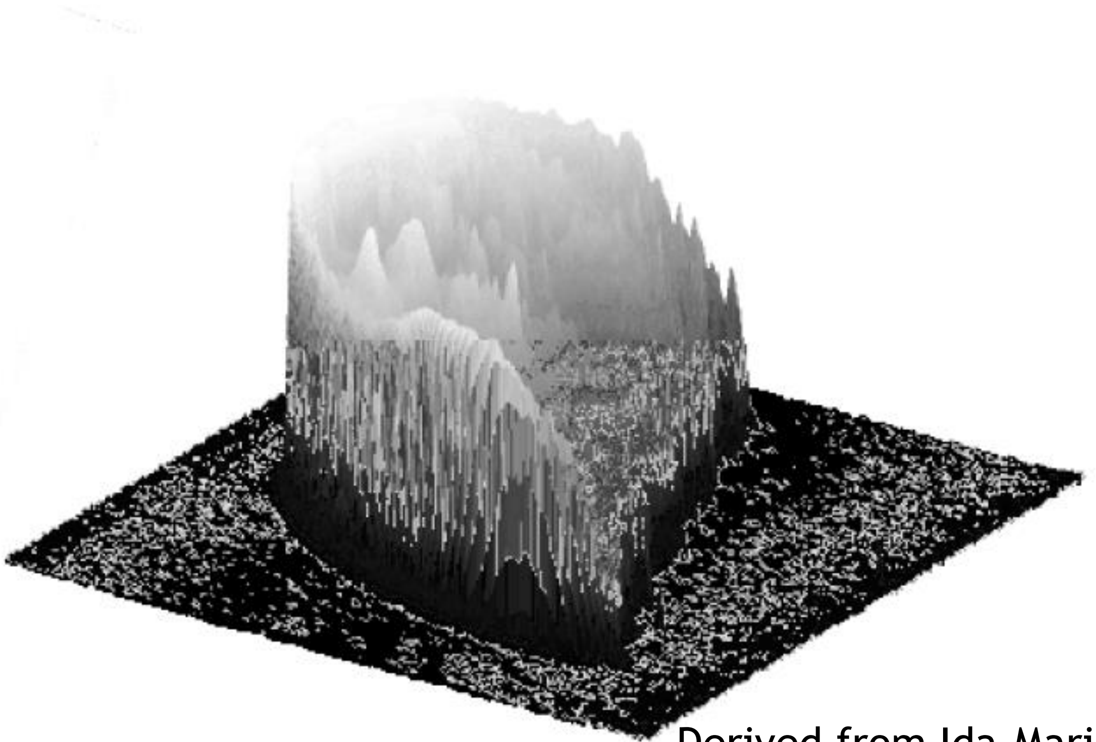
Mathematical Morphology

Grayscale Images and 3D

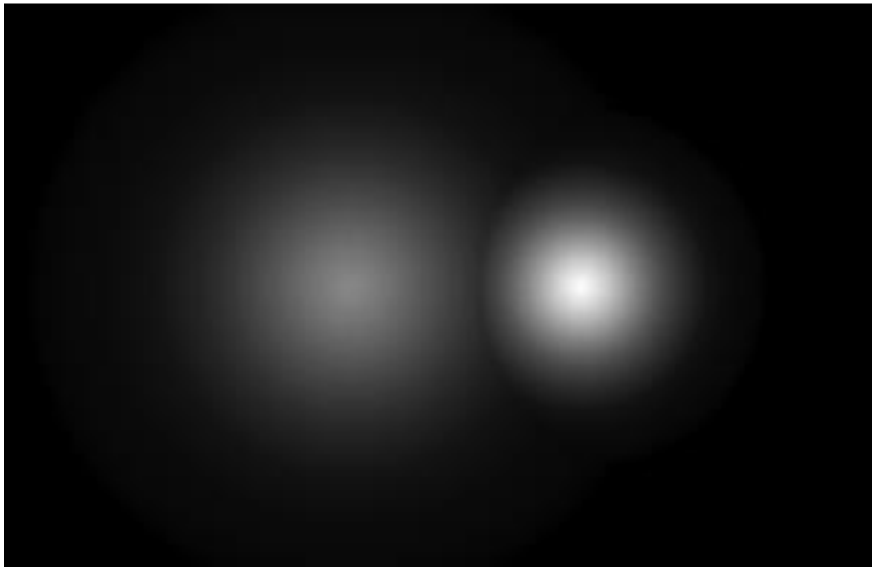
Extension to Grayscale Image

- A 2D grayscale image is treated as a 3D solid in space – a landscape – whose height above the surface at a point is proportional to the brightness of the corresponding pixel.

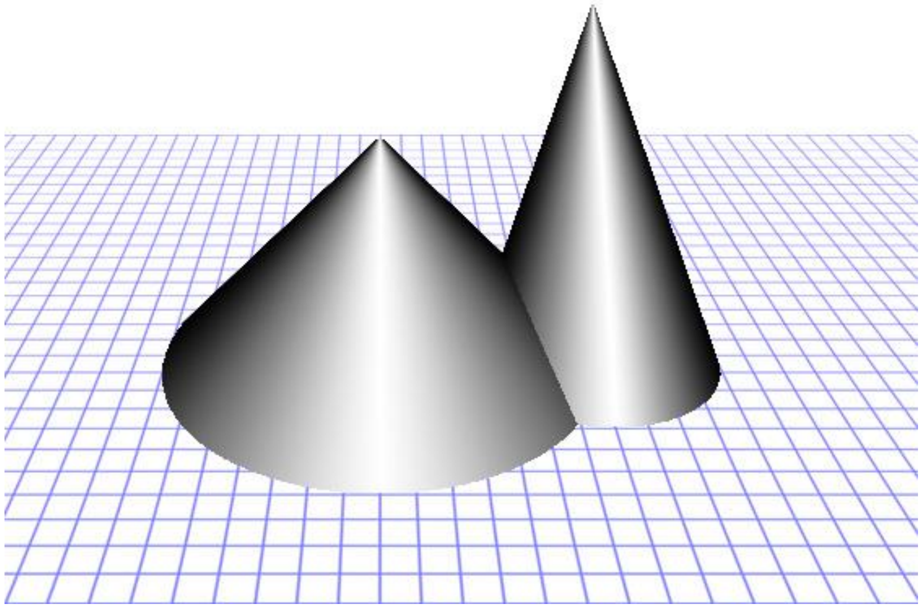


Derived from Ida-Maria Sintorn

Representation of Grayscale Images

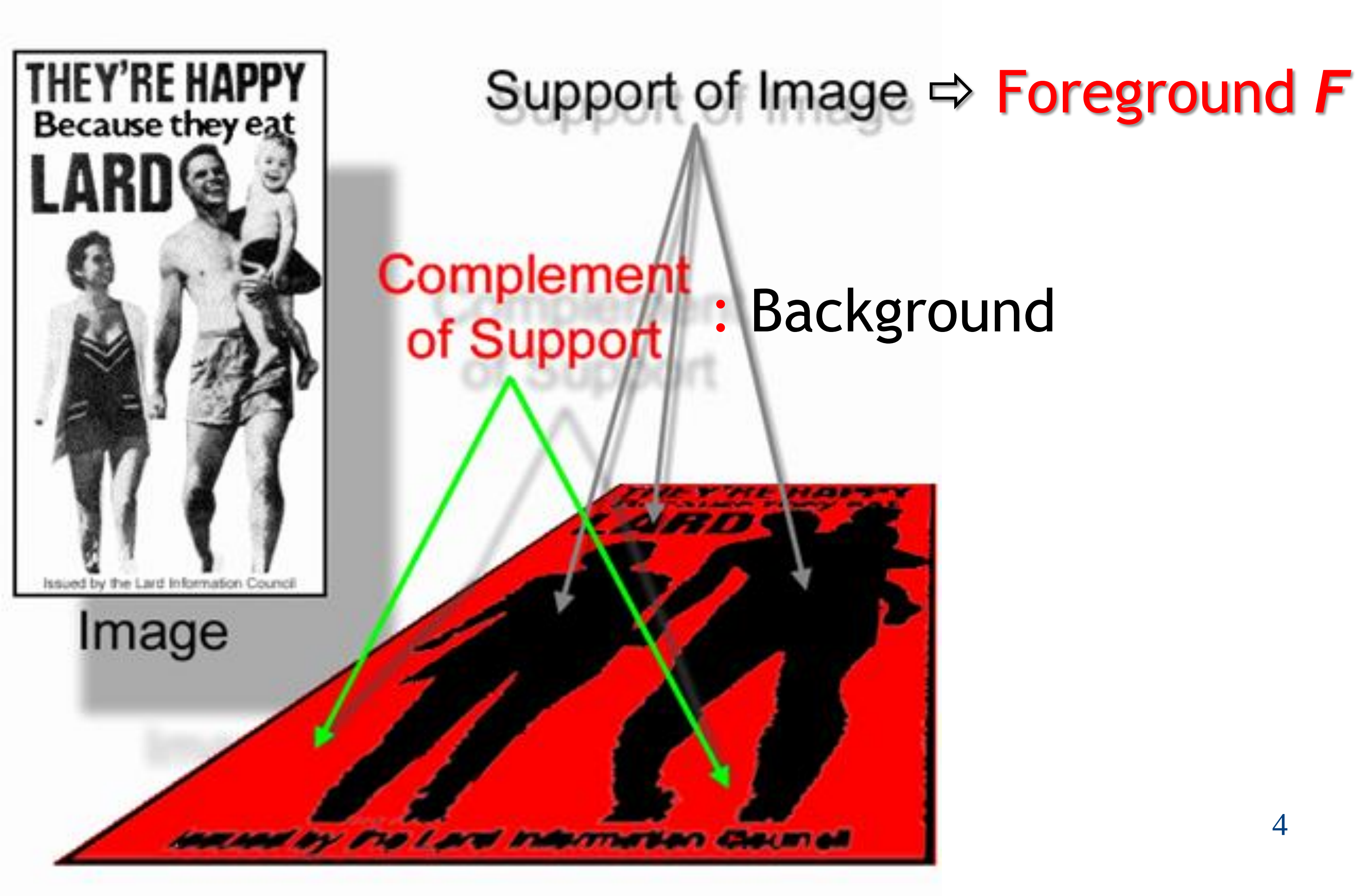


image



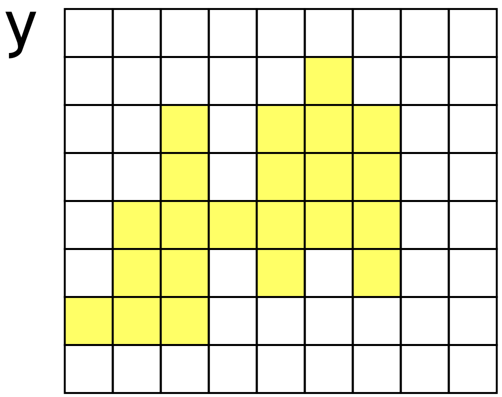
landscape

Support of an Image

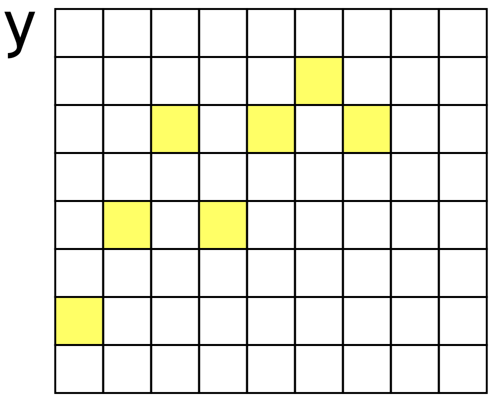


Umbra

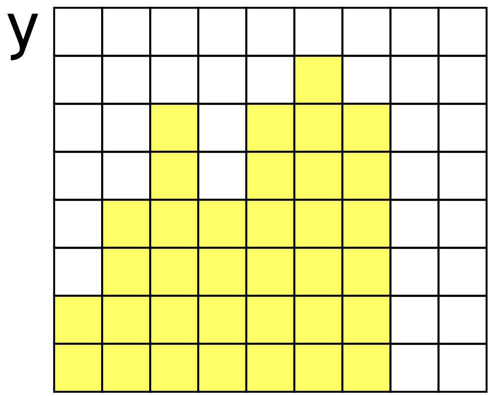
- This part of mathematical morphology is an extension to multidimensional 'images' in particular to grey level and color images
- $A \subseteq E^n$, $F \subseteq E^{n-1}$ (support of A), $x \in F$ (support element), $y \in E$ (intensity)
- Top of a set A (example for n=2): $T[A](x) = \max \{ y \mid (x, y) \in A \}$
- Umbra of f ($f: F \rightarrow E$): $U[f] = \{ (x, y) \in F \times E \mid y \leq f(x) \}$



Set A



Top of A



Umbra of A

Umbra - Proprieties

$$T[A] \subseteq A \subseteq U[A] \subseteq E^n$$

$$U[U[A]] \equiv U[A]$$

Umbra

- For each function $f:R^2 \rightarrow R$ it is possiblee to define an equivalent function $f:R^3 \rightarrow \{0, 1\}$
  - $g(x, y, z) = 1 \Leftrightarrow z \leq f(x, y) \Leftrightarrow z \in U[f(x, y)]$
  - $g(x, y, z) = 0 \Leftrightarrow z > f(x, y) \Leftrightarrow z \notin U[f(x, y)]$
- So the gray scale morphology is related to the mathematical morphology in 3D

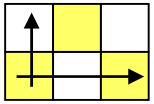
Dilation for grey level images

- $F, K \subseteq E^{n-1}$
- The dilation of image f and structural element k can be defined as:

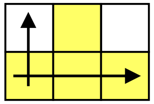
$$f \oplus k = T\{U[f] \oplus U[k]\}$$

- Tends to brighten the image, reduce dark regions
- From the computational view point this operation is equivalent to a convolution

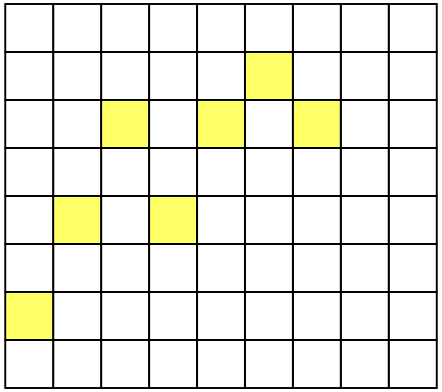
Dilation - Example



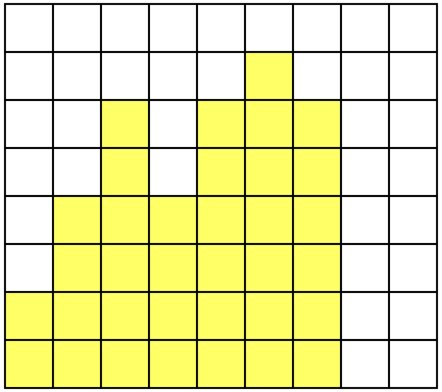
k



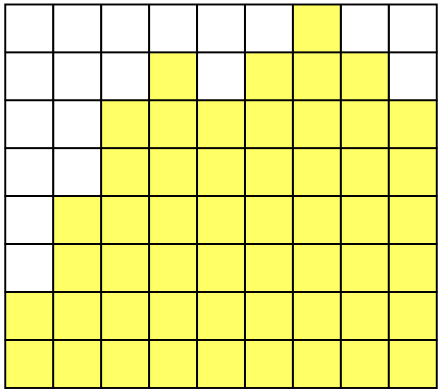
U[k]



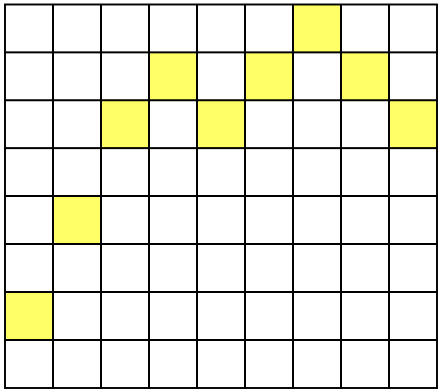
f



U[f]



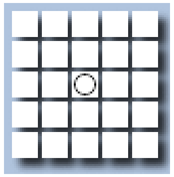
$U[f] \oplus U[k]$



$f \oplus k = T[U[f] \oplus U[k]]$

Dilation - Example

- magick lena.png -morphology dilate square:2 dilate.png



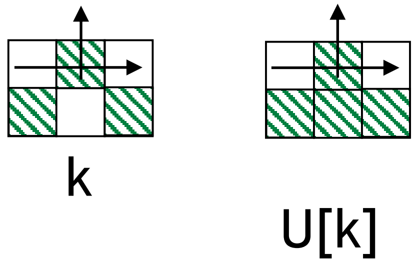
Grey scale erosion

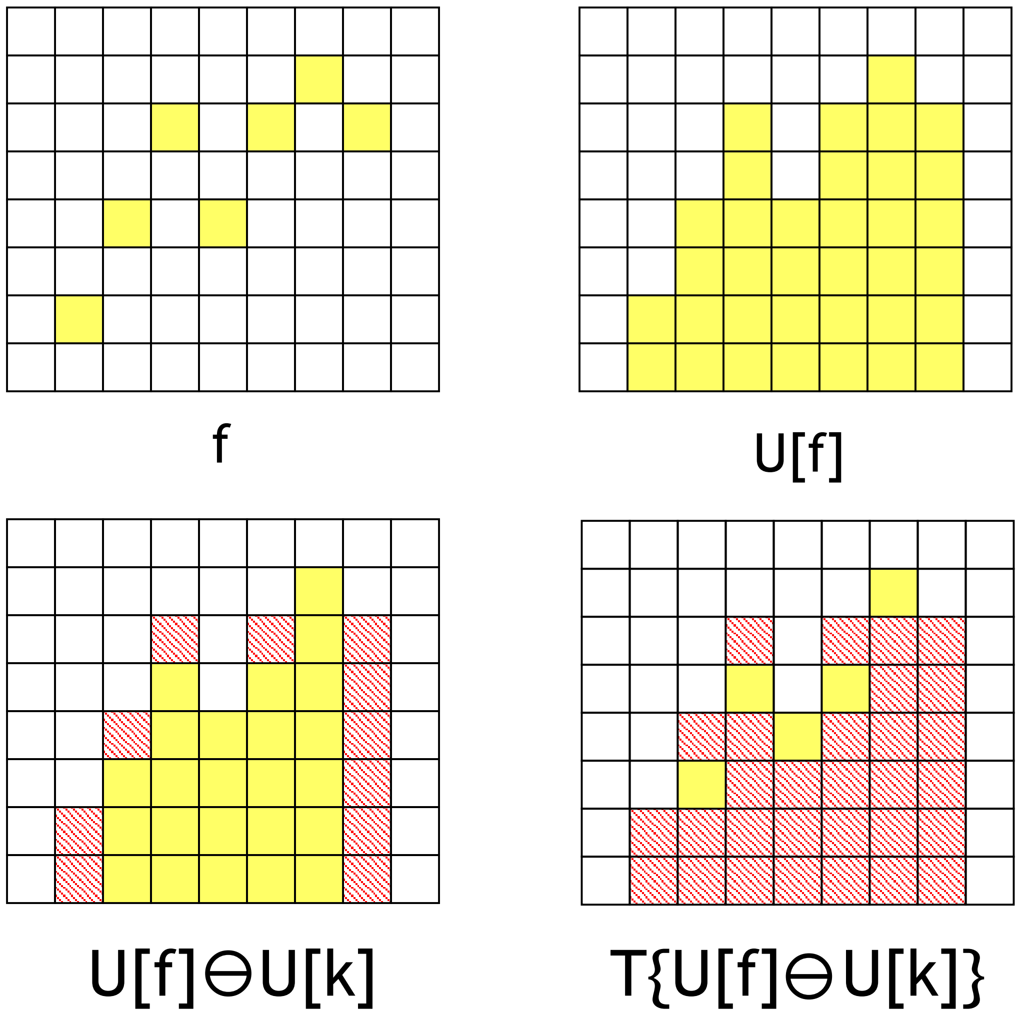
- $F, K \subseteq E^{n-1}$
- The erosion of image f and structural element k can be defined as:

$$f \ominus k = T\{U[f] \ominus U[k]\}$$

- Tends to darken the image, reduce bright regions
- From the computational view point this operation is equivalent to a convolution

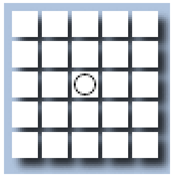
Erosion - Example





Erosion - Example

- magick lena.png -morphology erode square:2 erode.png

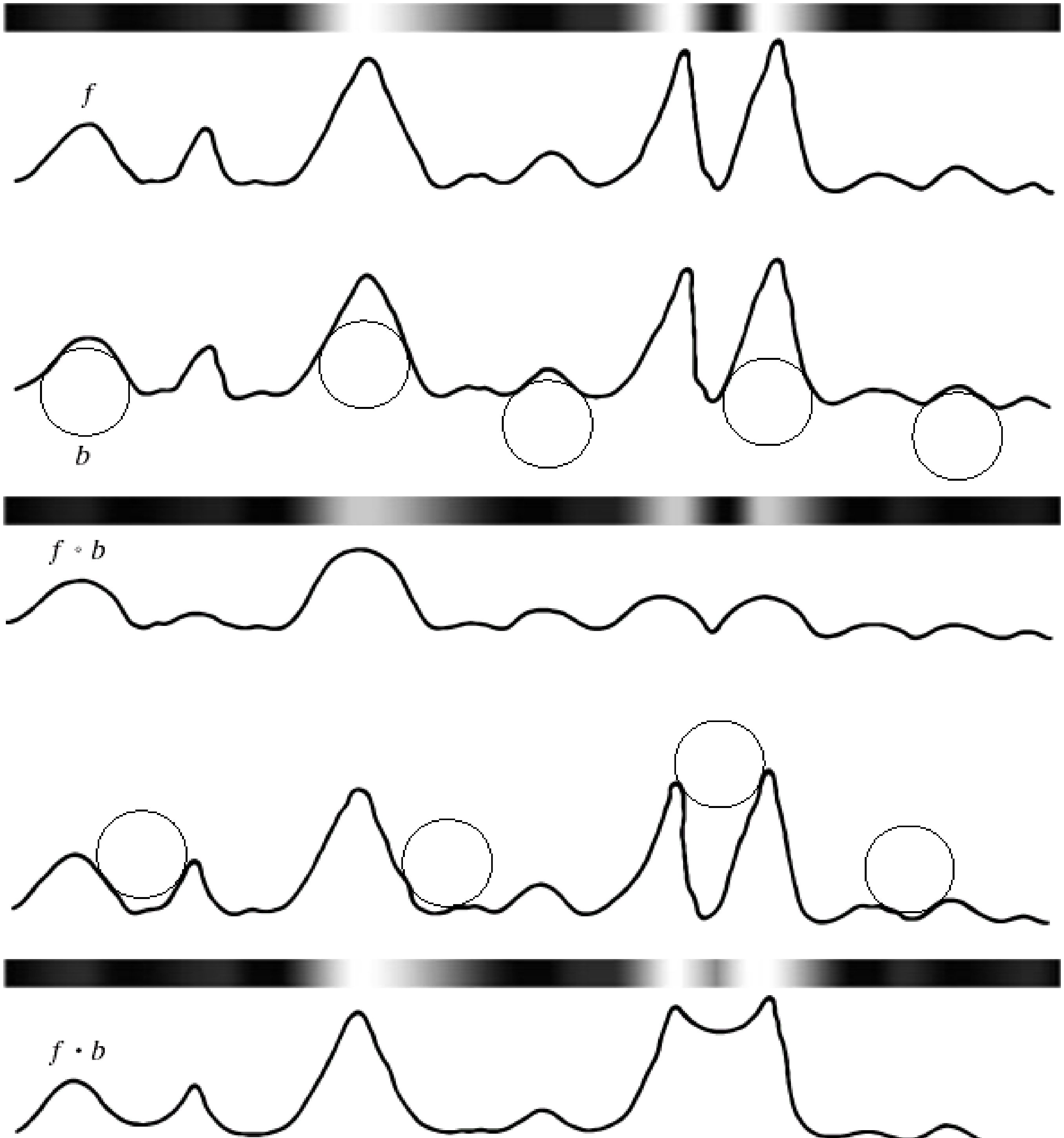


Opening and Closing

- **Opening** and **closing** of an image *f(x,y)* by a structuring element *b(x,y)* have the same form as their binary counterpart:

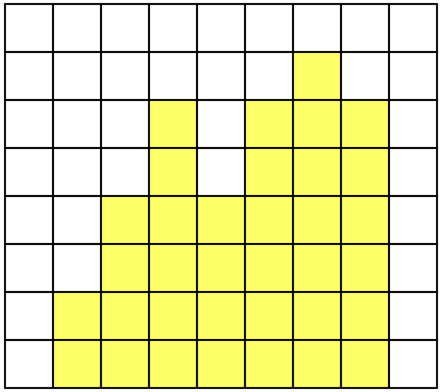
$$f \circ b = (f \ominus b) \oplus b \qquad\qquad f \bullet b = (f \oplus b) \ominus b$$

- Geometric interpretation:
- View the image as a 3-D surface map, and suppose we have a spherical structuring element.
  - **Opening**: roll the sphere against the *underside* of the surface, and take the *highest* points reached by any part of the sphere. Opening reduces bright details eliminating curvatures smaller than the specified SE.
  - **Closing**: roll the sphere *on top* of the surface, and take the *lowest* points reached by any part of the sphere. Closing reduces dark details eliminating curvatures smaller than the specified SE.
- Opening and closing are used often in combination as morphological filters for image smoothing and noise removal.

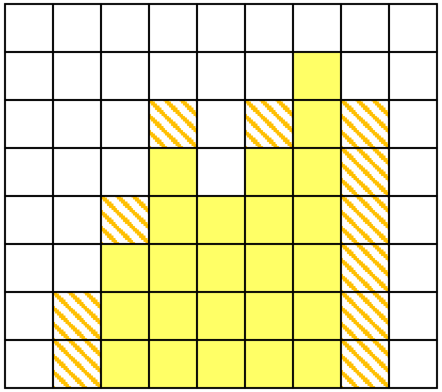


Gonzales-Woods

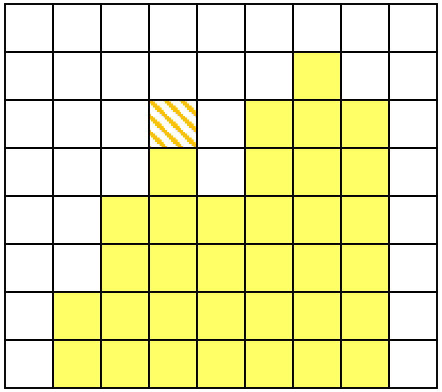
Opening - Example



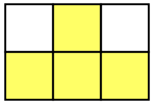
U[A]



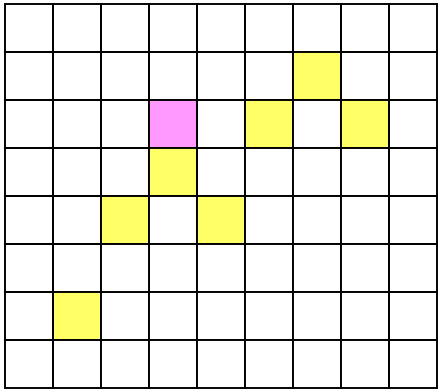
$U[A] \ominus U[k]$



$U[A] \ominus U[k] \oplus U[k]$



U[k]

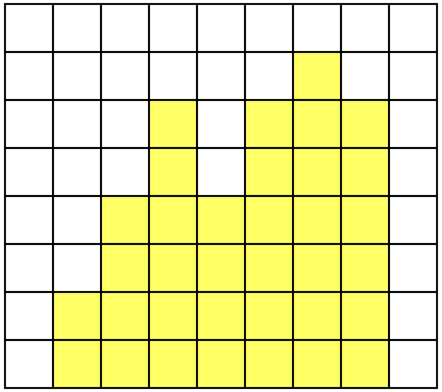


$A \circ k$

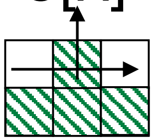
Opening - Example

- magick lena.png -morphology open square:2 dilate.png

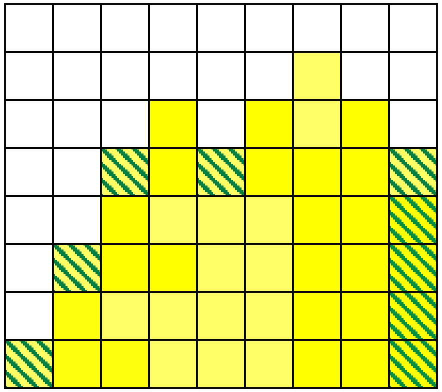
Closing - Example



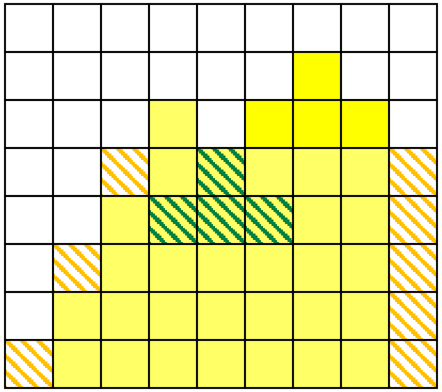
U[A]



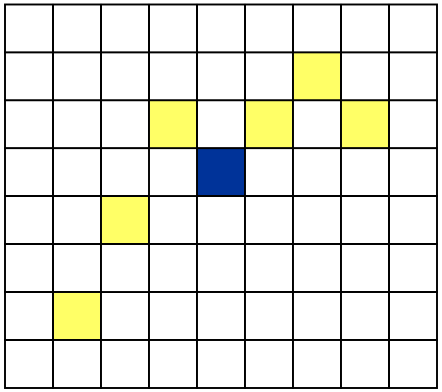
U[k]



$U[A] \oplus U[k]$



$[U[A] \oplus U[k]] \ominus U[k]$



$A \bullet k$

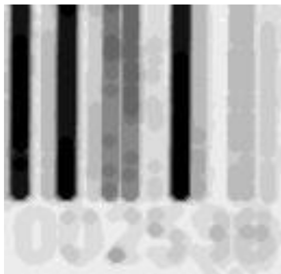
Closing - Example

- magick lena.png -morphology close square:2 dilate.png

Opening vs Closing on Gray Value Images

- Disk with radius 3 as structure element
- Opening: all the thin white bands have disappeared, only the broad one remains

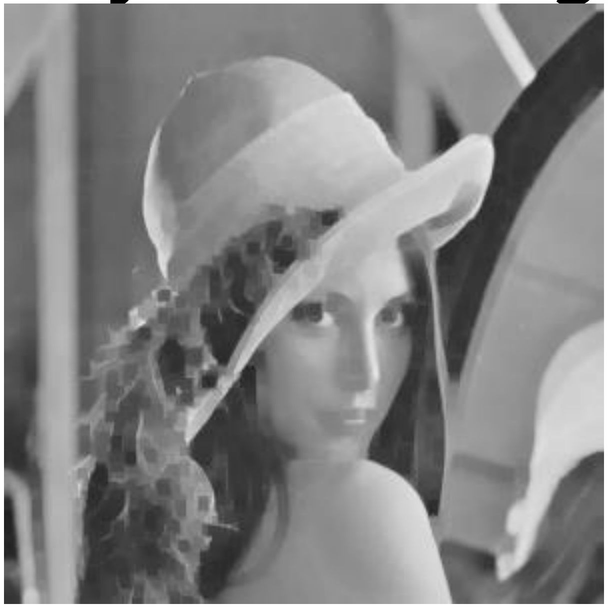
- Closing: all the valleys where the structure element does not fit have been filled, only the three broad black bands remain



Opening vs Closing on Gray Value Images

- 5x5 square structuring element

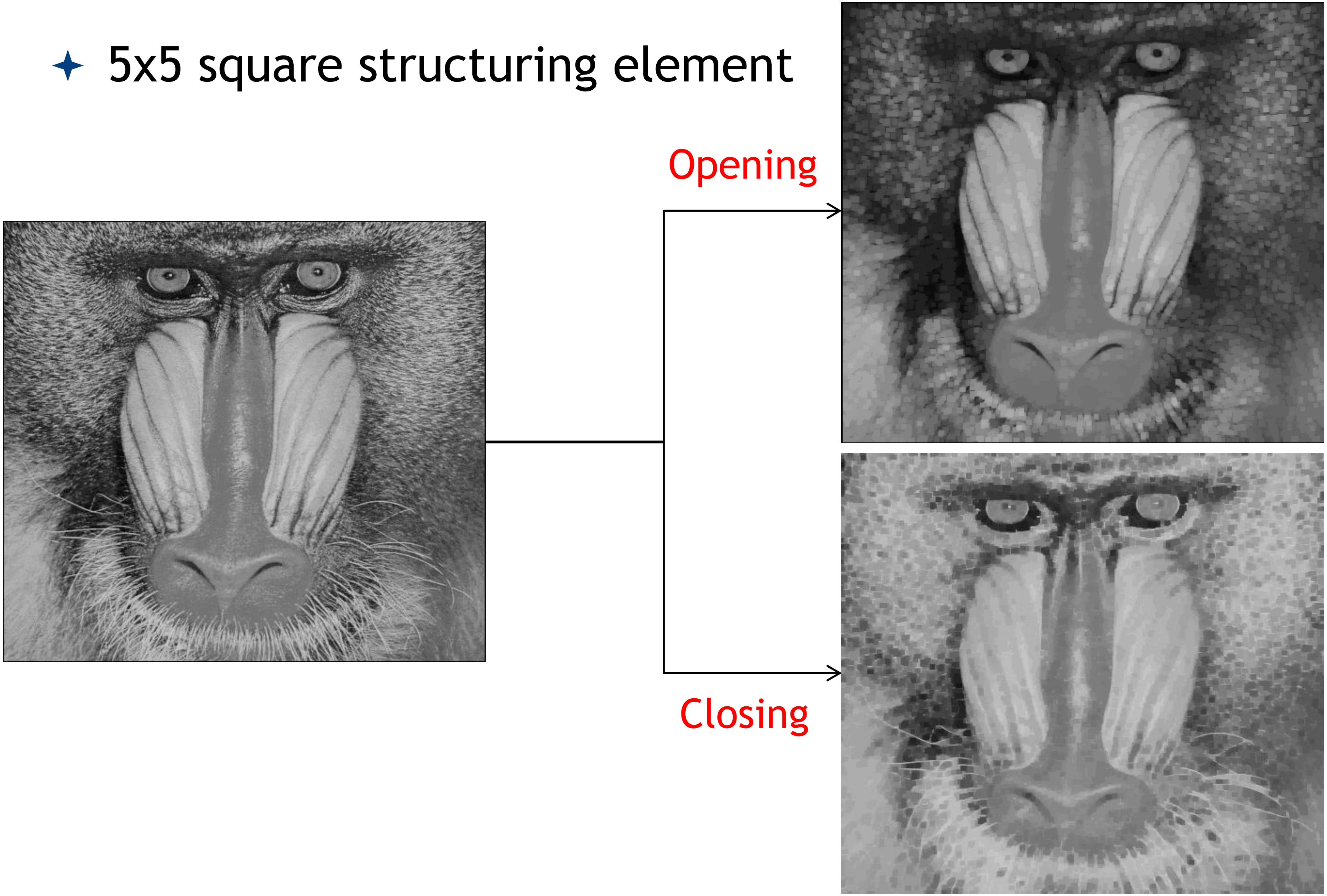
Closing



Opening

Opening vs Closing on Gray Value Images

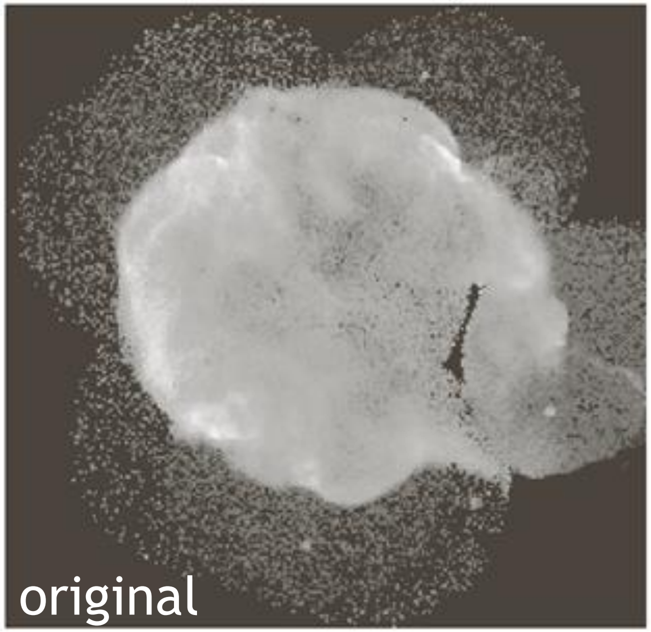
- 5x5 square structuring element

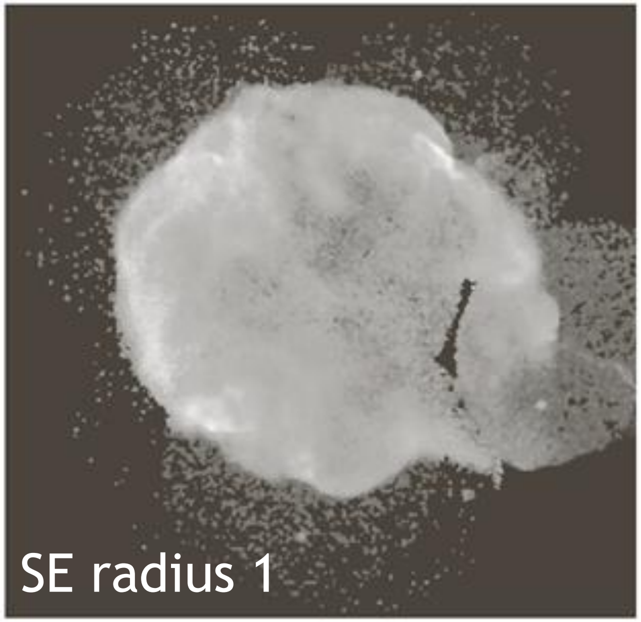


Morphological smoothing - Example

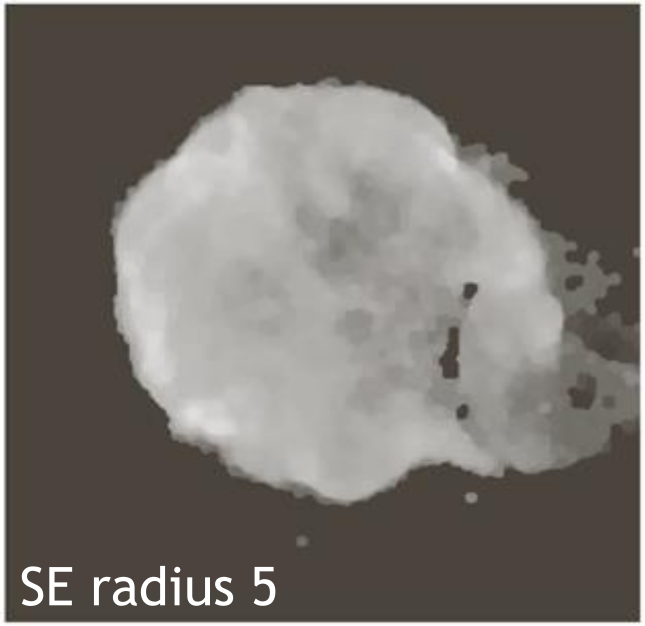
- Opening followed by closing

Morphological Smoothing: opening and closing





Cignus Loop supernova, taken by X-ray by NASA Hubble Telescope



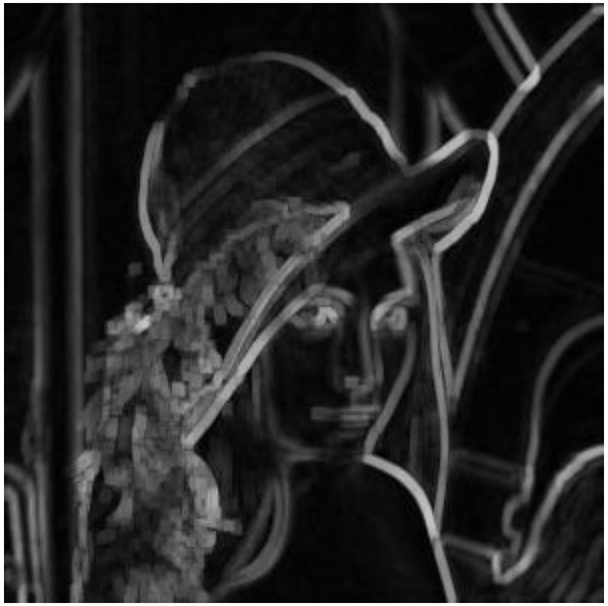
a	b
c	d

(a) 566 × 566 image of the Cygnus Loop supernova, taken in the X-ray band by NASA's Hubble Telescope. (b)–(d) Results of performing opening and closing sequences on the original image with disk structuring elements of radii, 1, 3, and 5, respectively. (Original image courtesy of NASA.)

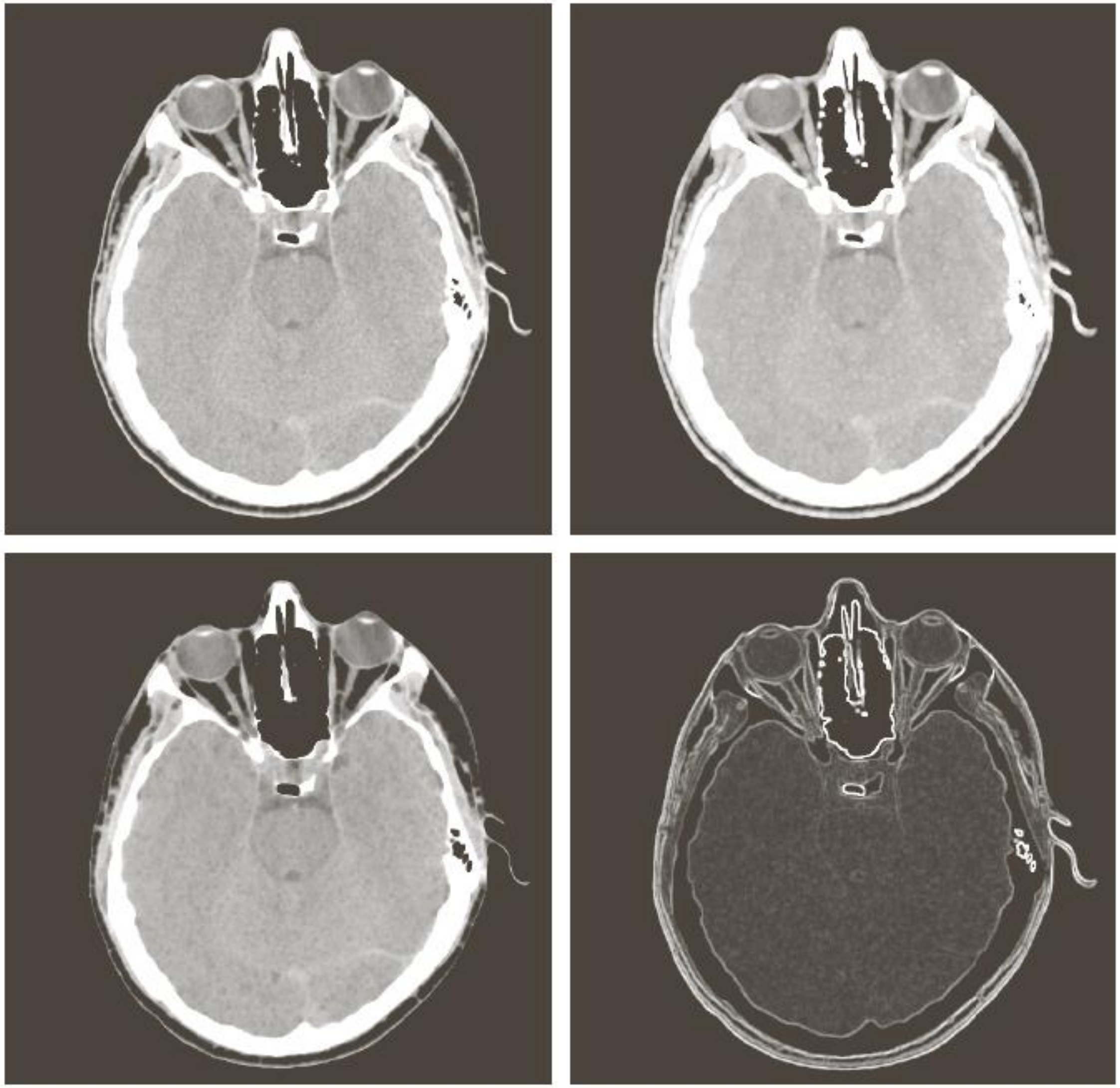
Gonzales-Woods

Morphological gradient - Example

- Difference between dilation and erosion
- magick lena.png -morphology edge square:2 edge.png
- The edges are enhanced and the contribution of the homogeneous areas are suppressed, thus producing a “derivative-like” (gradient) effect.



Morphological Gradient



a	b
c	d

(a) 512 × 512 image of a head CT scan.
(b) Dilation.
(c) Erosion.
(d) Morphological gradient, computed as the difference between (b) and (c).
(Original image courtesy of Dr. David R. Pickens, Vanderbilt University.)

Gonzales-Woods

Top-hat and Bottom-hat Transformations

- The top-hat transformation of a grayscale image f is defined as f minus its opening:

$$T_{hat}(f) = f - (f \circ b)$$

- The bottom-hat transformation of a grayscale image f is defined as its closing minus f:

$$B_{hat}(f) = (f \bullet b) - f$$

- One of the principal applications of these transformations is in removing objects from an image by using structuring element in the opening or closing operation

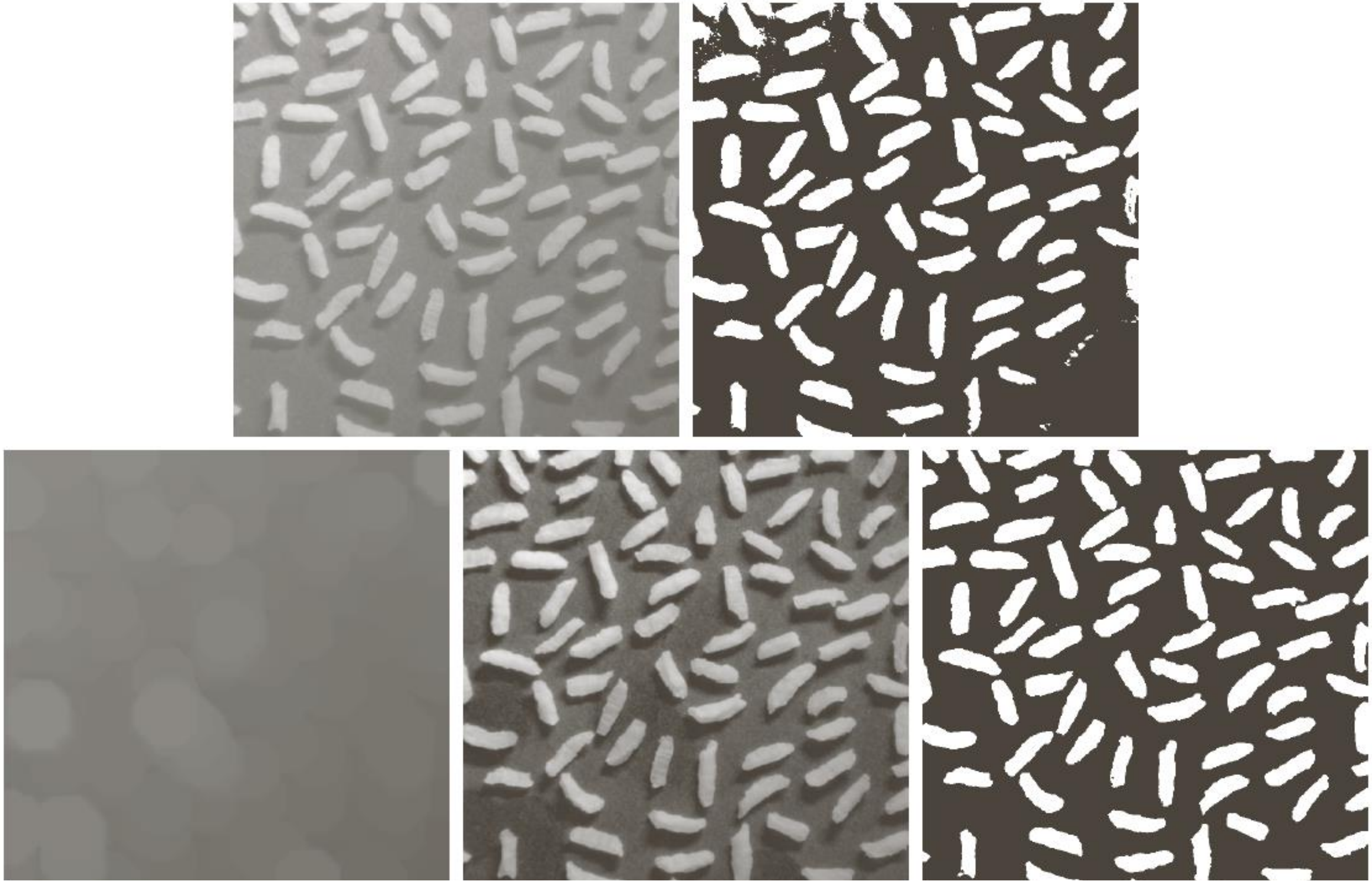
Top-hat transformation - Example

- Difference between original and opening
- magick lena.png -morphology TopHat square:2 top.png

Bottom-hat transformation - Example

- Difference between closing and original
- magick lena.png -morphology BottomHat square:2 bottom.png

Example of Using Top-hat Transformation in Segmentation



a b
c d e

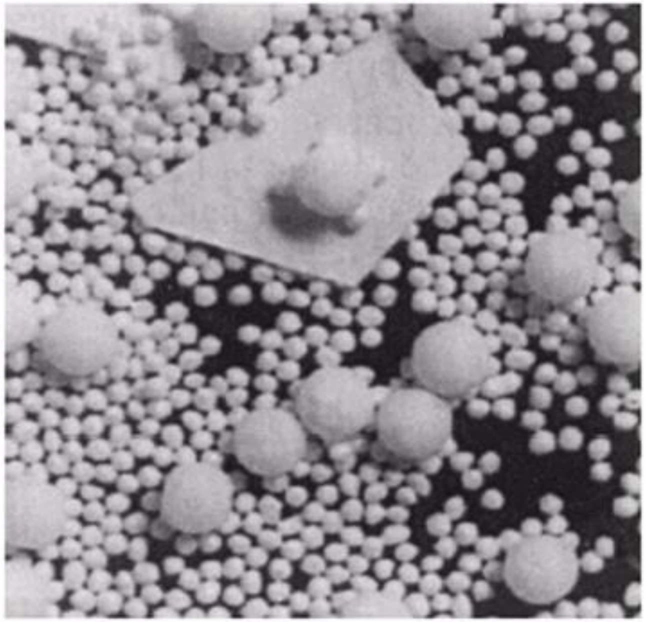
Using the top-hat transformation for *shading correction*. (a) Original image of size 600 × 600 pixels. (b) Thresholded image. (c) Image opened using a disk SE of radius 40. (d) Top-hat transformation (the image minus its opening). (e) Thresholded top-hat image.

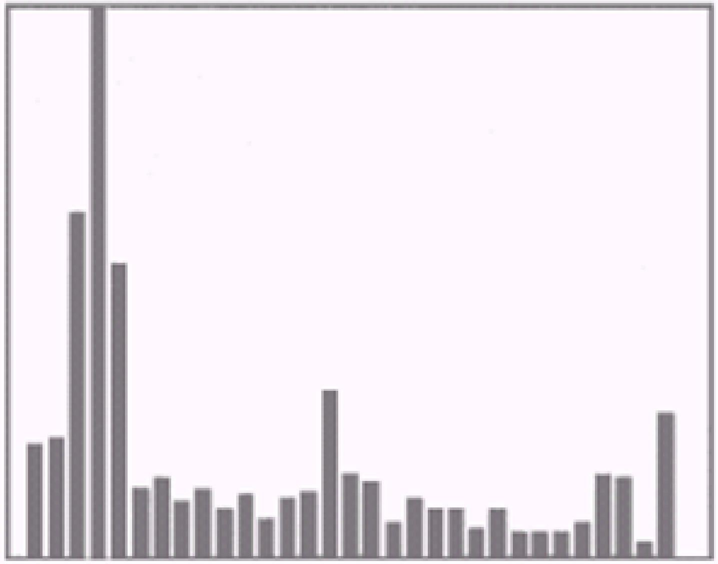
Gonzales-Woods

Granulometry

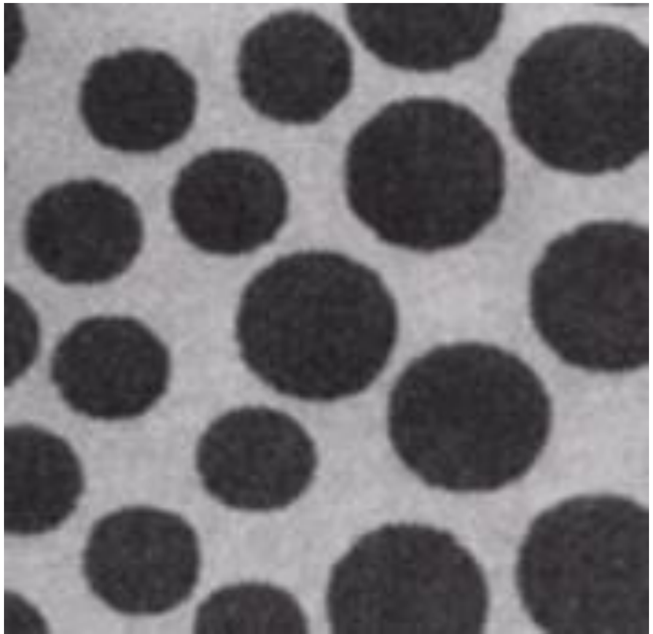
- ***Granulometry*** is a field that deals principally with **determining the size distribution of particles in an image**.
- a morphological approach to determine size distribution to construct a ***histogram*** of it is based on ***opening*** operations of particular size that have the most effect on regions of the input image that **contain particles of similar size**.
- For each opening, the sum (**surface area**) of the pixel values in the opening is computed
- This type of processing is **useful for describing regions with a predominant particle-like character**.

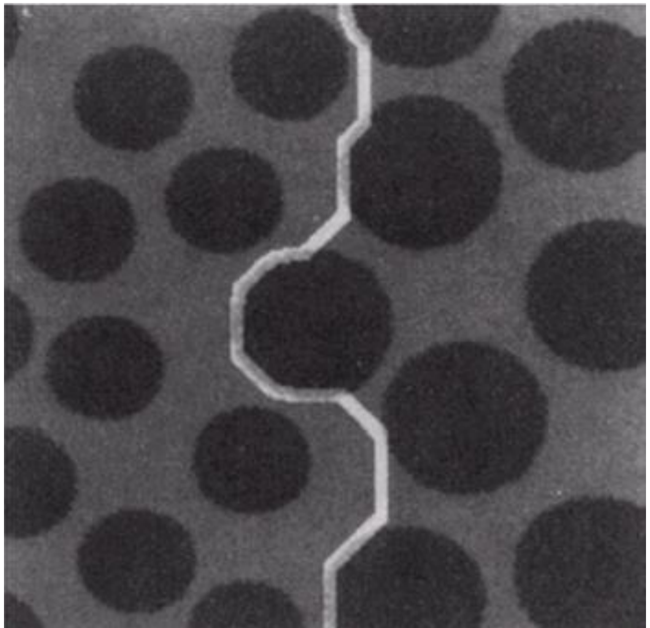
Granulometry and textural segmentation



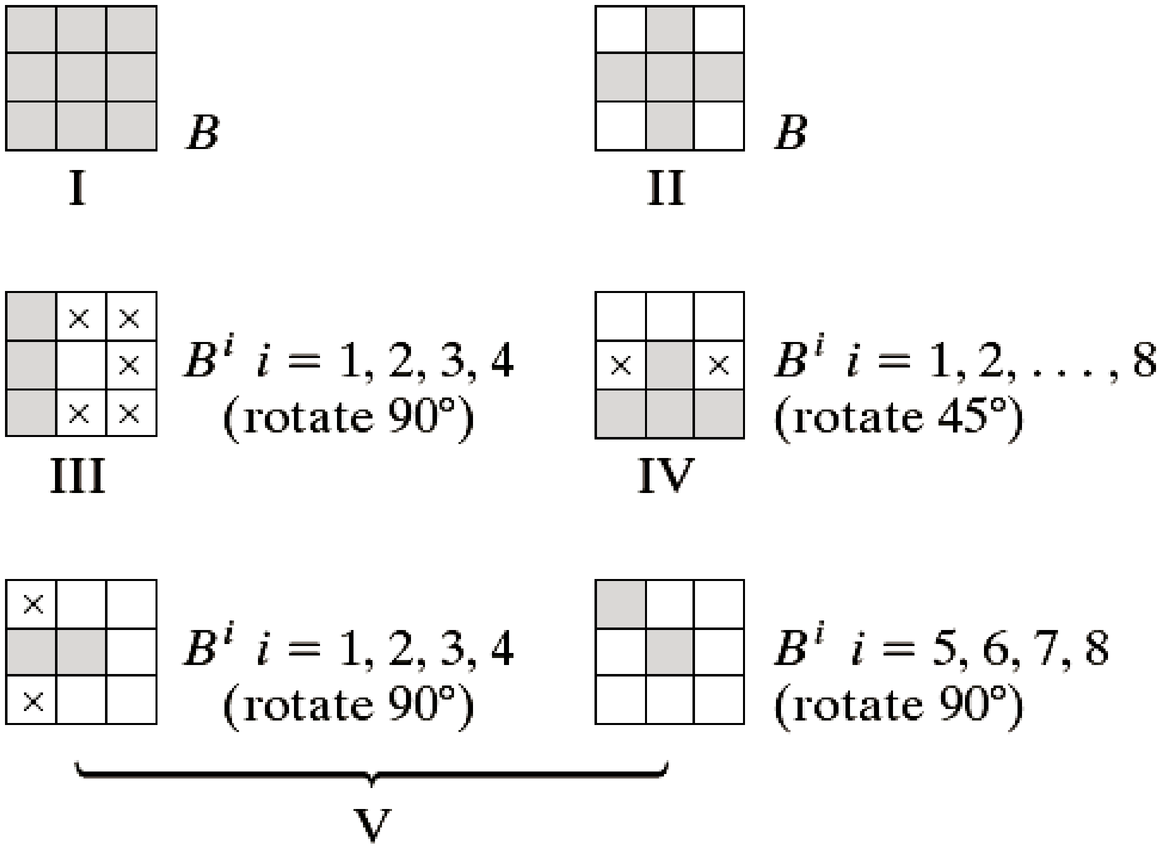


Mr. A. Morris, Leica Cambridge, Ltd.





Summary



- X's indicate «don't care» values

Gonzales-Woods

Summary

Operation	Equation	Comments
Translation	$(B)_z = \{w \mid w = b + z, \text{ for } b \in B\}$	Translates the origin of B to point z.
Reflection	$\hat{B} = \{w \mid w = -b, \quad \text{for } b \in B\}$	Reflects all elements of B about the origin of this set.
Complement	$A^c = \{w \mid w \notin A\}$	Set of points not in A.
Difference	$A - B = \{w \mid w \in A, w \notin B\} = A \cap B^c$	Set of points that belong to A but not to B.
Dilation	$A \oplus B = \left\{z \mid (\hat{B}_z) \cap A \neq \varnothing\right\}$	"Expands" the boundary of A. (I)
Erosion	$A \ominus B = \left\{z \mid (B)_z \subseteq A\right\}$	"Contracts" the boundary of A. (I)
Opening	$A \circ B = (A \ominus B) \oplus B$	Smoothes contours, breaks narrow isthmuses, and eliminates small islands and sharp peaks. (I)

(*Continued*)

Gonzales-Woods

Summary

Operation	Equation	Comments
Closing	$A \bullet B = (A \oplus B) \ominus B$	Smoothes contours, fuses narrow breaks and long thin gulfs, and eliminates small holes. (I)
Hit-or-miss transform	$A \circledast B = (A \ominus B_1) \cap (A^c \ominus B_2)$ $= (A \ominus B_1) - (A \oplus \hat{B}_2)$	The set of points (coordinates) at which, simultaneously, B_1 found a match ("hit") in A and B_2 found a match in A^c
Boundary extraction	$\beta(A) = A - (A \ominus B)$	Set of points on the boundary of set A. (I)
Hole filling	$X_k = (X_{k-1} \oplus B) \cap A^c$; $k = 1, 2, 3, \ldots$	Fills holes in A; X_0 = array of 0s with a 1 in each hole. (II)
Connected components	$X_k = (X_{k-1} \oplus B) \cap A$; $k = 1, 2, 3, \ldots$	Finds connected components in A; X_0 = array of 0s with a 1 in each connected component. (I)

(*Continued*)

Gonzales-Woods

Summary

Operation	Equation	Comments
Convex hull	$X_k^i = (X_{k-1}^i \circledast B^i) \cup A$; $i = 1, 2, 3, 4$; $k = 1, 2, 3, \ldots$; $X_0^i = A$; and $D^i = X_{\text{conv}}^i$	Finds the convex hull $C(A)$ of set A, where "conv" indicates convergence in the sense that $X_k^i = X_{k-1}^i$. (III)
Thinning	$A \otimes B = A - (A \circledast B)$ $= A \cap (A \circledast B)^c$ $A \otimes \{B\} =$ $((\ldots((A \otimes B^1) \otimes B^2)\ldots) \otimes B^n)$ $\{B\} = \{B^1, B^2, B^3, \ldots, B^n\}$	Thins set A. The first two equations give the basic definition of thinning. The last equations denote thinning by a sequence of structuring elements. This method is normally used in practice. (IV)
Thickening	$A \odot B = A \cup (A \circledast B)$ $A \odot \{B\} =$ $((\ldots(A \odot B^1) \odot B^2 \ldots) \odot B^n)$	Thickens set A. (See preceding comments on sequences of structuring elements.) Uses IV with 0s and 1s reversed.

Gonzales-Woods (*Continued*)

Summary

Operation	Equation	Comments
Skeletons	$S(A) = \bigcup_{k=0}^{K} S_k(A)$ $S_k(A) = \bigcup_{k=0}^{K} \{(A \ominus kB) - [(A \ominus kB) \circ B]\}$ Reconstruction of A: $A = \bigcup_{k=0}^{K} (S_k(A) \oplus kB)$	Finds the skeleton $S(A)$ of set A. The last equation indicates that A can be reconstructed from its skeleton subsets $S_k(A)$. In all three equations, K is the value of the iterative step after which the set A erodes to the empty set. The notation $(A \ominus kB)$ denotes the kth iteration of successive erosions of A by B. (I)
Pruning	$X_1 = A \otimes \{B\}$ $X_2 = \bigcup_{k=1}^{8} (X_1 \circledast B^k)$ $X_3 = (X_2 \oplus H) \cap A$ $X_4 = X_1 \cup X_3$	X_4 is the result of pruning set A. The number of times that the first equation is applied to obtain X_1 must be specified. Structuring elements V are used for the first two equations. In the third equation H denotes structuring element I.

Operation	Description
bothat	"Bottom-hat" operation using a 3 × 3 structuring element; use imbothat (see Section 9.6.2) for other structuring elements.
bridge	Connect pixels separated by single-pixel gaps.
clean	Remove isolated foreground pixels.
close	Closing using a 3 × 3 structuring element; use imclose for other structuring elements.
diag	Fill in around diagonally connected foreground pixels.
dilate	Dilation using a 3 × 3 structuring element; use imdilate for other structuring elements.
erode	Erosion using a 3 × 3 structuring element; use imerode for other structuring elements.
fill	Fill in single-pixel "holes" (background pixels surrounded by foreground pixels); use imfill (see Section 11.1.2) to fill in larger holes.
hbreak	Remove H-connected foreground pixels.
majority	Make pixel p a foreground pixel if at least five pixels in $N_8(p)$ (see Section 9.4) are foreground pixels; otherwise make p a background pixel.
open	Opening using a 3 × 3 structuring element; use function imopen for other structuring elements.
remove	Remove "interior" pixels (foreground pixels that have no background neighbors).
shrink	Shrink objects with no holes to points; shrink objects with holes to rings.
skel	Skeletonize an image.
spur	Remove spur pixels.
thicken	Thicken objects without joining disconnected 1s.
thin	Thin objects without holes to minimally connected strokes; thin objects with holes to rings.
tophat	"Top-hat" operation using a 3 × 3 structuring element; use imtophat (see Section 9.6.2) for other structuring elements.

Gonzales-Woods