and negatively in the formula
- ✓ For each biform variable x and for every pair of implicants D_i,D_j in the formula F, add the consensus of D_i and D_j with respect to the variable x and delete every subsumed implicants

Tison's method

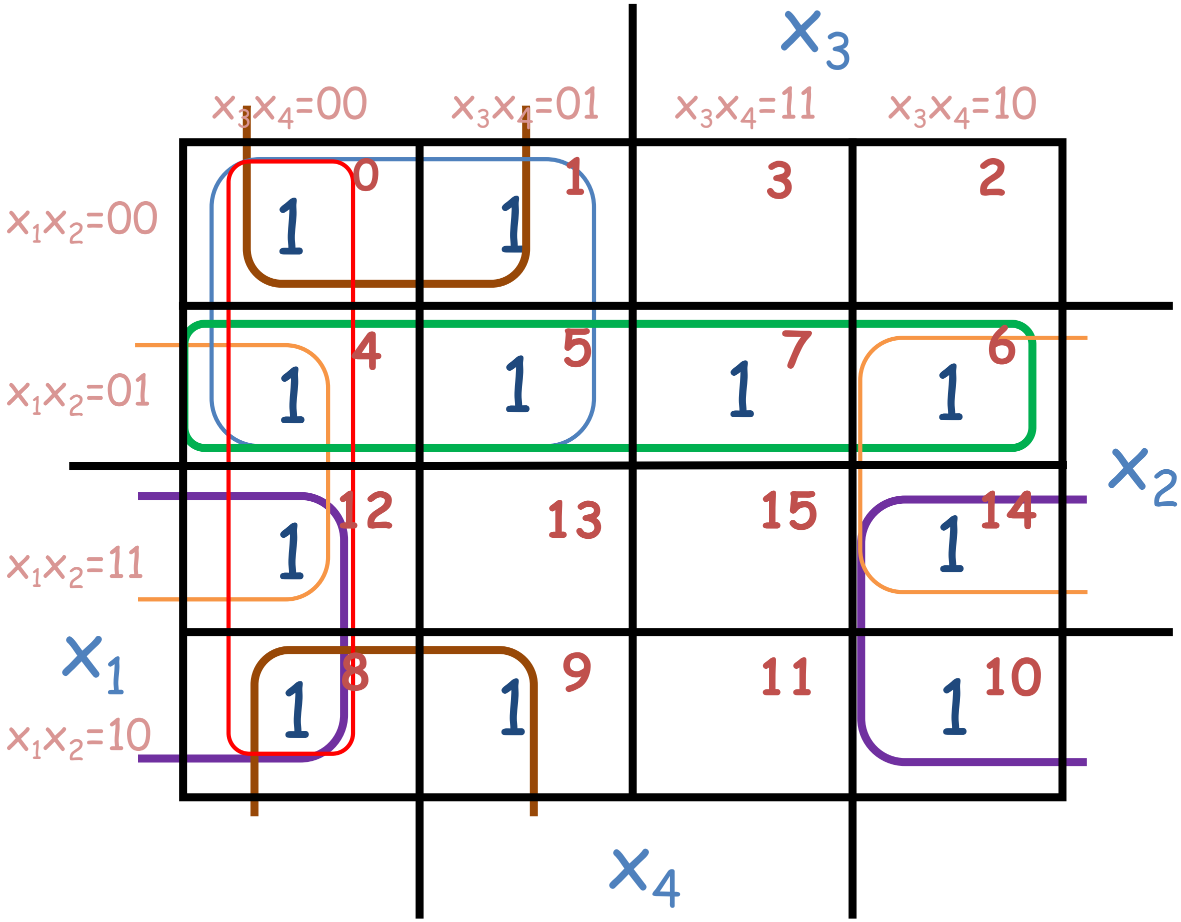
- When all possible consensi with respect to all possible biform variables are computed, and if all subsumed implicants are deleted, then *F* will contain **all and only** the prime implicants
- If a variable is a monoform one in a set, there is no consensus so that no resolution takes place with respect to that variable, and hence monoform variables are not considered
- The computational cost is influenced to a large extent by the order of the biform variable sequence used in resolution steps

Ordered implicant consensus method

- ✓ The number of implicants grows exponentially in general as the resolution process proceeds, which is the serious problem in computing time.
- ✓ The heuristic used is: the biform variable corresponding to the least number of consensi is considered first; and consensi with respect to this variable are performed.
- ✓ This helps the set of prime implicates to grow slowly at the early stages, which in turn reduces the computational cost

Example

✓ $F=x'_1x'_3x'_4+x'_1x'_2x'_3+x'_1x_2x_4+x_2x_3x'_4+x_1x'_3x'_4+x_1x'_2x'_3x_4+x_1x'_2x'_4$



$\sum_c=x'_1x_2+x'_2x'_3+x_1x'_4(+x'_1x'_3+x'_3x'_4+x_2x'_4)$

Example

- ✓ $F=x'_1x'_3x'_4+x'_1x'_2x'_3+x'_1x_2x_4+x_2x_3x'_4+x_1x'_3x'_4+x_1x'_2x'_3x_4+x_1x'_2x'_4$

  all variables are biformed, let us start by x_1

  Consensi to be added: $x'_3x'_4+x'_2x'_3x'_4+x'_2x'_3x_4+\sout{x'_2x'_3x'_4}$

  Eliminating subsamples:

  $F=x'_3x'_4+x'_2x'_3x'_4+x'_2x'_3x_4+x'_1x'_3x'_4+x'_1x'_2x'_3+x'_1x_2x_4+x_2x_3x'_4+x_1x'_3x'_4+x_1x'_2x'_3x_4+x_1x'_2x'_4$

- ✓ $F=x'_3x'_4+x'_2x'_3x_4+x'_1x'_2x'_3+x'_1x_2x_4+x_2x_3x'_4+x_1x'_2x'_4$

  let us go on by x_2

  Consensi to be added: $x'_1x'_3x_4+x_1x_3x'_4$

  Eliminating subsamples:

  $F=x'_1x'_3x_4+x_1x_3x'_4+x'_3x'_4+x'_2x'_3x_4+x'_1x'_2x'_3+x'_1x_2x_4+x_2x_3x'_4+x_1x'_2x'_4$

Example continued

✓ $F=x'_1x'_3x_4+x_1x_3x'_4+x'_3x'_4+x'_2x'_3x_4+x'_1x'_2x'_3+x'_1x_2x_4+$
$x_2x_3x'_4+x_1x'_2x'_4$

let us go on by x_3

Consensi to be added: $x_1x'_4+x_2x'_4$

Eliminating subsamples:

$+F=x_1x'_4+x_2x'_4+x'_1x'_3x_4+x_1x_3x'_4+x'_3x'_4+x'_2x'_3x_4+x'_1x'_2x'_3+$
$x'_1x_2x_4+x_2x_3x'_4+x_1x'_2x'_4$

✓ $F=x_1x'_4+x_2x'_4+\underline{x'_1x'_3x_4}+\underline{x'_3x'_4}+x'_2x'_3x_4+x'_1x'_2x'_3+x'_1x_2x_4$

let us go on by x_4

Consensi to be added:

$x_1x'_2x'_3+x'_1x_2x'_3+x'_1x_2+\underline{x'_1x'_3}+\underline{x'_2x'_3}+x'_1x_2x'_3$

Eliminating subsamples:

$\Sigma_C=x'_1x_2+x'_1x'_3+x'_2x'_3+x_1x'_4+x_2x'_4+x'_3x'_4$ Complete Sum

The optimum solution: the Petrick function

- ✓ Starting from the Complete Sum the minimal forms can be found through the Petrick function
- ✓ This function can be formulated on the basis of the set of implicants that cover each minterm
- ✓ Let us give an alphabetic name to the implicants:

  $\sum_c = x'_1 x_2 + x'_1 x'_3 + x'_2 x'_3 + x_1 x'_4 + x_2 x'_4 + x'_3 x'_4$

  A B C D E F

- ✓ Considering the sequence order $X_1\ X_2\ X_3\ X_4$ the implicant *A* covers the set of minterms {01★★} that is *A*={4,5,6,7},

  B= {0★0★}= {0, 1,4,5}; *C*={★00★}= {0,1,8,9}; D={1★★0}= {8,10,12,14}; E={★1★0}= {4,6,12,14}; F={★★00}= {0,4,8,12}.

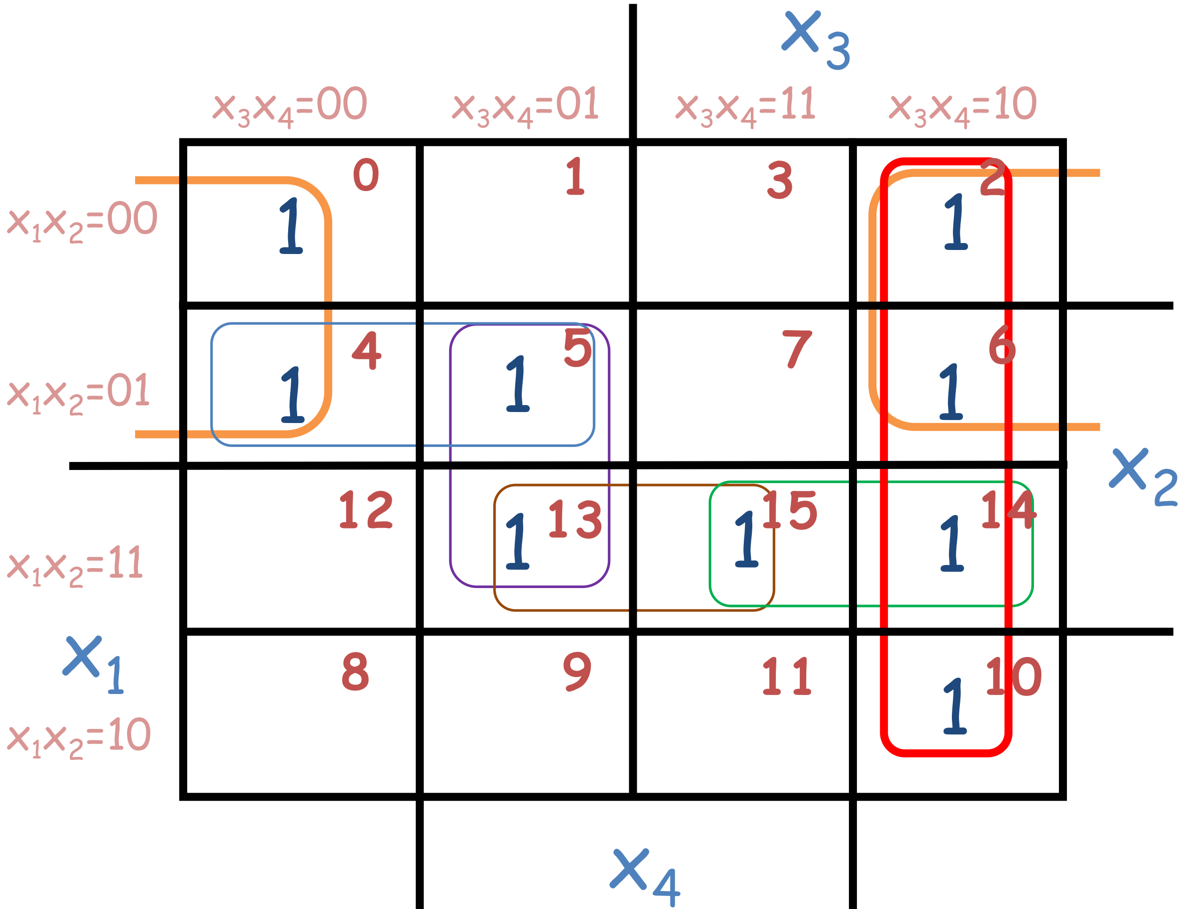
The optimum solution: the Petrick function

- The minterms m_0 can be covered by the implicants B or C or F; minterms m_1 by B or C; m_4 by A or B or E or F; m_5 by A or B; m_6 by A or E ; m_7 by A; m_8 by C or D or F; m_9 by C; m_{10} by D; m_{12} by D or E or F; m_{14} by D or E. Then:
- $P=(B+C+F)\,(B+C)\,(A+B+E+F)\,(A+B)\,(A+E)\,A\,(C+D+F)\,C\,D\,(D+E+F)\,(D+E) = ACD$
- The cover of the onset is then given by the implicants A, C and D so there is only one minimum boolean expression to the given formula, and is given by:

$$\sum_m = x'_1 x_2 + x'_2 x'_3 + x_1 x'_4 \quad \text{Minimum Sum}$$

Example

- ✓ $F=x'_1x'_3x'_4+x_1x_2x_3+x_2x'_3x_4+x'_2x_3x'_4+x_2x_3x'_4$



$\sum_c = x_3x'_4 + x'_1x'_4 + x_1x_2x_3 + x_2x'_3x_4 + x_1x_2x_4 + x'_1x_2x'_3$

Example

- ✓ $F=x'_1x'_3x'_4+x_1x_2x_3+x_2x'_3x_4+x'_2x_3x'_4+x_2x_3x'_4$

  all variables are biformed, let us start by x_1

  Consensi to be added: -

  let us go on by x_2

  Consensi to be added: $x_1x_3x'_4+x_3x'_4$

  Eliminating subsamples:

  $F=x_1x_3x'_4+x_3x'_4+x'_1x'_3x'_4+x_1x_2x_3+x_2x'_3x_4+x'_2x_3x'_4+x_2x_3x'_4$

- ✓ $F=x_3x'_4+x'_1x'_3x'_4+x_1x_2x_3+x_2x'_3x_4$

  let us go on by x_3

  Consensi to be added: $x'_1x'_4+x_1x_2x_4$

  Eliminating subsamples:

  $F=x'_1x'_4+x_1x_2x_4+x_3x'_4+x'_1x'_3x'_4+x_1x_2x_3+x_2x'_3x_4$

Example continued

- ✓ $F = x'_1x'_4 + x_1x_2x_4 + x_3x'_4 + x_1x_2x_3 + x_2x'_3x_4$

  let us go on by x_4

  Consensi to be added: $x'_1x_2x'_3 + x_1x_2x_3$

  Eliminating subsamples:

  $F = x'_1x_2x'_3 + x_1x_2x_3 + x'_1x'_4 + x_1x_2x_4 + x_3x'_4 + x_1x_2x_3 + x_2x'_3x_4$

Complete Sum

$\sum_c = x'_1x'_4 + x_3x'_4 + x'_1x_2x'_3 + x_1x_2x_3 + x_1x_2x_4 + x_2x'_3x_4$

The optimum solution: the Petrick function

- Let us give an alphabetic name to the implicants:

$$F = \underset{A}{x'_1x'_4} + \underset{B}{x_3x'_4} + \underset{C}{x'_1x_2x'_3} + \underset{D}{x_1x_2x_3} + \underset{E}{x_1x_2x_4} + \underset{F}{x_2x'_3x_4}$$

- Considering the sequence order $X_1\ X_2\ X_3\ X_4$ the implicant A covers the set of minterms {0,2,4,6}, B= {2,6,10,14}; C={4,5}; D={14,15}; E={13,15}; F={5,13}

0	2	4	5	6	10	13	14	15

- P=A(A+B)(A+C) (C+F) (A+B)B(E+F)(B+D)(D+E)
- P=AB(CE+DF+EF)=ABCE+ABDF+ABEF three solutions:

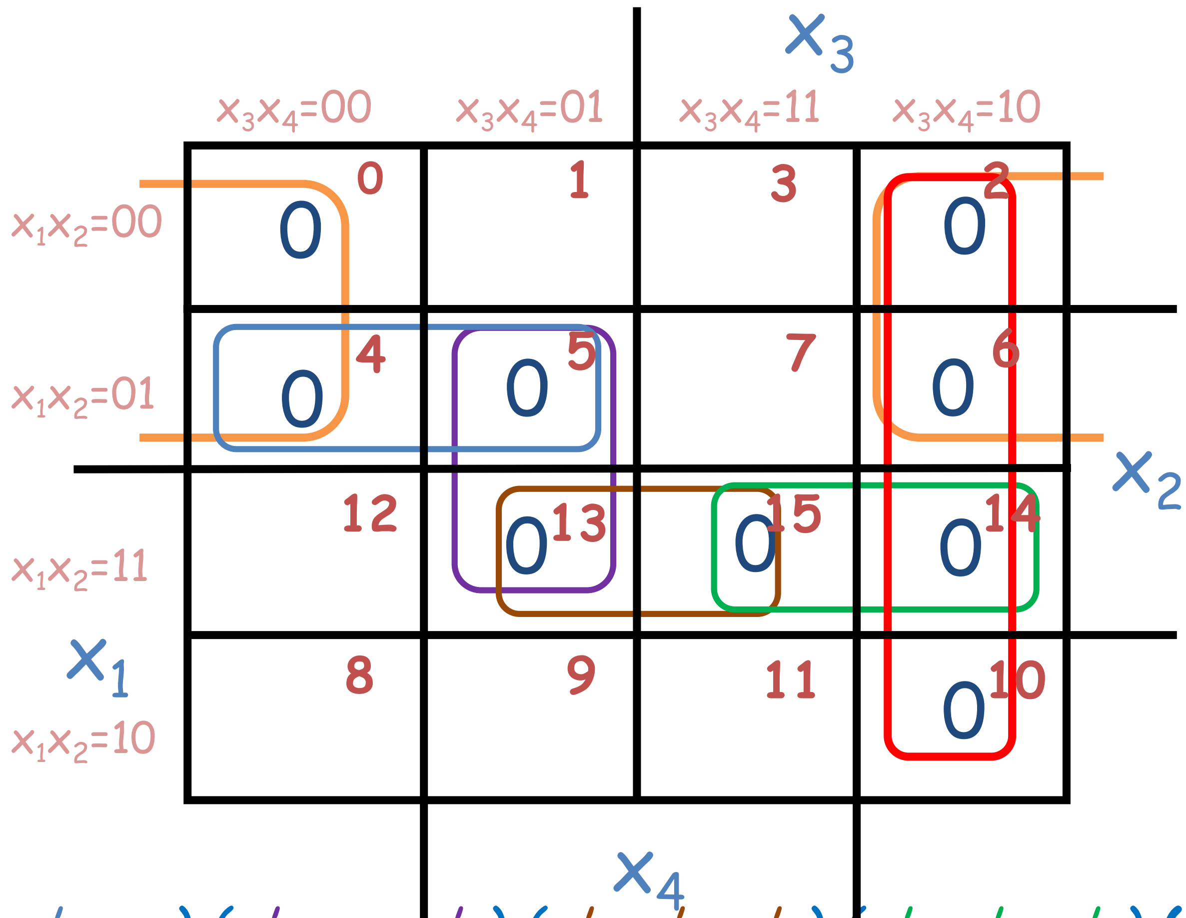
$$\sum m_{ABCE} = x'_1x'_4 + x_3x'_4 + x'_1x_2x'_3 + x_1x_2x_4 \qquad G=14,\ GE=17$$

$$\sum m_{ABDF} = x'_1x'_4 + x_3x'_4 + x_1x_2x_3 + x_2x'_3x_4 \qquad G=14,\ GE=17$$

$$\sum m_{ABEF} = x'_1x'_4 + x_3x'_4 + x_1x_2x_4 + x_2x'_3x_4 \qquad G=14,\ GE=17$$

Example with Maxterms

- ✓ $F=(x_1+x_3+x_4)\,(x'_2+x_3+x'_4)\,(x_2+x'_3+x_4)\,(x'_1+x'_2+x'_3)\,(x'_2+x'_3+x_4)$



$\prod_c = (x_1+x'_2+x_3)\,(x'_2+x_3+x'_4)\,(x'_1+x'_2+x'_4)\,(x'_1+x'_2+x'_3)\,(x'_3+x_4)\,(x_1+x_4)$

Example with Maxterms

- $F=(x_1+x_3+x_4)\,(x'_2+x_3+x'_4)\,(x_2+x'_3+x_4)\,(x'_1+x'_2+x'_3)\,(x'_2+x'_3+x_4)$

  all variables are biformed, let us start by x_1

  Consensi to be added: -

  let us go on by x_2

  Consensi to be added: $(x'_1+x'_3+x_4)\,(x'_3+x_4)$

  Eliminating subsamples:

  $F=(x'_1+x'_3+x_4)\,(x'_3+x_4)\,(x_1+x_3+x_4)\,(x'_2+x_3+x'_4)\,(x_2+x'_3+x_4)\,(x'_1+x'_2+x'_3)\,(x'_2+x'_3+x_4)$

- $F=(x'_3+x_4)(x_1+x_3+x_4)\,(x'_2+x_3+x'_4)(x'_1+x'_2+x'_3)$

Example continued

✓ F= $(x'_3+x_4)(x_1+x_3+x_4)\ (x'_2+x_3+x'_4)(x'_1+x'_2+x'_3)$

let us go on by x_3

Consensi to be added: $(x_1+x_4)\ (x'_1+x'_2+x'_4)$

Eliminating subsamples:

✓ F= $(x_1+x_4)(x'_1+x'_2+x'_4)(x'_3+x_4)(x_1+x_3+x_4)(x'_2+x_3+x'_4)\ (x'_1+x'_2+x'_3)$

✓ F= $(x_1+x_4)(x'_1+x'_2+x'_4)(x'_3+x_4)(x'_2+x_3+x'_4)(x'_1+x'_2+x'_3)$

let us go on by x_4

✓ Consensi to be added: $(x_1+x'_2+x_3)(x'_1+x'_2+x'_3)$

Eliminating subsamples:

F=$(x_1+x'_2+x_3)(x'_1+x'_2+x'_3)(x_1+x_4)(x'_1+x'_2+x'_4)(x'_3+x_4)(x'_2+x_3+x'_4)$
$(x'_1+x'_2+x'_3)$

Complete Product

$\Pi_c=(x_1+x'_2+x_3)(x'_1+x'_2+x'_3)(x_1+x_4)(x'_1+x'_2+x'_4)(x'_3+x_4)(x'_2+x_3+x'_4)$

The optimum solution: the Petrick function

- Let us give an alphabetic name to the implicates:

$\Pi_c=(x_1+x_4)(x'_3+x_4)(x_1+x'_2+x_3)(x'_1+x'_2+x'_3)(x'_1+x'_2+x'_4)(x'_2+x_3+x'_4)$

A B C D E F

- Considering the sequence order $X_1\ X_2\ X_3\ X_4$

  the implicate A covers the set of maxterms {0,2,4,6}, B= {2,6,10,14}; C={4,5}; D={14,15}; E={13,15}; F={5,13}

0 2 4 5 10 13 14 15

- P=A(A+B)(A+C) (C+F) (A+B) B (E+F)(B+D)(D+E)
- P=AB(CE+F)(D+E)=ABCE+ABDF+ABEF three solutions:

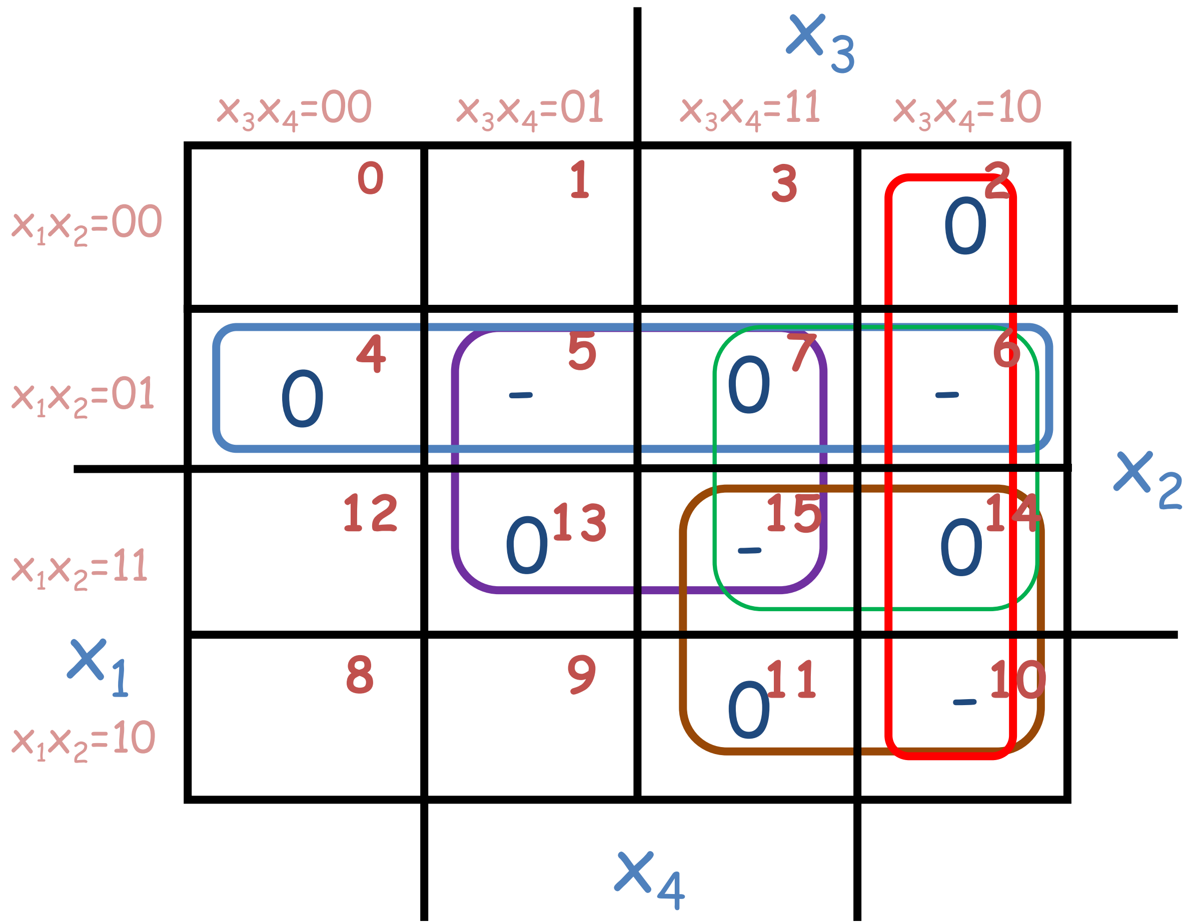
$\Pi_{M_{ABCE}}=(x_1+x_4)(x_3'+x_4)(x_1+x'_2+x_3)(x'_1+x'_2+x'_4)$ G=14, GE=18

$\Pi_{M_{ABDF}}=(x_1+x_4)(x_3'+x_4)(x'_1+x'_2+x'_3)(x'_2+x_3+x'_4)$ G=14, GE=18

$\Pi_{M_{ABEF}}=(x_1+x_4)(x_3'+x_4)(x'_1+x'_2+x'_4)(x'_2+x_3+x'_4)$ G=14, GE=18

Example with Don't Care

- ✓ $F(X_1, X_2, X_3, X_4) = \prod_M(2,4,7,11,13,14) \prod_d(5, 6, 10, 15)$



$\prod_C = (x_1+x'_2)\ (x'_2+x'_4)\ (x'_1+x'_3)\ (x'_3+x_4)\ (x'_2+x'_3)$

Example with Don't Care

- ✓ $F(X_1,X_2,X_3,X_4) = \prod_M(2,4,7,11,13,14) \prod_d(5,6,10,15)$

CONSIDER BOTH MAXTERMS AND DCs

- ✓ $F = (x_1+x_2+x'_3+x_4)(x_1+x'_2+x_3+x_4)(x_1+x'_2+x'_3+x'_4)$
$(x'_1+x_2+x'_3+x'_4)(x'_1+x'_2+x_3+x'_4)(x'_1+x'_2+x'_3+x_4)$
$\mathbf{(x_1+x'_2+x_3+x'_4)(x_1+x'_2+x'_3+x_4)(x'_1+x_2+x'_3+x_4)}$
$\mathbf{(x'_1+x'_2+x'_3+x'_4)}$

  all variables are biformed, let us start by x_1

  Consensi to be added: $(x_2+x'_3+x_4)(x'_2+x'_3+x'_4)(x'_2+x_3+x'_4)(x'_2+x'_3+x_4)$

  Eliminating subsamples:

- ✓ $F = (x_2+x'_3+x_4)(x_1+x'_2+x_3+x_4)(x'_2+x'_3+x'_4)(x'_1+x_2+x'_3+x'_4)(x'_2+x_3+x'_4)$
$(x'_2+x'_3+x_4)$

  let us go on by x_2

  Consensi to be added: $(x'_3+x_4)(x'_1+x'_3+x'_4)$

Example continued

Eliminating subsamples:

$F = (x'_3+x_4)(x'_1+x'_3+x'_4)(x_1+x'_2+x_3+x_4)(x'_2+x'_3+x'_4)(x'_2+x_3+x'_4)$

✓ $F = (x'_3+x_4)(x'_1+x'_3+x'_4)(x_1+x'_2+x_3+x_4)\underline{(x'_2+x'_3+x'_4)}\underline{(x'_2+x_3+x'_4)}$

let us go on by x_3

Consensi to be added: $(x_1+x'_2+x_4)$ ~~$(x'_1+x'_2+x'_4)$~~ $(x'_2+x'_4)$

Eliminating subsamples:

✓ $F = (x_1+x'_2+x_4)(x'_2+x'_4)(x'_3+x_4)(x_1+x'_2+x_3+x_4)(x'_2+x'_3+x'_4)(x'_1+x'_3+x'_4)(x'_2+x_3+x'_4)$

✓ $F = (x_1+x'_2+x_4)(x'_2+x'_4)(x'_3+x_4)(x'_1+x'_3+x'_4)$

let us go on by x_4

Consensi to be added: $(x_1+x'_2)(x'_2+x'_3)(x'_1+x'_3)$

Eliminating subsamples:

$\Pi_c = (x_1+x'_2)(x'_2+x'_3)(x'_2+x'_4)(x'_3+x_4)(x'_1+x'_3)$ Complete Product

The optimum solution: the Petrick function

- Let us give an alphabetic name to the implicates:

$$\prod_C = (x_1+x'_2)\,(x'_2+x'_4)\,(x'_1+x'_3)\,(x'_3+x_4)\,(x'_2+x'_3)$$

A B C D E

- Considering the sequence order $X_1\, X_2\, X_3\, X_4$ the implicate A covers the set of maxterms {4,5,6,7}, B={5,7,13,15}; C={10,11,14,15}; D={2,6,10,14}; E={6,7,14,15}
- **NOTE THAT ONLY MAXTERMS NEED TO BE COVERED!**

2 4 7 11 13 14

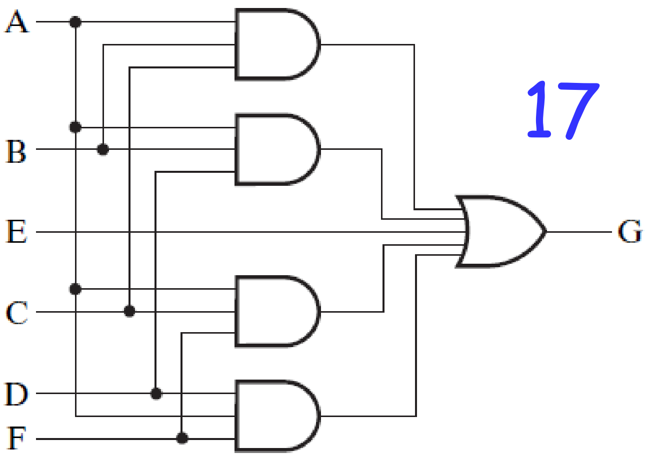
- P=DA(A+B+E) CB (C+D+E)=ABCD one solution:

$$\prod_{ABCD} = (x_1+x'_2)\,(x'_2+x'_4)\,(x'_1+x'_3)\,(x'_3+x_4)$$ G=12, GE=16

Multiple-Level Optimization

- ✓ Multiple-level circuits: circuits that are not two-levels (with or without input and/or output inverters)
- ✓ Multiple-level circuits can have reduced gate input cost compared to two-level (SOP and POS) circuits
- ✓ Multiple-level optimization is performed by applying transformations to circuits represented by equations while evaluating cost

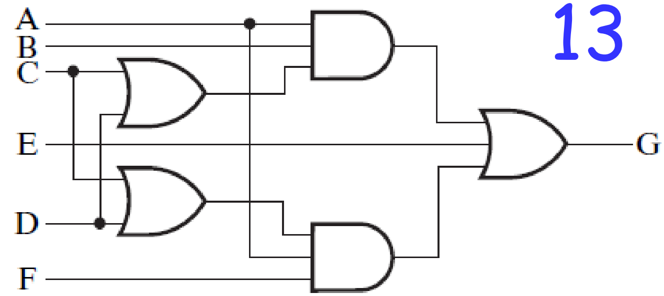
Multi level circuits – circuit analysis



17

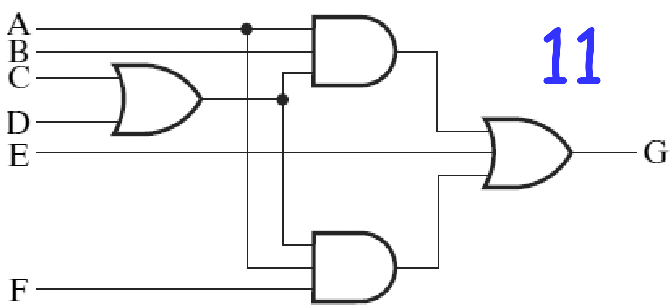
(a)

G = ABC+ABD+E+ACF+ADF



13

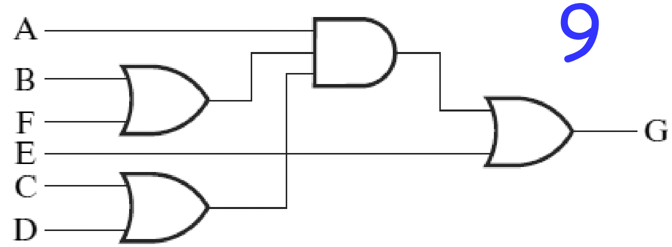
G = AB(C+D)+E+AF(C+D)



11

(c)

G = (C+D)(AB+AF)+E



9

(d)

G =A(C+D)(B+F)+E

Transformation Examples

- ✓ Algebraic Factoring

$$F = \bar{A}\,\bar{C}\,\bar{D} + \bar{A}\,B\,\bar{C} + ABC + AC\bar{D} \qquad G = 16$$

- Factoring:

$$F = \bar{A}\,(\bar{C}\,\bar{D} + B\bar{C}) + A\,(BC + C\bar{D}) \qquad G = 18$$

- Factoring again:

$$F = \bar{A}\,\bar{C}\,(B + \bar{D}) + AC\,(B + \bar{D}) \qquad G = 12$$

- Factoring again:

$$F = (\bar{A}\,\bar{C} + AC)\,(B + \bar{D}) \qquad G = 10$$

Transformation Examples

- ✓ **Decomposition** $F = (\bar{A}\,\bar{C} + AC)(B+\bar{D})$
  - The terms $B+\bar{D}$ and $\bar{A}\,\bar{C}+AC$ can be defined as new functions **E** and **H** respectively, decomposing F:

$$F = E\,H,\ E = B + \bar{D},\ \text{and}\ H = \bar{A}\bar{C} + AC \qquad G = 10$$

- ✓ Substitution of **E** into F
  - Returning to F just before the final factoring step:

$$F = \bar{A}\,\bar{C}\,(B + \bar{D}) + AC\,(B + \bar{D}) \qquad G = 12$$

  - Defining $E = B + \bar{D}$, and substituting in F:

$$F = \bar{A}\,\bar{C}\,E + ACE \qquad G = 10$$

  - This substitution has resulted in the same cost as the decomposition

Transformation Examples

✓ Elimination

Beginning with two functions: $X = B + C \quad Y = A + B$

$Z = \overline{A}\,X + C\,Y$ $\quad G = 10$

- Eliminating X and Y from Z:

$Z = \overline{A}\,(B + C) + C\,(A + B)$ $\quad G = 10$

- "Flattening" (Converting to SOP expression):

$Z = \overline{A}\,B + \overline{A}\,C + AC + BC$ $\quad G = 12$

- This has increased the cost, but has provided a new SOP expression for two-level optimization.
- Two-level Optimization

$Z = \overline{A}\,B + C$ $\quad G = 4$

- Increasing gate input count G temporarily can result in a final solution with a smaller G

Transformation Examples

✓ Extraction

- Beginning with two functions:

$E = \bar{A}\,\bar{B}\,\bar{D} + \bar{A}BD$

$H = \bar{B}C\bar{D} + BCD$ $\qquad G = 16$

- Finding a common factor and defining it as a function:

$F = \bar{B}\,\bar{D} + BD$

- We perform extraction by expressing E and H as the three functions:

$E = \bar{A}F,\ H = CF$ $\qquad G = 10$

- The reduced cost G results from the sharing of logic between the two output functions

Basic Transformations

- ✓ Factoring - finding a factored form from SOP or POS expression
- ✓ Decomposition - expression of a function as a set of new functions
- ✓ Substitution of G into F - expression function F as a function of G and some or all of its original variables
- ✓ Elimination of G into F - Inverse of substitution
- ✓ Extraction - decomposition applied to multiple functions simultaneously