Deep Learning

10 –Reinforcement Learning

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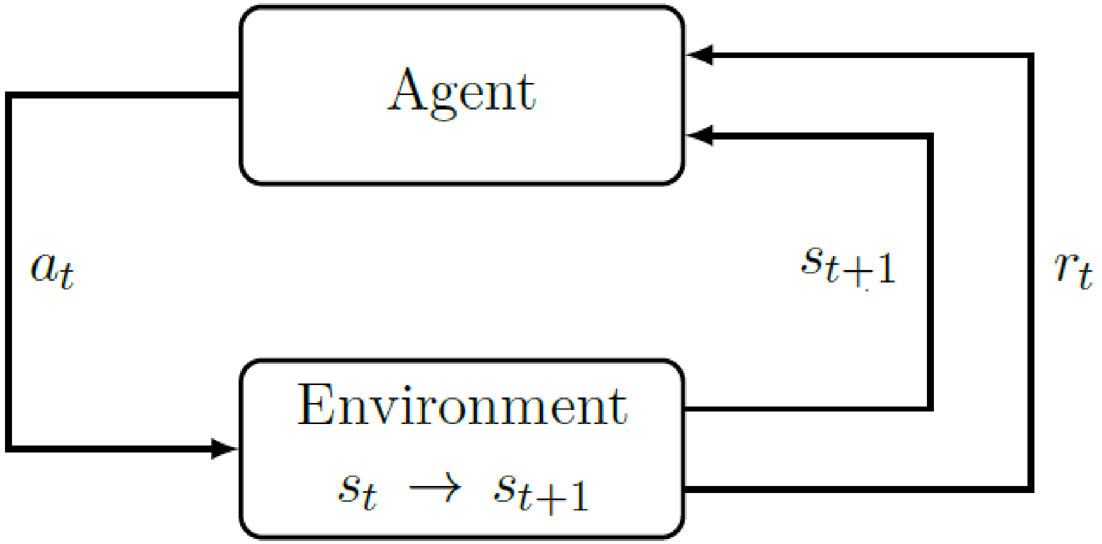
This presentation can be downloaded at:
http://vision.unipv.it/DL

Basic assumptions

[image from: https://arxiv.org/pdf/1811.12560.pdf]



The **Environment**: is in *state* s_t — time

An **Agent** observes *state* s_t and performs *action* a_t

The **Environment** *state* transitions from $s_t \rightarrow s_{t+1}$

The **Agent** receives *reward* r_t

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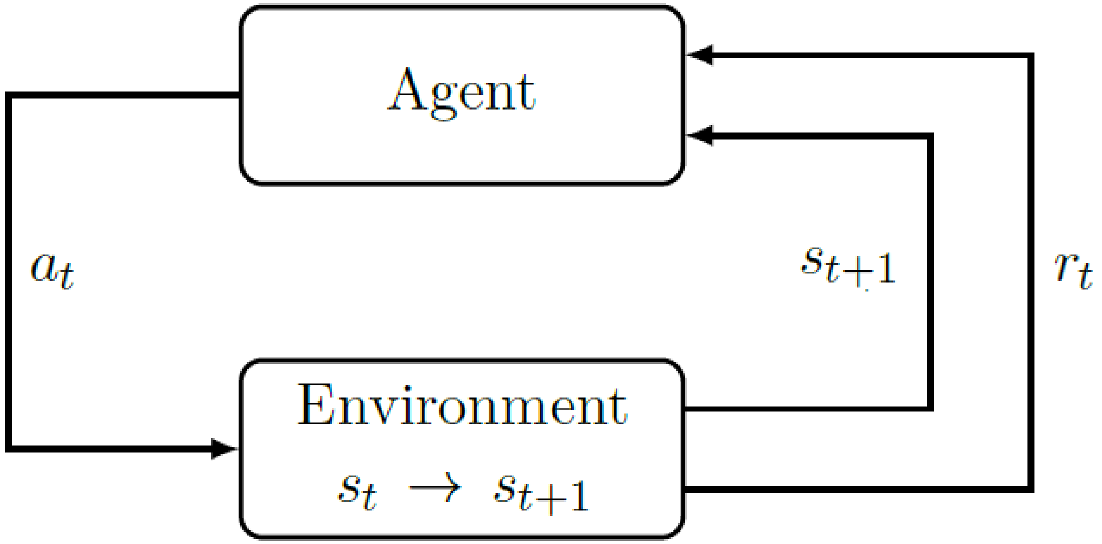

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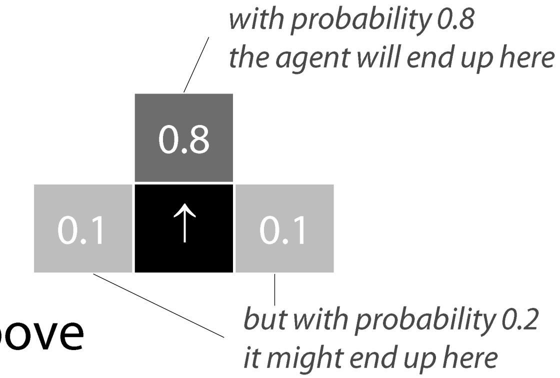
Cumulative reward: $R := \sum_{t=0}^{\infty} r_t$

An example: *gridworld*

	1	2	3	4
1	-0.02	-0.02	-0.02	1
2	-0.02		-0.02	-1
3	-0.02	-0.02	-0.02	-0.02

The *state* of the agent is the position on the grid: e.g. (1,1), (3,4), (2,3)

At each time step, the agent can *move* one box in the directions ←↑↓→



The effect of each move is somewhat stochastic, however: for example, a move ↑ has a slight probability of producing a different (*and perhaps unwanted*) effect

Entering each state yields the *reward* shown in each box above

There are two *absorbing states*: entering either the green or the red box means exiting the *gridworld* and completing the game

- What is the best (*i.e. maximally rewarding*) movement policy?

Markov Decision Process (MDP)

	1	2	3	4
1	-0.02	-0.02	-0.02	1
2	-0.02		-0.02	-1
3	-0.02	-0.02	-0.02	-0.02

Formalization and abstraction of the gridworld example

Markov Decision Process: $< \mathcal{S}, \mathcal{A}, r, P, \gamma >$

A set of *states* : $\mathcal{S} = \{s_1, s_2, \dots\}$

A set of *actions* : $\mathcal{A} = \{a_1, a_2, \dots\}$

A *reward function* : $r : \mathcal{S} \to \mathbb{R}$

A *transition probability distribution* : $P(S_{t+1} \mid S_t, A_t)$ (also called a *model*)

Markov property: the transition probability depends only on the previous state and action

$$P(S_{t+1} \mid S_t, A_t) = P(S_{t+1} \mid S_t, A_t, S_{t-1}, A_{t-1}, S_{t-2}, A_{t-2}, \dots)$$

A *discount factor* : $0 \leq \gamma < 1$

Markov Decision Process (MDP): policies and values

The agent is supposed to adopt a *deterministic* <u>*policy*</u> : $\pi : \mathcal{S} \to \mathcal{A}$

In other words, the agent always chooses its *action* depending on the *state* alone

Given a policy π , the ***state value function*** is defined, for each state s as:

$$V^\pi(s) := \mathbb{E}[r(S_t) + \gamma r(S_{t+1}) + \gamma^2 r(S_{t+2}) + \ldots \mid \pi, S_t = s]$$

Note the role of the *discount factor*: a value $\gamma < 1$ means that that future rewards could be weighted less (by the agent) than immediate ones

Note also that all states S_t must be described by *random variables* :
i.e. the policy is deterministic but the state transition is not

Note also that when the reward is *bounded*, i.e. $r(S) \leq r_{\max}$

$$\sum_{t=0}^{\infty} \gamma^t \, r(S_t) \;\leq\; r_{\max} \sum_{t=0}^{\infty} \gamma^t \;=\; r_{\max} \frac{1}{1-\gamma}$$

for $\gamma < 1$ this is the *geometric series*

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In the *gridworld* example:

- The set of states is finite
- The set of actions is finite
- For every policy, each entire story is <u>finite</u>

  Sooner or later the agent will fall into one of the absorbing states

Bellman equations

By working on the definition of value function:

$$\begin{aligned} V^\pi(s) &:= \mathbb{E}[r(S_t) + \gamma r(S_{t+1}) + \gamma^2 r(S_{t+2}) + \ldots \mid \pi, S_t = s] \\ &= \mathbb{E}[r(S_t) + \gamma(r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots) \mid \pi, S_t = s] \\ &= r(s) + \gamma \mathbb{E}[r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots \mid \pi, S_t = s] \\ &= r(s) + \gamma \sum_{s'} P(s' \mid s, \pi(s)) \cdot \mathbb{E}[r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots \mid \pi, S_{t+1} = s'] \\ &= r(s) + \gamma \sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^\pi(S_{t+1}) \end{aligned}$$

This means that in a Markov Decision Process:

$$V^\pi(s) = r(s) + \gamma \sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^\pi(S_{t+1})$$

This is true for any *state*, so there is one such equation for each of those

If the set of states is finite, there are exactly $|S|$ *(linear) Bellman equations for* $|S|$ *variables:*
in general, for any deterministic policy , V^π *can be computed analytically*

Optimal policy – Optimal value function

- Basic definitions

$$V^*(s) := \max_\pi V^\pi(s),\ \forall s \in S$$
$$\pi^*(s) := \text{argmax}_\pi V^\pi(s),\ \forall s \in S$$

Property: for every MDP, there exists such an optimal deterministic policy (*possibly non-unique*)

With Bellman Equations:

$$\max_\pi V^\pi(s) = r(s) + \gamma \max_\pi \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^\pi(S_{t+1}) \right)$$
$$V^*(s) = r(s) + \gamma \max_\pi \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^*(S_{t+1}) \right)$$
$$= r(s) + \gamma \max_a \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, a) \cdot V^*(S_{t+1}) \right)$$

Therefore:

$$\pi^*(s) = \text{argmax}_a \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, a) V^*(S_{t+1}) \right)$$

Computing V^* *directly from these equations is unfeasible, however*

There are in fact $|A|^{|S|}$ *possible strategies*

However, once V^* *has been determined,* π^* *can be determined as well*

Optimal value function: value iteration

- Value iteration algorithm

  Initialize: $V(s) := r(s), \ \forall s \in S$
  Repeat:

  Note that there is no policy: all actions must be explored

  1) For every state, update: $V(s) := r(s) + \gamma \max_a \sum_{s'} P(s' \mid s, a) V(s')$

  Theorem: for every fair way (*i.e. giving an equal chance*) of visiting the states in S, this algorithm converges to V^*

Value iteration and optimal policy

	1	2	3	4
1	-0.02	-0.02	-0.02	1
2	-0.02		-0.02	-1
3	-0.02	-0.02	-0.02	-0.02

Initialize states
(e.g. using rewards as initial values)

Iterate and compute V^*

	1	2	3	4
1	0.86	0.90	0.93	1
2	0.82		0.69	-1
3	0.78	0.75	0.71	0.49

V^*

Define the optimal policy as:

$$\pi^*(s) := \operatorname{argmax}_a \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, a) \cdot V^*(S_{t+1})\right)$$

	1	2	3	4
1	→	→	→	1
2	↑		↑	-1
3	↑	←	←	←

π^*

Optimal policy: policy iteration

- Policy iteration algorithm

Initialize $\pi(s), \forall s \in S$ at random
Repeat:

1) For each state, compute: $V(s) := V^{\pi}(s)$
2) For each state, define: $\pi(s) := \text{argmax}_a \sum_{s'} P(s' \mid s, a) V(s')$

This step is computationally expensive: either solve the equations or use value iteration (with fixed policy π)

Theorem: for every fair way (*i.e. giving an equal chance*) of visiting the states in S, this algorithm converges to π^*

As with the value iteration algorithm, this algorithm uses partial estimates to compute new estimates.
It is also greedy, in the sense that it exploits its current estimate $V^{\pi}(s)$

Policy iteration converges with very few number of iterations,
but every iteration takes much longer time than that of *value iteration*

The tradeoff with value iteration is the action space:
when action space is large and state space is small, policy iteration could be better

Offline vs. *Online* learning

- *Value iteration* and *policy iteration* are offline algorithms

  The *model* , i.e. the Markov Decision Process is known

  What needs to be learn is the optimal policy π^*

  In the algorithms, *visiting states* just means considering: there is no agent actually playing the game.

- Different conditions: *learning by doing* …

  Suppose the *model* (i.e. the MDP) is NOT known, or perhaps known only in part

  Then the agent must learn by doing…

Action value function

An analogous of the value function V^π

Given a policy π, the ***action value function*** is defined, for each pair (s, a) as:

$$Q^\pi(s,a) := \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot V^\pi(S_{t+1})$$

$$= \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot \mathbb{E}[r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots \mid \pi, S_{t+1}]$$

$$= \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \mathbb{E}[\gamma r(S_{t+2}) + \ldots \mid \pi, S_{t+1}]]$$

$$= \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \gamma Q^\pi(S_{t+1}, \pi(S_{t+1}))]$$

In other words, $Q^\pi(s,a)$ is the expected value of the reward in S_{t+1}
by taking action a in state s and then following policy π from that point on

Following a similar line of reasoning, the ***optimal*** *action value function* is

$$Q^*(s,a) = \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \gamma \max_{a'} Q^*(S_{t+1}, a')]$$

Q-Learning

- Q-learning algorithm (ε-*greedy version*)

Initialize $\hat{Q}(s,a)$ at random, put the agent is in a random state s

Repeat:

1) Select the action $\text{argmax}_a \hat{Q}(s,a)$ with probability $(1-\varepsilon)$ otherwise, select a at random
2) The agent is now in state s' and has received the reward r
3) Update $\hat{Q}(s,a)$ by

$$\Delta \hat{Q}(s,a) = \alpha[r + \gamma \max_{a'} \hat{Q}(s',a') - \hat{Q}(s,a)]$$

Exponential Moving Average (see later …)

$\mathcal{A}$

Q-Learning

- Q-learning algorithm

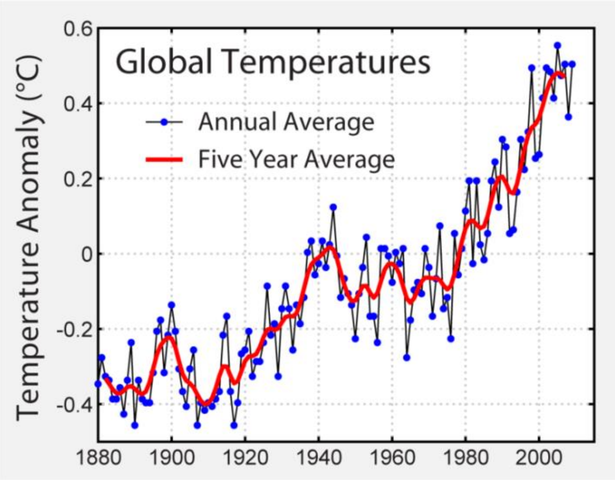
Theorem (Watkins, 1989): in the limit of that each action is played infinitely often and each state is visited infinitely often and $\alpha \to 0$ as experience progresses, then

$$\hat{Q}(s,a) \to Q^*(s,a)$$

with probability 1

The Q-learning algorithm bypasses the MDP entirely,
in the sense that the optimal strategy is learnt without learning the model $P(S_{t+1} \mid S_t, A_t)$

An aside: *moving averages*



[image from wikipedia]

Following non-stationary phenomena

- Average

  Definition: $$\overline{v}_T := \frac{1}{T}\sum_{k=1}^{T} v_k$$

  Running implementation:

$$\overline{v}_T = \frac{1}{T}(v_T + \sum_{k=1}^{T-1} v_k) = \frac{1}{T}(v_T + (T-1)\overline{v}_{T-1})$$

$$= \overline{v}_{T-1} + \frac{1}{T}(v_T - \overline{v}_{T-1}) = \frac{1}{T}\, v_T + (1 - \frac{1}{T})\, \overline{v}_{T-1}$$

"the weight of newer observations diminishes with time"

- Simple Moving Average (SMA)

$$\overline{v}_{T,n} := \frac{1}{n}\sum_{k=T-n}^{T} v_k$$

- Exponential Moving Average (EMA)

$$\overline{v}_{T,\alpha} := \alpha\, v_T + (1-\alpha)\,\overline{v}_{T-1,\alpha},\ \ \alpha \in [0,1]$$

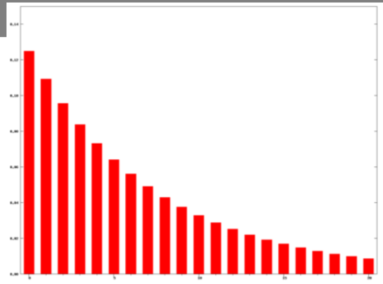
"the weight of newer observations remains constant"

- Exponential Moving Average (EMA)

$$\overline{v}_{T,\alpha} := \alpha\, v_T + (1-\alpha)\,\overline{v}_{T-1,\alpha},\ \ \alpha \in [0,1]$$

$$(1-\alpha)^{\Delta_t}$$

"the weight of older observations diminishes with time"



[image from wikipedia]

Expanding:

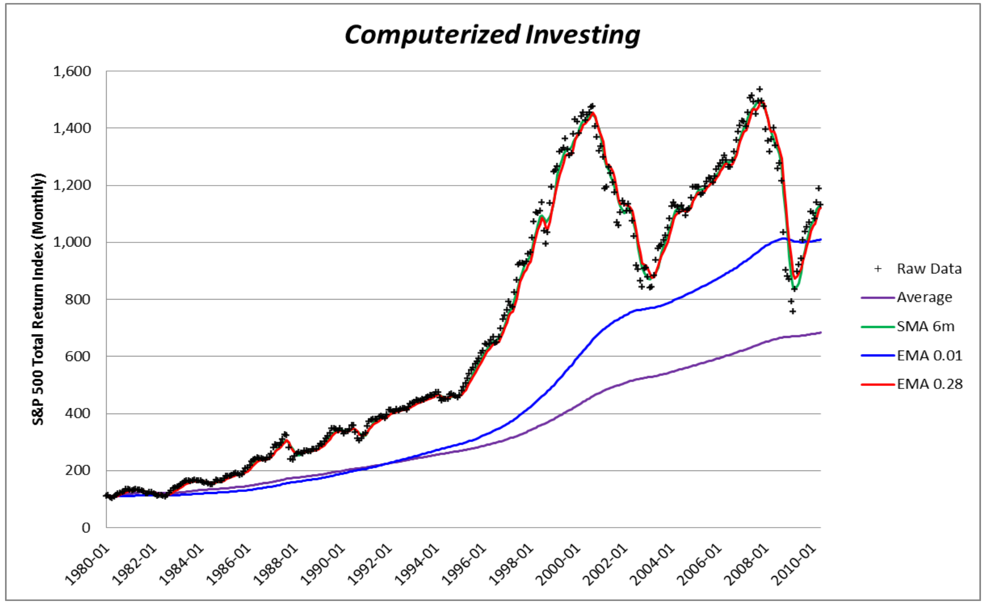
$$\begin{aligned}\overline{v}_{t,\alpha} &= \alpha\, v_t + (1-\alpha)\,\overline{v}_{t-1,\alpha} \\ &= \alpha\, v_t + (1-\alpha)(\alpha\, v_{t-1} + (1-\alpha)\overline{v}_{t-2,\alpha}) \\ &= \alpha\, v_t + (1-\alpha)(\alpha\, v_{t-1} + (1-\alpha)(\alpha\, v_{t-2} + (1-\alpha)\overline{v}_{t-3,\alpha})) \\ &= \alpha\,(v_t + (1-\alpha)\, v_{t-1} + (1-\alpha)^2\, v_{t-2}) + (1-\alpha)^3\, \overline{v}_{t-3,\alpha}\end{aligned}$$

The weight of past contributions decays as

$$(1-\alpha)^{\Delta_t}$$

A SMA with n previous values is approximately equal to an EMA with

$$\alpha = \frac{2}{n+1}$$



Q-Learning revisited

- Q-learning algorithm (ε-*greedy version*)

off-policy

Initialize $\hat{Q}(s,a)$ at random, put the agent is in a random state s

Repeat:

1) Select the action $a = \text{argmax}_a \hat{Q}(s,a)$ with probability $(1-\varepsilon)$
   otherwise, select a at random
2) The agent is now in state s' and has received the reward r
3) Update $\hat{Q}(s,a)$ by

$$\Delta\hat{Q}(s,a) = \alpha[r + \gamma \max_{a'} \hat{Q}(s',a') - \hat{Q}(s,a)]$$

By rewriting step 3)

$$\hat{Q}(s,a) = \hat{Q}(s,a) + \Delta\hat{Q}(s,a) = \hat{Q}(s,a) + \alpha[r + \gamma \max_{a'} \hat{Q}(s',a') - \hat{Q}(s,a)]$$

$$= \underbrace{\alpha[r + \gamma \max_{a'} \hat{Q}(s',a')] + (1-\alpha)\hat{Q}(s,a)}_{\text{Exponential Moving Average}}$$

compare with (see before):

$$Q^*(s,a) = \underbrace{\sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \gamma \max_{a'} Q^*(S_{t+1},a')]}_{\text{Expectation}}$$

SARSA

- SARSA algorithm (ε-*greedy version*)

on-policy

Initialize $\hat{Q}(s,a)$ at random, put the agent is in a random state s
Repeat:

1) Select the action $a = \text{argmax}_a \hat{Q}(s,a)$ with probability $(1-\varepsilon)$
otherwise, select a at random
2) The agent is now in state s' and has received the reward r
3) Select the action $a' = \text{argmax}_a \hat{Q}(s',a)$ with probability $(1-\varepsilon)$
otherwise, select a' at random
4) Update $\hat{Q}(s,a)$ by

$$\Delta\hat{Q}(s,a) = \alpha[r + \gamma\hat{Q}(s',a') - \hat{Q}(s,a)]$$

No more 'max' here

Q-learning is a an *off-policy* algorithm: each update involves $\max_{a'} \hat{Q}(s',a')$
(i.e. *exploration* is not taken into account)

SARSA is a an *on-policy* algorithm: each update involves $\hat{Q}(s',a')$
(which involves the next policy action, *exploration* included)

SARSA vs Q-Learning

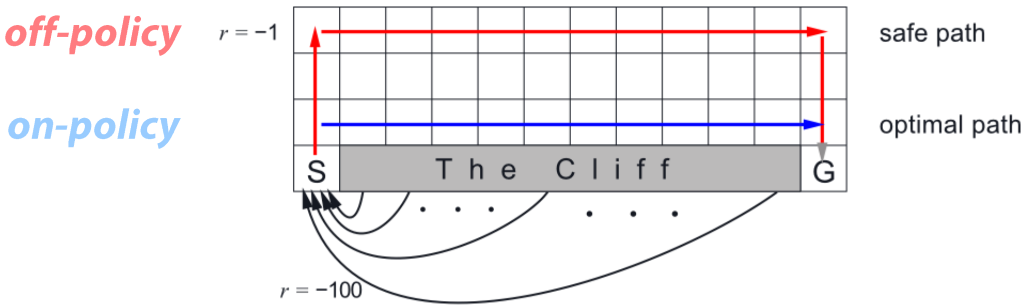
- Cliff World

  'S' is the start

  'G' is the goal

  Each white box has $r = -1$

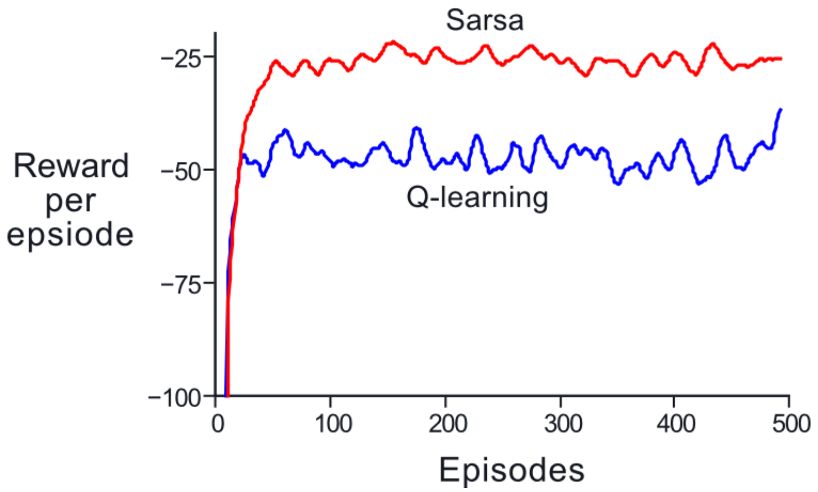
  'The Cliff' region has $r = -100$ and entails going back to 'S'



- Experimental Results

  SARSA finds a sub-optimal but safer path since its learning takes into account the ε risk of going off the cliff

  Q-learning finds the optimal path but, occasionally, it falls off the cliff during learning due to the ε-greedy strategy



Reinforcement Learning Methods

[image from: https://arxiv.org/pdf/1811.12560.pdf]