A Growing Self-Organizing Network for Reconstructing Curves and Surfaces

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Abstract—Self-organizing networks such as Neural Gas, Growing Neural Gas and many others have been adopted in actual applications for both dimensionality reduction and manifold learning. Typically, the goal in these applications is obtaining a good estimate of the topology of a completely unknown subspace that can be explored only through an unordered sample of input data points. In the approach presented here, the dimension of the input manifold is assumed to be known in advance. This prior assumption can be harnessed in the design of a new, growing self-organizing network that can adapt itself in a way that, under specific conditions, will guarantee the effective and stable recovery of the exact topological structure of the input manifold.

I. INTRODUCTION

In the original Self-Organizing Map (SOM) algorithm by Teuvo Kohonen [1] a lattice of connected units is progressively made to assume a configuration that is representative of an input data distribution. During the learning process, the weight vectors – i.e. positions in the input space associated to each unit – are adapted by finding the unit that best matches each input and moving it ‘closer’ to that input together with a subset of neighboring units, to an extent that decreases with the distance on the lattice from the best matching unit. As the adaptation progresses, the SOM tends to represent the topology of the input data distribution in the sense that it maps inputs that are ‘close’ in the input space to units that are neighbors in the lattice.

In the Neural Gas (NG) algorithm [2], the topology of the network of units is not fixed, as it is with SOMs, but is learnt from the input distribution as part of the adaptation process. Martinetz and Schulten [3] have shown that, under certain conditions, the Neural Gas algorithm tends to construct a restricted Delaunay graph, namely a triangulation with remarkable topological properties to be discussed later. They deem the structure constructed by the algorithm a topology representing network (TRN).

Besides the thread of subsequent developments in the field of neural networks, the work by Martinetz and Schulten have also raised a considerable interest in the community of computational topology and geometry. Several theoretical results that followed in this direction are currently the basis of quite popular methods for curve and surface reconstruction in computer graphics [4], although these methods have little or nothing in common with neural network algorithms.

Perhaps the most relevant of these theoretical results, for the purposes of what follows, is the assessment of the possibility to reconstruct, from an input point sample drawn from a manifold, a structure that is homeomorphic to the manifold itself (see definitions below) via the competitive Hebbian rule of the NG algorithm. Homeomorphism, as we will see, is a stronger condition than being just restricted Delaunay and highly desirable, too. The result in point says that the critical aspect is the density of units in the network, which must be adequate – in a very precise sense – to the features of the input manifold.

A key difficulty, however, in making this result effective is that the density threshold involved can be explicitly computed, at a non-negligible cost, only when the input manifold is known. The main contribution of this work is the design of a new kind of growing, self-organizing neural network that progressively adapts the density of its units to an unknown input manifold until, under certain condition to be analyzed in detail, the structure becomes stably homeomorphic to it. The assessment of the new algorithm is supported by experimental evidence, which also shows its effectiveness for a large class of inputs and suggests its potential suitability.
II. RELATED WORK

The Neural Gas algorithm [2] adopts the so-called competitive Hebbian rule as the basic method for establishing connections among units: for each input, a connection is added, if not already present, between the closest and second-closest units, according to the metric of choice. In order to cope with the mobility of units during the leaning process, an aging mechanism is also introduced: at each input, the age of the connection between the closest and second-closest units, if already present, is set to zero, while the age of other connections is increased by one. Connections whose age exceeds a given threshold will eventually be removed.

As proven in [5], the NG algorithm obeys a stochastic gradient descent on a function based on the average of the geometric quantization error. As known, this is not true of a SOM [6], whose learning dynamics does not minimize an objective function of any sort. This property of NG relies on how the units in the network are adapted: at each input, the units in the network are first sorted according to their distance and then adapted by an amount that depends on their ranking.

A well-known development of the NG algorithm is the Growing Neural Gas (GNG) [7]. In GNG, as the name suggests, the set of units may grow (and shrink) during the adaptation process. Each unit in a GNG is associated to a variable that stores an average value of the geometric quantization error. At fixed intervals, the unit with the largest average error is detected and a new unit is created between the unit itself and the neighbor unit having the second-largest average error. In this way, the set of units grows progressively until a maximum threshold is met. Units in GNG are also removed, when they become unconnected as a result of connection aging. Adaptation in the GNG algorithm is limited to the winning unit, i.e. the one being closest to the input, and its connected neighbors. This is meant to avoid the expensive operation required in the NG algorithm to establish the ranking of each unit. By this, the time complexity $O(N \log N)$ of each NG iteration reduces to linear time in the number of units $N$. This saving comes at a price, however, as the convergence assessment of NG described in [5] does not apply anymore. The GNG algorithm is almost identical to the Dynamic Cell Structure by Bruske and Sommer [8]. The GNG algorithm has also been applied to surface reconstruction and mesh optimization [9], [10].

The Grow-When-Required (GWR) algorithm [11] is a development of GNG. In a GWR, the error variable associated to each GNG unit is replaced by a firing counter $f$ that decreases exponentially each time the unit is winner. When $f$ gets below a certain threshold $T_f$, the unit is deemed habituated and its behavior changes: an habituated unit being closest to the input is adapted only if the distance between the two is below a certain threshold $T$, otherwise a new unit is created. This means that, in general, the network continues to grow until the input data sample is completely included in the union of balls of radius $R$ centered in each unit.

III. TOPOLOGICAL INTERLUDE

This section describes a conceptual pathway that summarizes the relevant theoretical results and leads to the fundamental Theorem 3.5. All basic definitions given here will be necessarily quite concise; further information can be found in textbooks such as [12] and [13].

Two (topological) spaces $X$ and $Y$ are said to be homeomorphic, denoted as $X \approx Y$, if there is a function $f : X \rightarrow Y$ that is bijective, continuous and has a continuous inverse. A (sub)space $M \subseteq \mathbb{R}^d$ is a closed, compact $k$-manifold if every point $x \in M$ has a neighborhood (i.e. an open set including it) that is homeomorphic to $\mathbb{R}^k$. In other words, a $k$-manifold is a (sub)space that locally ‘behaves’ like $\mathbb{R}^k$.

The Voronoi cell of a point $p$ in a set of points $L$ is:

$$V_p = \{ x \in \mathbb{R}^d \mid \|x - p\| \leq \|x - q\|, \forall q \in L, q \neq p \}$$

The intersection of two Voronoi cells in $\mathbb{R}^d$ may be either empty or a linear face of dimension $d - 1$. Likewise, the intersection of $n$ such cells is either empty or a linear face of dimension $d - n + 1$, with a minimum of 0. For example, the intersection of $d$ cells is either empty or an edge (i.e. dimension 1) and the intersection of $d + 1$ cells is either empty or a single point (i.e. dimension 0). The Voronoi cell together with the faces of all dimensions form the Voronoi complex $\mathcal{V}(L)$. Fig. 2(a) shows an example of a Voronoi complex for a set of points in $\mathbb{R}^2$. From now on, we will call landmarks the points in $L$. 
A finite set of landmarks \( L \) in \( \mathbb{R}^d \) is said to be non degenerate if no \( d + 2 \) Voronoi cells have a non-empty intersection.

The Delaunay graph of a finite set \( L \) is dual to the Voronoi complex, in the sense that two landmarks \( p \) and \( q \) in \( L \) are connected in the Delaunay graph iff the intersection of the two corresponding Voronoi cells is not empty. The Delaunay simplicial complex \( \mathcal{D}(L) \) is defined by extending the above idea to simplices of higher dimensions: a face \( \sigma \) of dimension \( n \) is in \( \mathcal{D}(L) \) iff the intersection of the Voronoi cells corresponding to the \( n+1 \) landmarks in \( \sigma \) is non-empty. If the set \( L \) is non-degenerate, \( \mathcal{D}(L) \) will contain simplices of dimension at most \( d \). Fig. 2(b) shows the Delaunay graph corresponding to the Voronoi complex in Fig. 2(a).

A. Restricted Delaunay complex

The concept of restricted Delaunay graph was already defined in [3], albeit with a slightly different terminology. Let \( M \subseteq \mathbb{R}^d \) be a manifold of dimension \( k \). The restricted Voronoi cell of \( p \) w.r.t a manifold \( M \) is:

\[
V_{p, M} = V_p \cap M
\]

The restricted Delaunay graph of \( L \) with respect to \( M \) is a graph where landmarks \( p \) and \( q \) are connected iff \( V_{p, M} \cap V_{q, M} \neq \emptyset \). Fig. 2(c) shows the restricted Delaunay graph with respect to a closed curve. Note that the restricted Delaunay graph is a subset of the Delaunay graph for the same set of landmarks. The restricted Delaunay simplicial complex \( \mathcal{D}_M(L) \) is the simplicial complex obtained from the restricted Voronoi complex \( \mathcal{V}_M(L) \).

As the example in Fig. 2(c) shows, the restricted Delaunay graph is not necessarily homeomorphic to the closed curve \( M \). For instance, landmark \( q \) has only one neighbor, instead of two, whereas \( p \) has four. This means that the piecewise-linear curve \( \mathcal{D}_M(L) \) does not represent the non-intersecting, closed curve \( M \) correctly. The critical aspect, as we will see, is the density of \( M \).

![Fig. 3: The medial axis (dashed line) of a closed curve and the local feature size at a point \( x \).](image)

B. Homeomorphism and \( \varepsilon \)-sample

The medial axis of \( M \) is the closure of the set of points that are the centers of maximal balls, called medial balls, whose interiors are empty of any points from \( M \). The local feature size at a point \( x \in M \) is the distance of \( x \) from the medial axis. The global feature size of a compact manifold

\[
\ellfs(x) \leq \frac{\|x-p\|}{2} \quad \text{for all } x \in M, \quad p \in L
\]

is the distance of \( x \) to the nearest landmark. The critical aspect is that such positive values exist. The effect of theorem 3.1 are reported in the literature. It will suffice for our purposes that such positive values exist. The effect of the above theorem is exemplified in Fig. 2(d): as the density of \( L \) increases, the restricted Delaunay graph becomes homeomorphic to the curve.

C. Witness complex

The notion of a witness complex is tightly related to the competitive Hebbian rule described in [3] and, at least historically, descends from the latter.

Given a set of landmarks \( L \subseteq \mathbb{R}^d \), a weak witness of a simplex \( \sigma \subseteq L \) is a point \( x \in \mathbb{R}^d \) for which \( \|x - p\| \leq \|x - q\| \) for all \( p \in \sigma \) and \( q \in L - \sigma \). A strong witness is a weak witness for which \( \|x - p\| = \|x - q\| \) for all \( p, q \in \sigma \). Note in passing that all points belonging to the intersection of two Voronoi cells \( V_p \) and \( V_q \) are strong witnesses for the edge \((p, q)\). The set of all weak witnesses for this same edge is also called the second order Voronoi cell of the two points \( p \) and \( q \) [5].

The witness complex \( \mathcal{W}(L) \) is the simplicial complex that can be constructed from the set of landmarks \( L \) with a specific set of witnesses \( W \).
Theorem 3.2: Let $L \subset \mathbb{R}^d$ be a set of landmarks. If every face of a simplex $\sigma \subseteq L$ has a weak witness in $\mathbb{R}^d$, then $\sigma$ has a strong witness $\mathbb{R}^d$ [15].

As a corollary, this theorem implies that $C^W(L) \subseteq D(L)$ and in the limit the two complexes coincide, when $W \equiv \mathbb{R}^d$.

Theorem 3.2 does not extend to restricted Delaunay complexes, namely when the witnesses belong to a manifold $M$. A counterexample is shown in Fig. 4(a), which describes the witness complex constructed from the same $L$ as in Fig. 2(d) using the whole $M$ as the set of witnesses. The second-order Voronoi regions (in blue) for the connections (in orange) which do not belong to the restricted Delaunay graph, do intersect the closed curve. This means that $M$ contains weak witnesses for connections for which $M$ itself contains no strong witnesses, as the definition of $D_M(L)$ would require.

Theorem 3.3: If $M$ is a compact smooth manifold without boundary of dimension 1 or 2, there exists a $\varepsilon > 0$ such that if $L$ is an $\varepsilon$-sample of $M$, the following implication holds: if every face of a simplex $\sigma \subseteq W$ has a weak witness, then $\sigma$ has a strong witness [16].

Like before, this implies that $C^W_M(L)$, i.e. the witness complex restricted to $M$, is included in $D_M(L)$ and the two coincide in the limit, when $W \equiv M$. As a corollary, Theorem 3.3 also implies that $D_M(L) \approx M$. Specific, viable values for $\varepsilon$ are reported in [16], but once again it will suffice to our purposes that such positive values exist.

The effect of theorem 3.3 is described in Fig. 4(c) and (d): by further increasing the density of $L$, the witness complex $C^W_M(L)$ coincides with $D_M(L)$, when $W$ is the entire curve. Fig. 4(c) shows that, as the density of $L$ increases, the second-order Voronoi regions for the violating connections ‘move away’ from $M$ and tend to aggregate around the medial axis.

The other side of the coin is represented by:

Theorem 4.4: For manifolds of dimension greater than 2, no positive value of $\varepsilon$ can guarantee that every face of a simplex $\sigma \in C^W_M(L)$ having a weak witness also has a strong witness [17].

D. Finite sets of witnesses and noise

As fundamental as it is, Theorem 3.3 is not per se sufficient for our purposes. First, it assumes the potential coincidence of $W$ with $M$, which would require the set of witnesses to be infinite, and, second, its proof requires that $L \subset M$, which is not necessarily true with vector quantization algorithms and/or in the presence of noise.

An $\varepsilon$-sparse sample $L$ is such that the pairwise distance between any two points of $L$ is greater than $\varepsilon$. A $\delta$-noisy sample $L$ is such that no points in $L$ are farther than $\delta$ from $M$.

Theorem 3.5: If $M$ is a compact smooth manifold without boundary of dimension 1 or 2, there exist positive values of $\varepsilon$ and $\delta$ such that if $W$ is a $\delta$-noisy $\delta$-sample of $M$ and $L$ is a $\delta$-noisy, $\varepsilon$-sparse ($\varepsilon + \delta$)-sample of $M$ then $C^W_M(L) \subseteq D_M(L)$ [18].

Actually, a stronger result holds in the case of curves, stating that $C^W_M(L)$ and $D_M(L)$ will coincide. For surfaces this guarantee cannot be enforced due to a problem that is described in Fig. 5. Even if $L$ is non-degenerate, in fact, it may contain landmarks that are arbitrarily close to a co-circular configuration, thus making the area of the second-order Voronoi region in the midst so small that no density condition on $W$ could ensure the presence of an actual witness. This means that, in a general case, a few connections will be missing from $C^W_M(L)$.

Fig. 6: The star, closure and link of a landmark in simplicial complexes of dimensions 1 (above) and 2 (below).

IV. SELF-ORGANIZING ADAPTIVE MAP (SOAM)

The intuitive idea behind the SOAM algorithm introduced here, which is derived from the GWR algorithm [11], is pretty simple. In passing and as already implied, we assume that the units in the network are each associated to a landmark, or weight vector. Neighborhoods like those of landmarks $p$ and $q$ in Fig. 2(c) are true symptoms of the fact that the network structure is not homeomorphic to the input manifold $M$. These singularities can be easily detected, provided that the expected kind of neighborhood – which depends on the dimension of $M$, is known.

The self-organizing network reacts to detected singularities by just creating more units, until eventually it reaches a density that makes all singularities disappear. Provided that both the input manifold and the point sample drawn from it have the required properties, the theoretical results presented guarantee that such a bounded density exists. Remarkably, density adaptations need only be local, as the target density is function of the local feature size of $M$.

A. Topology-driven state transitions

The state of each unit in a SOAM is determined by the value of its firing counter, as with the GWR algorithm, and the topology of its neighborhood.
Given a unit $i$ with associated landmark $p_i$, its neighborhood in a simplicial complex $C(L)$ is defined as the star $\text{St}(p_i)$, consisting of $p_i$ together with all simplices having $p_i$ as one of their vertices. The closure of a star $\text{Cl}(\text{St}(p_i))$ is obtained by adding all simplices in $C(L)$ having a non-null intersection with $\text{St}(p_i)$. The link is defined as (see Fig. 6)

$$\text{Lk}(p_i) = \text{Cl}(\text{St}(p_i)) - \text{St}(p_i)$$

The eight possible states for the units of a SOAM are:
- **active**: The default state of any newly-created unit.
- **habituated**: The value of the firing counter $f_i$ (see algorithm) of the unit is greater than a predefined threshold $T_f$.
- **connected**: The unit is habituated and all the units in its link are habituated as well.
- **half-disk**: The link of the unit is homeomorphic to a half-sphere.
- **disk**: The link of the unit is homeomorphic to a sphere.
- **boundary**: The unit is an half-disk and all its neighbors are regular (see below).
- **patch**: The unit is a disk and all its neighbors are regular.
- **singular**: The unit is over-connected, i.e. its link contains a sphere plus some other units and connections.

The actual combinatorial predicates to be applied to the links depend on the expected dimension of the input manifold $M$. For instance, in Fig. 6 the two links above and below on the right are each homeomorphic to a sphere of dimensions 1 and 2, respectively.

In the case of expected dimension 2, a unit whose link is that of a disk but contains three units only is a tetrahedron and, by definition, is considered singular as well.

Two further, derived state definitions are convenient for describing the SOAM algorithm:
- **regular**: The unit is in one of these states: half-disk, disk, boundary or patch.
- **stable**: In this version of the algorithm, only the patch state is deemed stable.

Fig. 7: Topology-driven state transitions for units in a SOAM. The same state colors are used in other figures.

Fig. 8: The overall model of habituation and dishabituation of insertion thresholds, with different time constants $\tau$.

**B. Adaptive insertion thresholds**

The GWR algorithm adopts a model for the exponential decay of the firing counter $f_i$ that has been derived from the biophysics of the habituation of neural synapses [11]. The equation that rules the model is:

$$h(t) = H - \frac{1}{\alpha} \cdot (1 - e^{(-\alpha \cdot t)/\tau})$$

with $\alpha = 1/(H - h_{min})$, where $h(t)$ is the value being adapted, at time $t$, $H$ and $h_{min}$ are the maximum and minimum values, respectively, and $\tau$ is a suitable time constant. The reverse model, of dishabituation, is also considered here:

$$h(t) = H - \frac{1}{\alpha} \cdot e^{(-\alpha \cdot t)/\tau}$$

In the SOAM algorithm the overall model of habituation and dishabituation is also applied to local insertion thresholds $r_i$ of units. More precisely, as shown in Fig. 8, the value of $r_i$ for singular units decays exponentially to a value $r_{min}$, while the state persists, whereas the value of $r_i$ of regular units grows asymptotically to the initial value $R$.

Fig. 9: A SOAM reconstructing the Stanford bunny. In the final structure, all 435 units are in the stable state.
C. The SOAM algorithm

Let $U$ be the set of network units. Initially, $U$ contains two units only, each with an associated landmark $p_1$ assigned at random, according to $P(\xi)$. The set $C$ of connections is empty. All firing counters $f_i$ and insertion thresholds $r_i$ of newly created units are initialized to $F$ and $R$, respectively.

1) Receive the next sample $\xi$ from the input stream.
2) Determine the two units $b$ and $s$ whose landmarks are closest and second-closest to $\xi$, respectively

$$b = \arg \min_{i \in U} ||\xi - p_i||$$
$$s = \arg \min_{i \in U-b} ||\xi - p_i||$$

3) Add the connection $(b, s)$ to $C$, if not already present, otherwise set its age to 0.
4) Unless unit $b$ is stable, increase by one the age of all its connections. Remove all connections of unit $b$ whose age exceeds a given threshold $T_{age}$. Then, remove all units that remain isolated.
5) If unit $b$ is at least habituated and $||\xi - p_b|| > r_b$
   - Create a new unit $v$ and add it to $U$
   - Set the new landmark $p_v = (p_b + \xi)/2$
   - Add two new connections $(b, v)$ and $(v, s)$ to $C$
   - Remove the connection $(b, s)$ from $C$
6) Adapt the firing counters of $b$ and of all units $n(b)$ that are connected to $b$

$$\Delta f_b = (\alpha_b \cdot (F - f_b) - 1)/\tau_f$$
$$\Delta f_{n(b)} = (\alpha_b \cdot (F - f_{n(b)}) - 1)/\tau_{f,n}$$

7) Update the state of $b$, according to the value of $f_b$ and the topology of its neighborhood of connected units.
8) If the unit $b$ is singular, adapt its insertion threshold

$$\Delta r_b = (\alpha_r \cdot (R - r_b) - 1)/\tau_{r, hab}$$

Otherwise, if the unit $b$ is regular, adapt its insertion threshold in the opposite direction

$$\Delta r_b = (\alpha_r \cdot (R - r_b))/\tau_{r, dis}$$

9) Unless $b$ is stable, adapt its landmark and those of all connected units $n(b)$

$$\Delta p_b = \eta_b \cdot f_b \cdot (\xi - p_b)$$
$$\Delta p_{n(b)} = \eta_{b,n} \cdot f_{n(b)} \cdot (\xi - p_{n(b)})$$

Otherwise, if $b$ is stable, adapt only the landmark of $b$ itself

$$\Delta p_b = \eta_{stable} \cdot f_b \cdot (\xi - p_b)$$

where, typically, $\eta_{stable} \ll \eta_b$.
10) If further inputs are available, return to step (1), unless some termination criterion has been met.

Fig. 10: A SOAM reconstructing the heptoroid, which has genus 22. The final structure includes 10,103 units and has been obtained after some 8M input signals.

V. Experimental evidence

Practical experiments have been performed to assess the properties of the SOAM algorithm, in particular with surfaces, as this is the most complex case. The test set adopted includes 32 triangulated meshes, obtained from the AIM@SHAPE repository [19], having a genus (i.e. the number of tori whose connected sum is homeomorphic to the given surface [13]) ranging from 0 to 65. All meshes were rescaled to make each bounding box have a major size of 256. In experimental runs, mesh vertices were used as input points and selected at random with uniform distribution.

Homeomorphism between a stable SOAM structure and a triangulated mesh can be easily verified: two closed, triangulated surfaces are homeomorphic if they are both orientable (or non-orientable) and have the same Euler characteristic [13]:

$$\chi = \text{vertex} - \text{edges} + \text{faces}$$

For completeness, also a few tests have been performed with non-orientable manifolds (e.g. the Klein bottle), directly sampled from parametric expressions.

Fig. 11: Typical growth and stabilization of a SOAM during the reconstruction of the Stanford bunny. Each line charts the number of units in the state corresponding to the color.
A. Topological convergence

For the whole test set of triangulated meshes and in the absence of noise, it has been possible to make the SOAM reach a condition where all units were stable and the resulting structure was homeomorphic to the input mesh.

A typical experimental run of a SOAM is charted in Fig. 11, which corresponds the reconstruction process in Fig. 9. The input mesh in this case includes 72,027 vertices. As we can see, the number of stable units (dark red line) in the SOAM grows progressively until a level of equilibrium is reached and maintained from that point on. In contrast, singular units (blue line) tend to disappear, although occasional onsets may continue for a while, for reasons that will be explained in short. Needless to say, in the most complex test cases, a similar level of equilibrium was reached only after a much larger number of signals, but essentially the same behaviour has been observed in all successful runs.

B. Edge flipping

When constructing a stable witness complex, the aging mechanism adopted in the NG algorithm [2] and its successors may represent a problem. The aging mechanism, in fact, is meant to cope with the mobility of landmarks during the adaptation process: any connection becoming unsupported by witnesses will age rapidly and eventually be removed. The drawback is that, in general, the probability of removing a connection due to aging is inversely proportional to that of sampling a supporting witness, which can be small without being zero. In borderline cases, e.g. when four landmarks approach co-circularity, connections are repeatedly removed and created in a sort of ‘blinking’ behavior.

In step 4 of the algorithm, the aging mechanism is simply stopped for connections between stable units. From that point on, any individual witness for an extra connection will be stopped for connections between stable and created in a sort of ‘blinking’ behavior. A possible explanation is that the mobility of landmarks makes it highly improbable the occurrence of false positives, intended as stable network configurations that do not represent M.

The values $\eta_b = 0.05$ and $\eta_{b,n} = 0.0005$ are relatively high, as it emerged that an increased mobility of landmarks accelerates the process. Actually, this higher mobility also proved to be the way the problem described in Fig. 5 could be avoided or greatly limited. Only with much smaller values of $\eta_b$ and $\eta_{b,n}$, in fact, persistent small ‘holes’ were observed in the network. A possible explanation, to be fully investigated yet, is that the mobility of landmarks acts like a ‘simulation of simplicity’ [20], that is, a perturbation that makes low-probability configurations, like co-circularity, ineffective.

Time constants for firing counters, $\tau_f = 3.33$ and $\tau_{f,n} = 14.33$, and the habituation threshold $T_f = 0.243$ were chosen initially and never changed afterwards, as their effect is altogether limited. This is true also for the values $\tau_{r,dis} = 9$, $T_{age} = 30$ and $\eta_{stable} = 0.0002$. In contrast, the value of the time constant $\tau_{r,hab} = 3$ has a more relevant impact, as lower values may produce an excessive density and greater ones may slow down the process significantly.

D. When adaptation fails

Two kinds of symptoms may signal the failure of the adaptation process:

a) Persistence of connected units
b) Persistence of both singular and connected units

The causes of a) could be twofold. In the light of Theorem 3.5, either the input data set is not a $\delta$-sample of $M$, i.e. not dense enough, or the value of $r_{min}$ is too low and the SOAM eventually became too dense.

Barring defects in $M$ (see also Fig. 13), symptom b) is typically due to noise. In agreement with the same theorem above, experiments show that the effect of noise is ‘on-off’: if the input data set is $\delta$-noisy, i.e. the noise is bounded and low, the SOAM continues to converge albeit at a much slower speed; beyond that point, symptom b) will almost surely occur. Given that the actual noise threshold depends on
the global feature size of $M$, its value may be very difficult to compute beforehand. The symptoms deriving from it, however, are very easily detected and hence an experimental estimation should be feasible, in many cases.

VI. CONCLUSIONS AND FUTURE WORK

The design of the SOAM algorithm relies heavily on the theoretical corpus presented. On the other hand, the algorithm retains the inherent simplicity of its predecessors, in that it does not require complex computations beyond scalar products and combinatorial predicates. Through these, the SOAM yields a reliable reconstruction of an unknown manifold with stronger topological guarantees.

True, many theoretical aspects still remain to be clarified. For one, the guarantee that a SOAM will eventually reach a stable configuration for any non-defective input data set remains to be proved. Nevertheless, in the author’s opinion, a strong point of the SOAM algorithm is that it produces clear signs of a convergence – or lack thereof – whose correctness is verifiable. In addition, experiments also seem to suggest that the algorithm is less exposed to known reconstruction problems [18] than could be expected.

About future developments, the current limitations in the dimension of the manifold might be overcome along the lines suggested in [21], although this would require switching to the weighted version of the Delaunay complex. Another great potential of the SOAM algorithm may relate to tracking non-stationary input manifolds, which is the reason why, in the design of the algorithm, care has been taken to keep all transitions reversible.

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