Deep Learning

A course about theory & practice

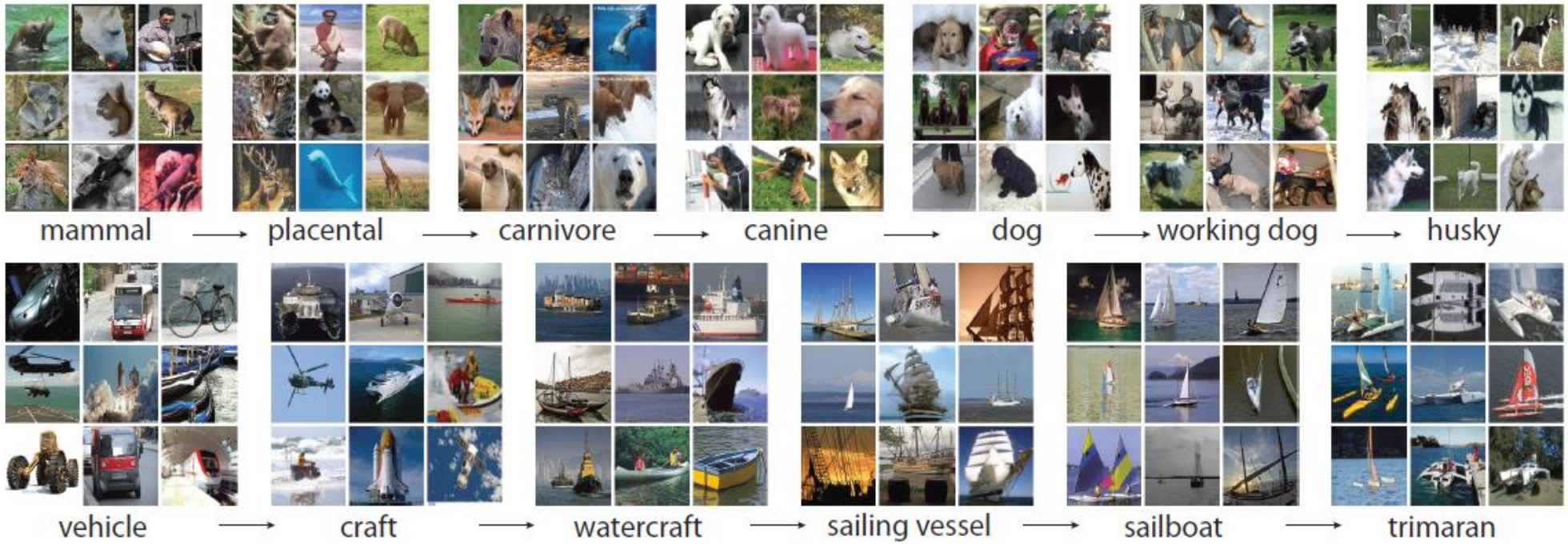
UNIVERSITÀ DI PAVIA

Deep Convolutional Neural Networks

Marco Piastra

ImageNet Challenge

- The ImageNet Large Scale Visual Recognition Challenge



1,461,406 full resolution images

Complex and multiple textual annotation,
hierarchy of 1000 object classes along several dimensions

The image classification challenge is run annually since 2010

[figures from www.nvidia.com]

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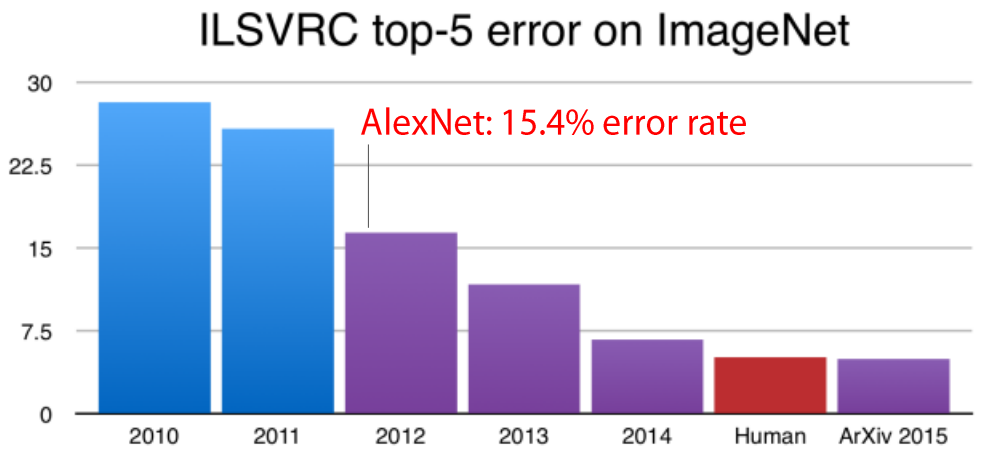

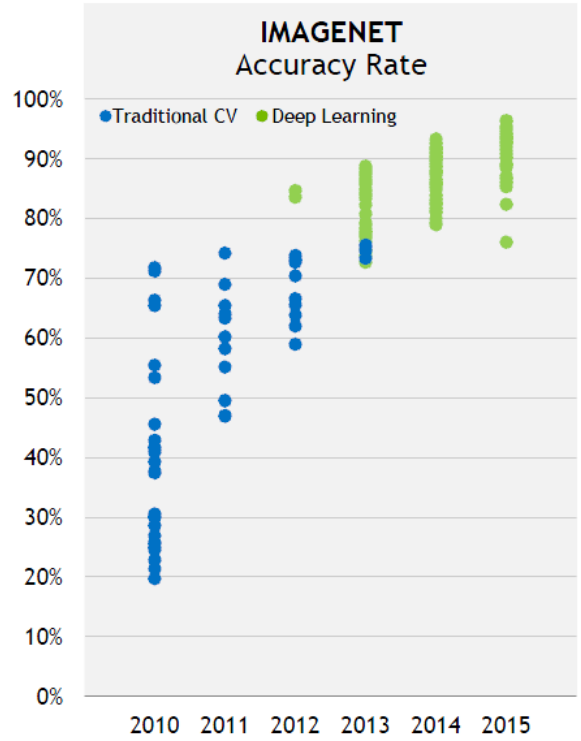

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The Mother of all DCNNs

Deep Convolutional Neural Network (DCNN)

- ***AlexNet*** [Krizhevsky, Sutskever & Hinton, 2012]

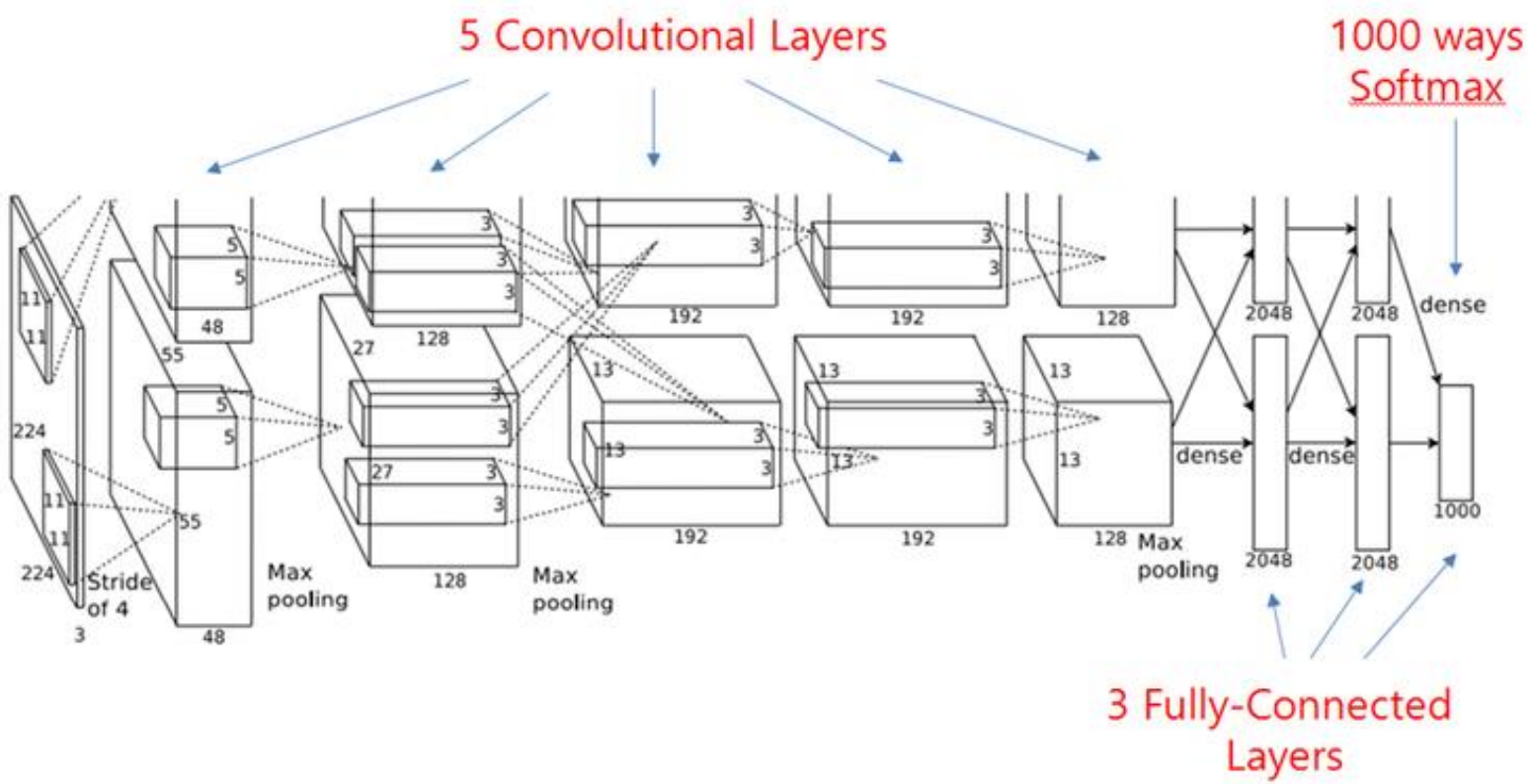


Image from [Krizhevsky, Sutskever & Hinton, 2012]

The Mother of all DCNNs

Deep Convolutional Neural Network (DCNN)

- ***AlexNet*** [Krizhevsky, Sutskever & Hinton, 2012]

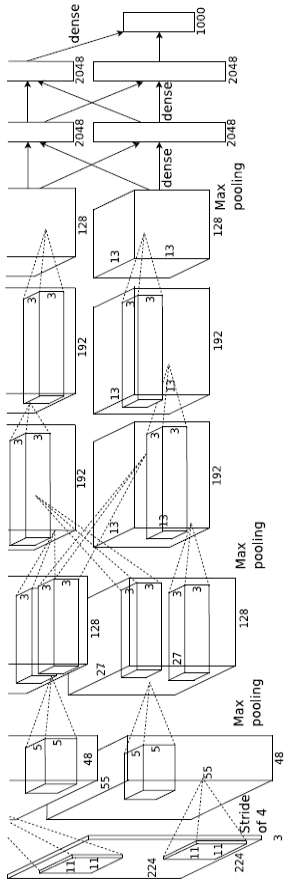
This layer is ***Softmax***

These layers are ***fully connected (FC)*** FF neural networks

This layer is ***Max Pooling***

These layers are ***convolutional*** with ***ReLU*** as non-linearity

This layer is ***Local Response Normalization (LRN)***



DCNN Building Blocks (layerwise)

Convolutional Layer

- ***Convolution operation***

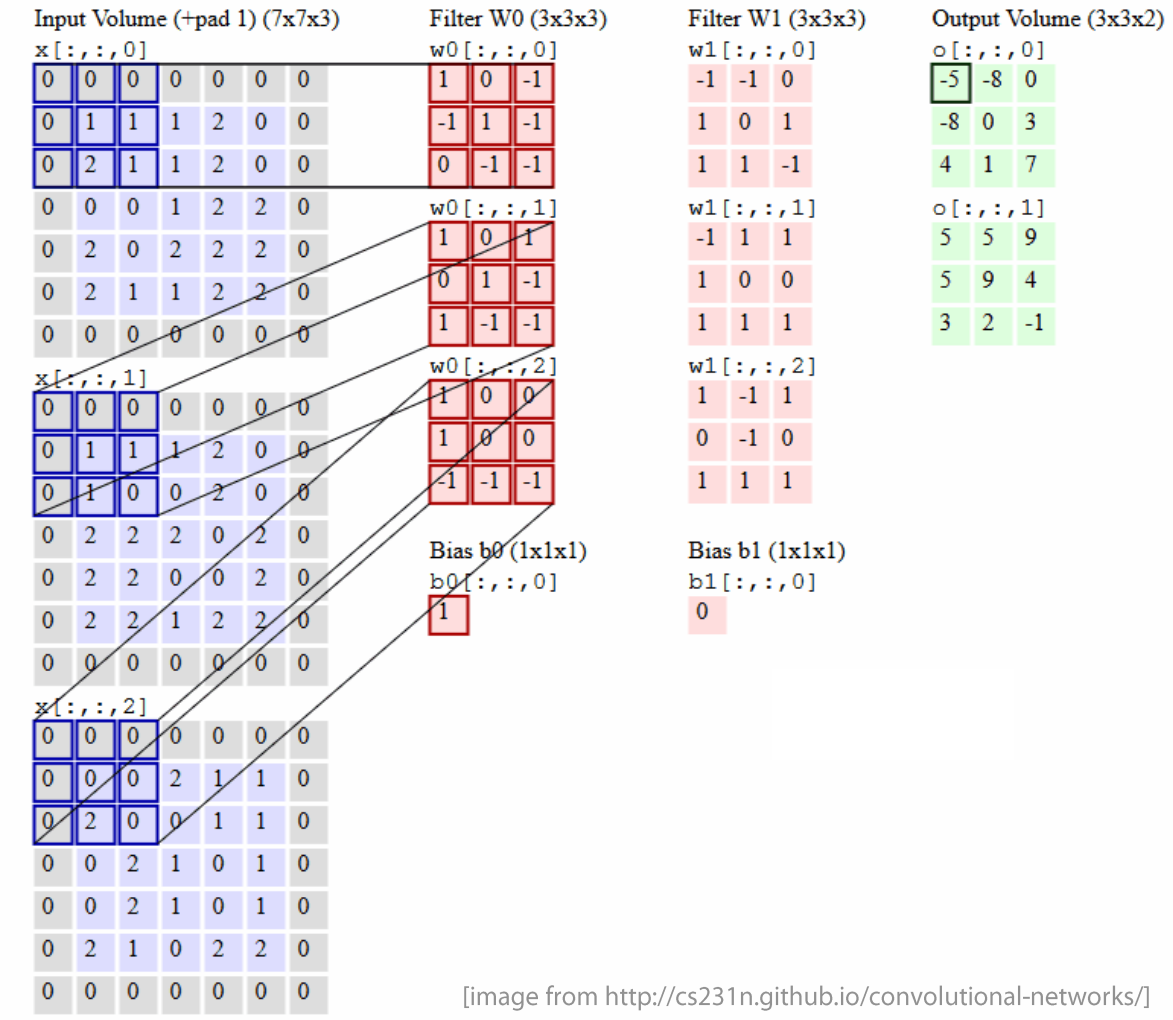
  A ***convolution filter***
  is a square (or cubic) matrix

  It is first centered on a pixel
  of the input image

  It produces a scalar value:
  the dot product
  between the filter
  and the image region
  around the pixel

  By mapping the same
  procedure on all pixels
  of the input image,

  a new image is produced
  (i.e. a *feature map*)



[image from http://cs231n.github.io/convolutional-networks/]

Convolutional Layer

- **Convolution operations (on images)**

A ***convolution filter***
is a square (or cubic) matrix

In symbols

$$\boldsymbol{Y}_i = \boldsymbol{W}_i * \boldsymbol{X}$$

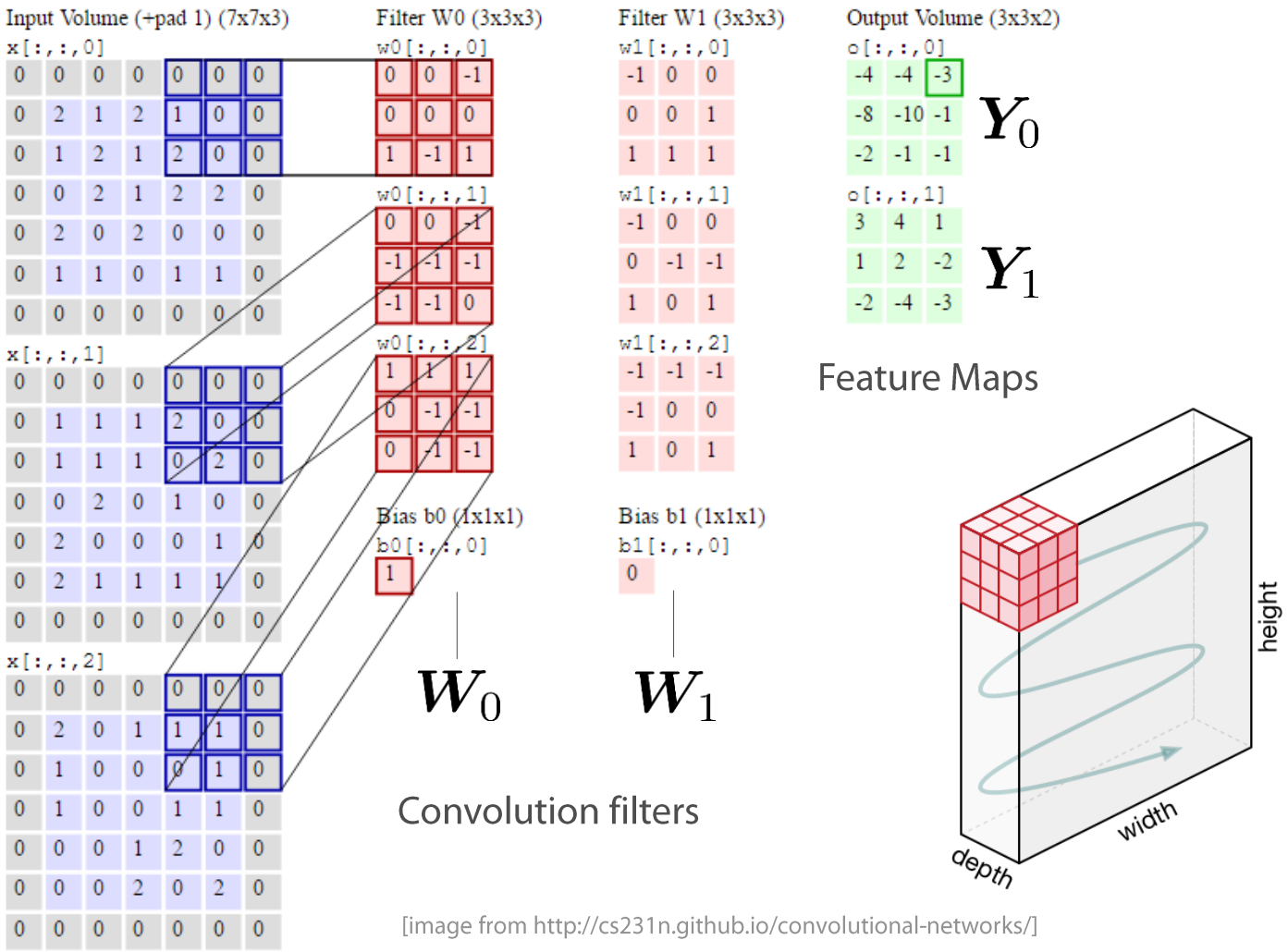
i-th feature map

convolution operator

where:

$\boldsymbol{X}$

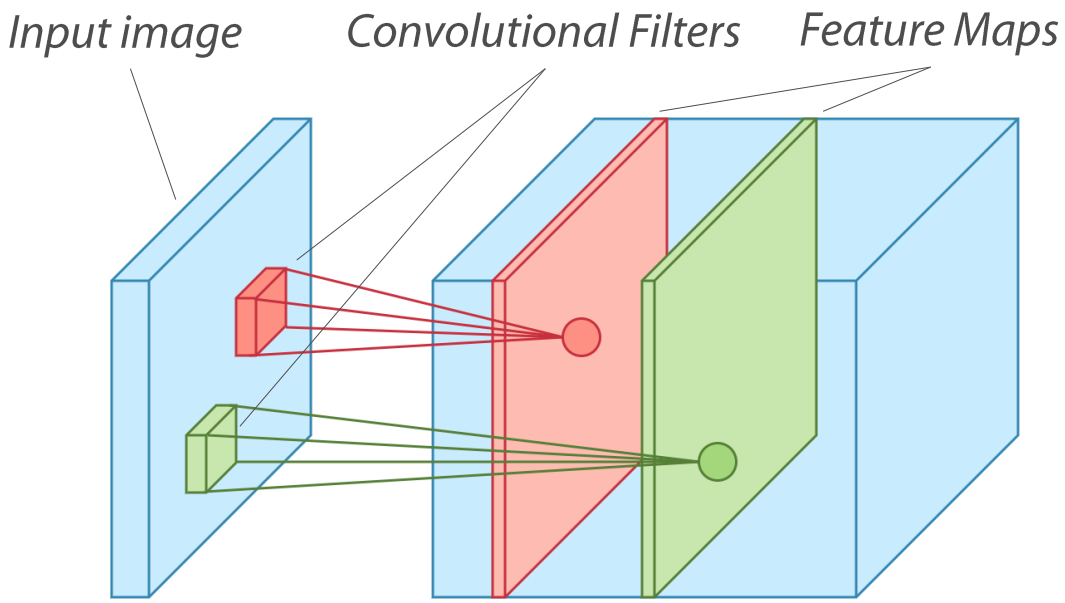
Input image (e.g. RGB)



[image from http://cs231n.github.io/convolutional-networks/]

Convolutional Layer

- **Convolution network (first layer)**



$$\boldsymbol{X} * [\boldsymbol{W}_0, \cdots, \boldsymbol{W}_h] = [\boldsymbol{Y}_0, \cdots, \boldsymbol{Y}_h]$$

[image adapted from http://cs231n.github.io/convolutional-networks/]

Convolutional Layer

- ***Convolution operation with non-linearity***

  The linear form for convolution

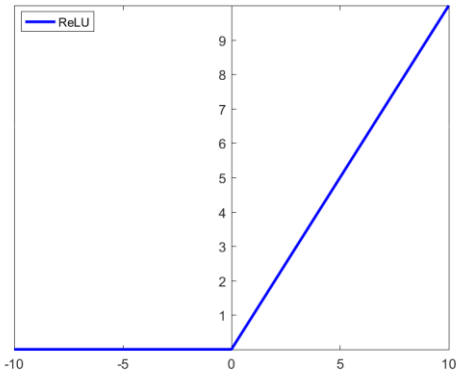
$$\boldsymbol{Y}_i := \boldsymbol{W}_i * \boldsymbol{X}$$

  A non-linearity is added in convolutional neural networks

$$\boldsymbol{Y}_i := \mathrm{ReLU}(\boldsymbol{W}_i * \boldsymbol{X})$$

  Applied elementwise to all matrix components

$$y = \max(0, x)$$



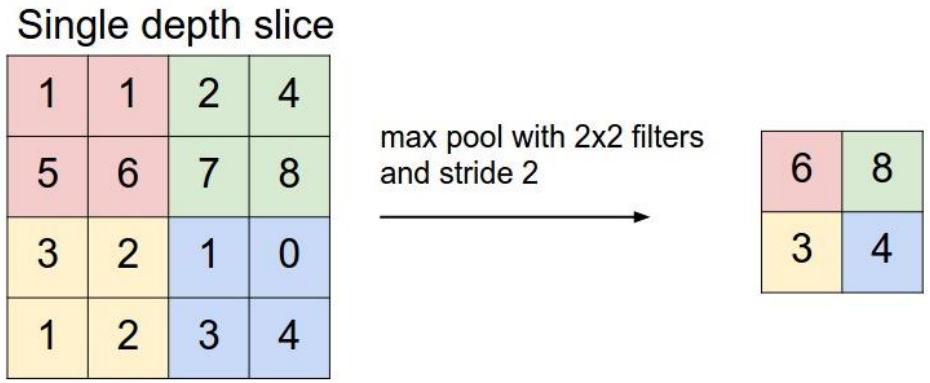
ReLU

Why ReLU?
To be seen later on, when discussing training

Max Pooling Layer

- ***Max Pooling operation***

  Returns the maximum value in a pre-defined region of its input

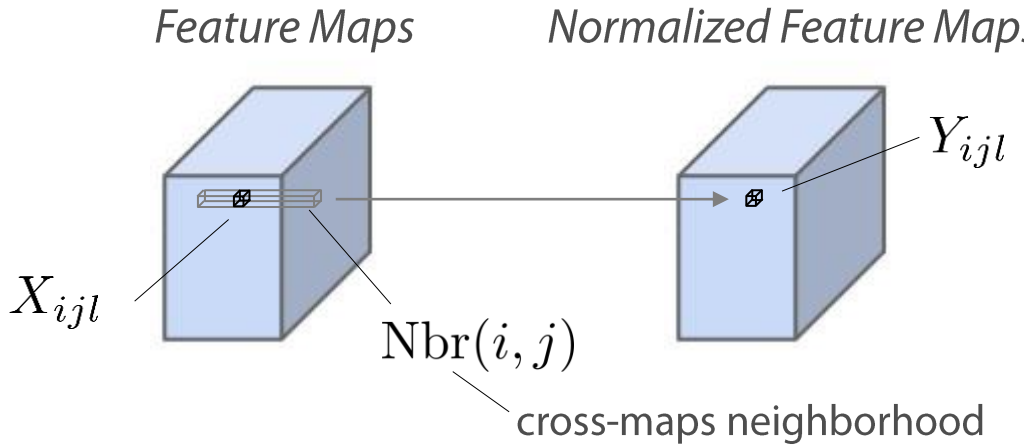


Local Response Normalization Layer

- ***Local Response Normalization (LRN)***

Rationale:

compensating the tendency
of ReLU to produce
large values in output



$$[\boldsymbol{X}_0, \cdots, \boldsymbol{X}_h] \qquad [\boldsymbol{Y}_0, \cdots, \boldsymbol{Y}_h]$$

Two variants:

- *across feature* maps
  (i.e. as in figure)
- *within feature* map
  (i.e. with neighboring pixels)

$$Y_{ijl} := \frac{X_{ijl}}{\left(a + \alpha \sum_{k \in \mathrm{Nbr}(i,j)} (X_{ijk})^2\right)^{\beta}}$$

note the summation

where a, α, β are fixed *hyperparameters*

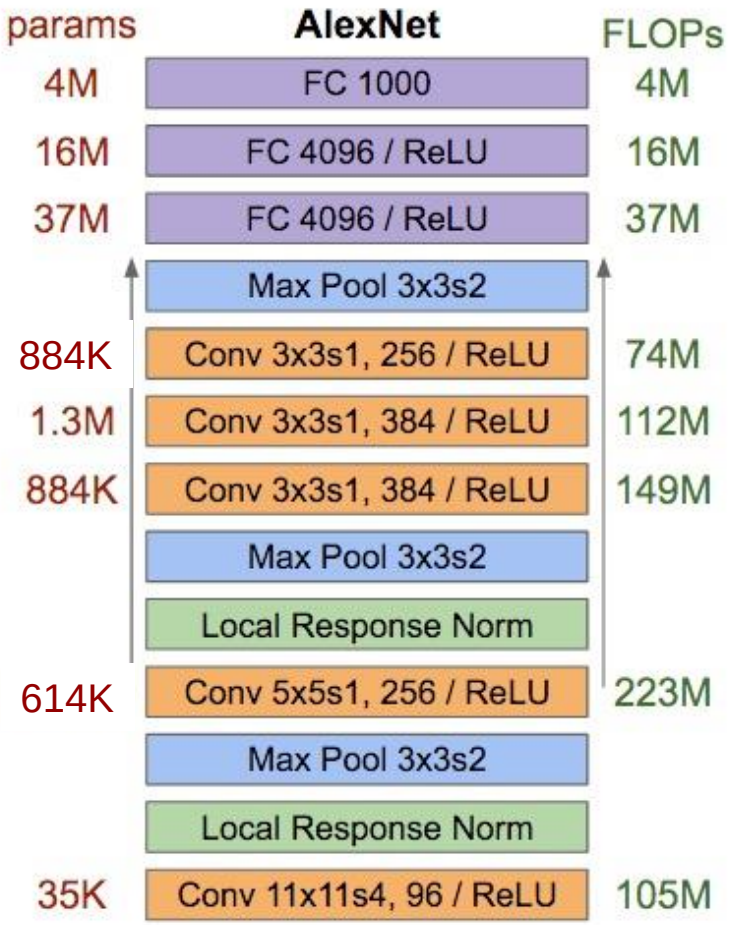
AlexNet Architecture

- ***AlexNet*** [Krizhevsky, Sutskever & Hinton, 2012]
  - number of parameters, per layer
    in red on the left
  - number of *floating point operations*, (FLOP) per layer in single forward pass
    in green on the right

Higher layers have more parameters but the bulk of the computation takes place at lower layers

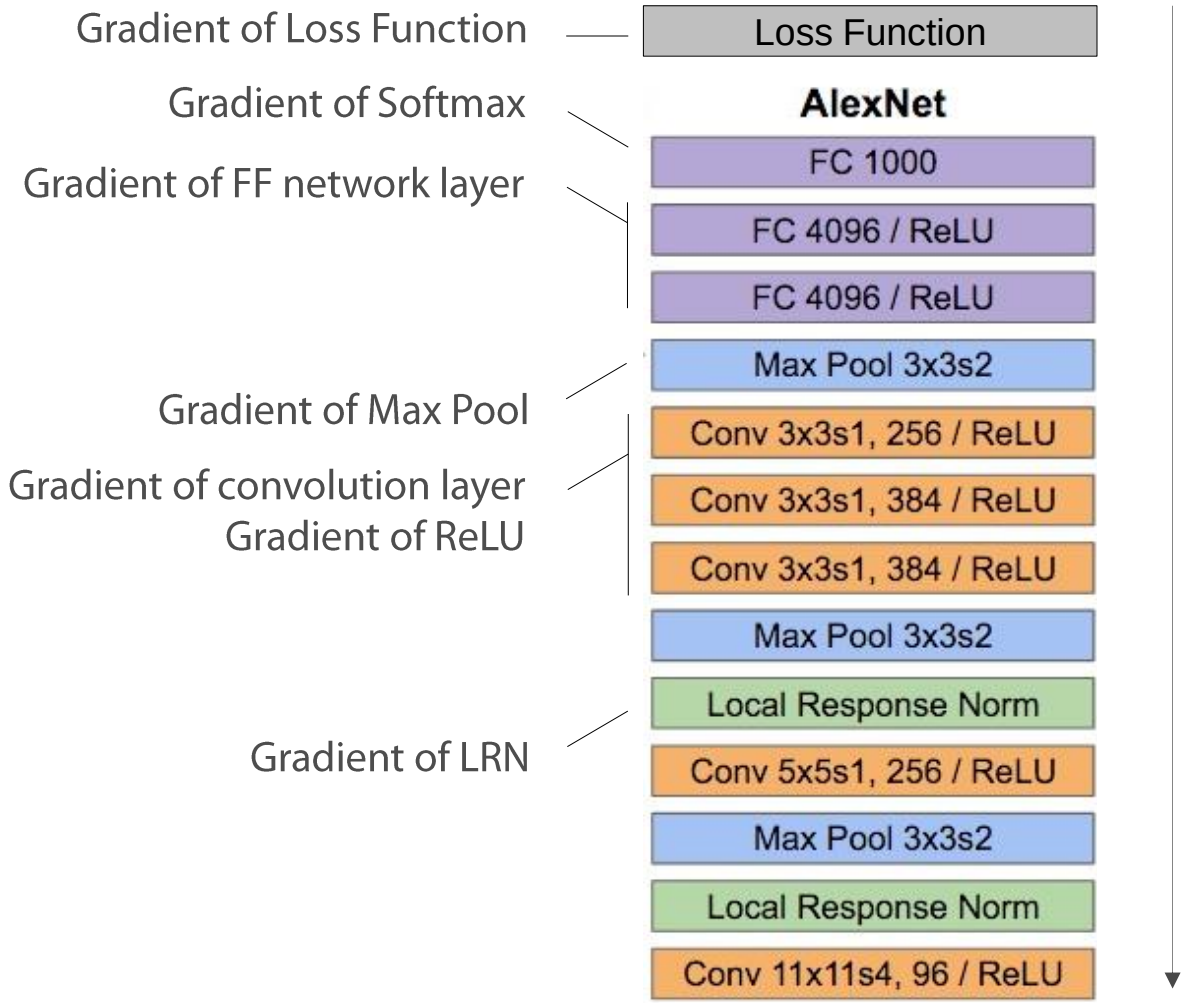
Totals:

- around 64M parameters
- around 837M FLOPs for a single pass



AlexNet Gradient

- ***Computing gradients (backward propagation)***



Convolutional Layer Gradient

- **Gradient of *convolutional layer***

Define

$$\boldsymbol{Y} = (\boldsymbol{W} * \boldsymbol{X})$$

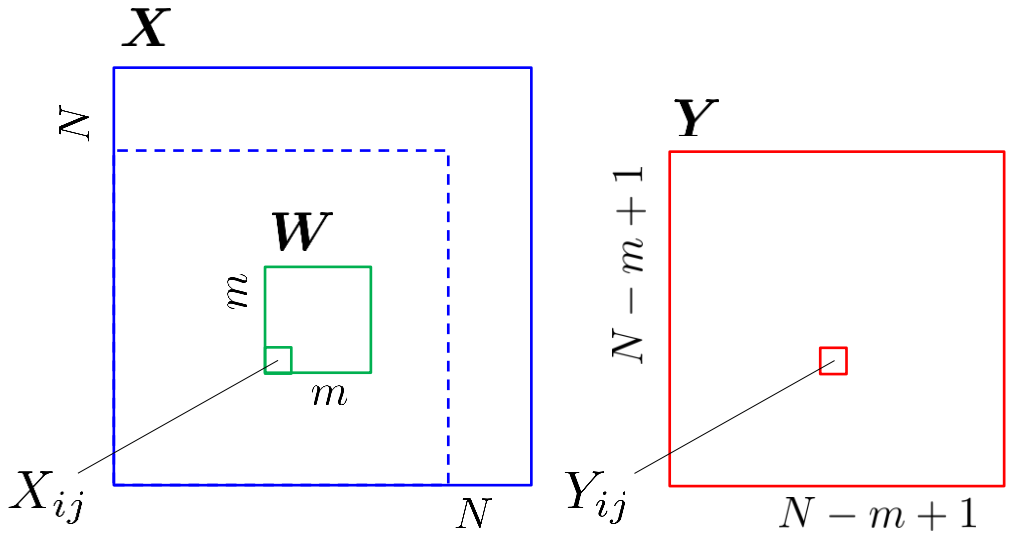
and, for convenience, assume that

$$\boldsymbol{W} \in \mathbb{R}^{m \times m}, \quad \boldsymbol{X} \in \mathbb{R}^{N \times N}$$

the input image is square

$$Y_{ij} = \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} W_{ab} X_{(i+a)(j+b)}$$

the convolution operator is 'centered' in the lower left corner



All matrices in this example are indexed as images: i.e. the lower left corner is $\circ\!\!\circ$

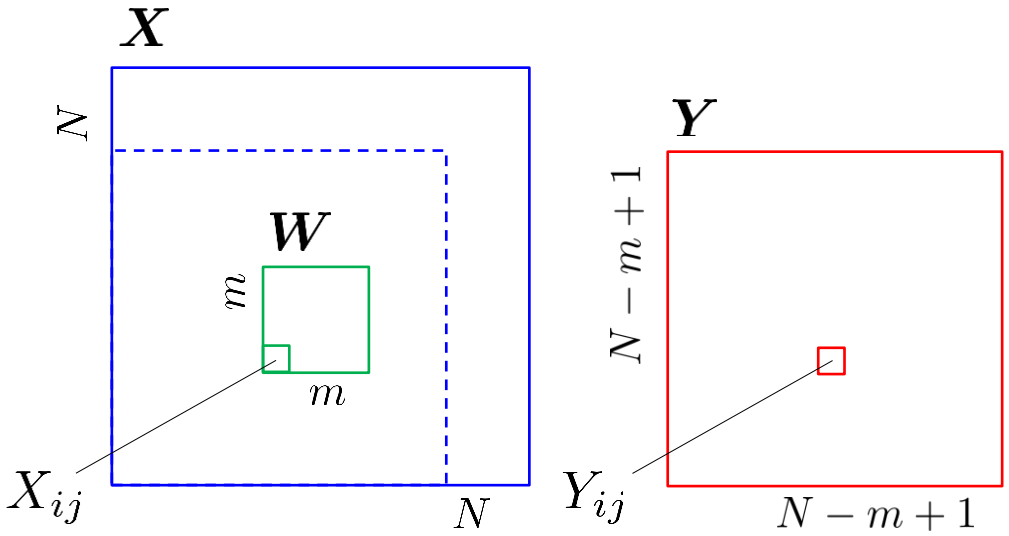
(In general m *is odd and the convolution is 'centered' in the centroid of the filter)*

Convolutional Layer Gradient

- **Gradient of *convolutional layer***

$$\boldsymbol{Y} = (\boldsymbol{W} * \boldsymbol{X})$$

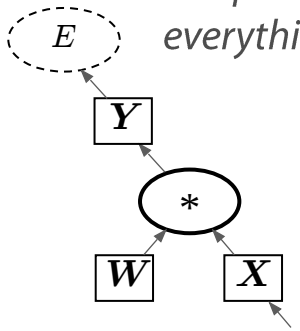
$$Y_{ij} = \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} W_{ab} X_{(i+a)(j+b)}$$



Consider the pseudo-graph

The flow is bottom-up, in this example

The pseudo-node E represents everything that stands above



Another subgraph stands below (not represented)

Convolutional Layer Gradient

- **Gradient of *convolutional layer***

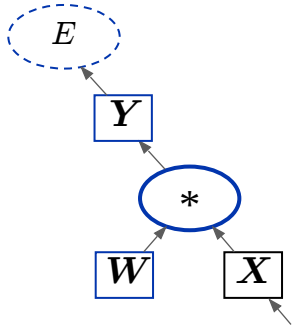
$$\boldsymbol{Y} = (\boldsymbol{W} * \boldsymbol{X})$$

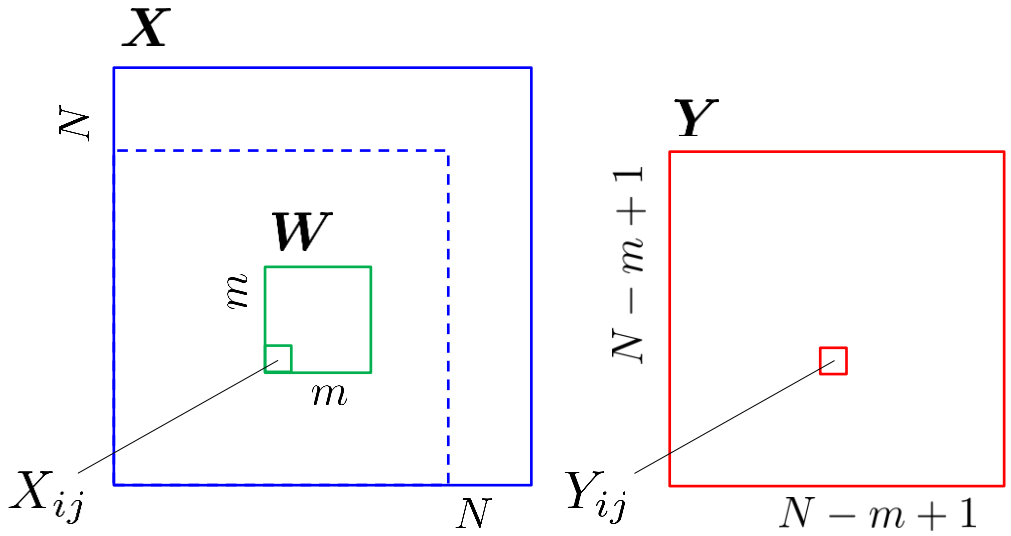
$$Y_{ij} = \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} W_{ab} X_{(i+a)(j+b)}$$

Case 1:

$$\frac{\partial}{\partial \boldsymbol{W}} E(\boldsymbol{Y})$$

i.e. the 'end of the chain'





Convolutional Layer Gradient

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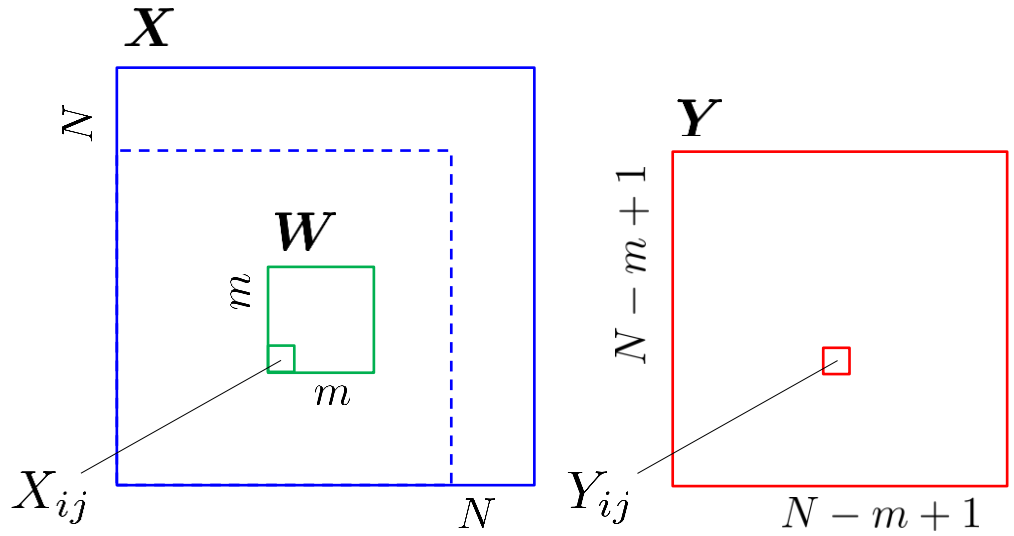

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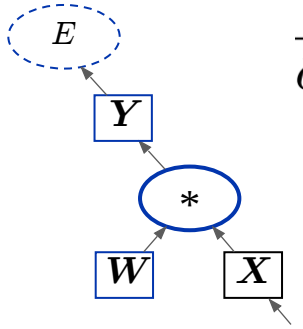


$$\frac{\partial}{\partial W_{lk}} E(\boldsymbol{Y}) = \sum_{i=0}^{N-m} \sum_{j=0}^{N-m} \frac{\partial E(\boldsymbol{Y})}{\partial Y_{ij}} \frac{\partial Y_{ij}}{\partial W_{lk}}$$

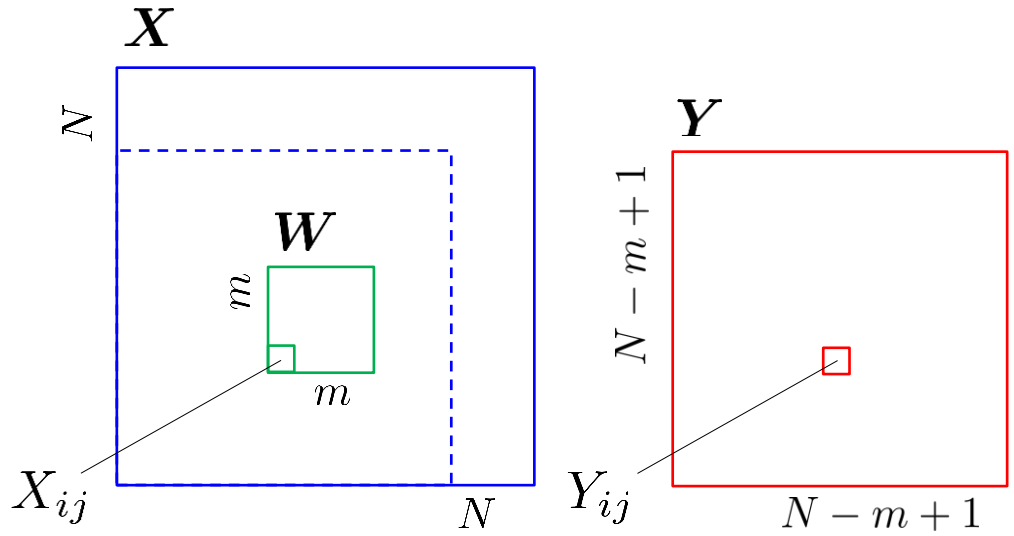
by applying the chain rule
in extended version (see also Episode 1)

Convolutional Layer Gradient

- **Gradient of *convolutional layer***

$$\boldsymbol{Y} = (\boldsymbol{W} * \boldsymbol{X})$$

$$Y_{ij} = \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} W_{ab} X_{(i+a)(j+b)}$$



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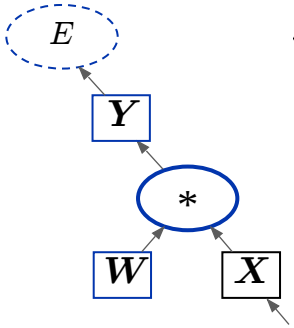

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$$\partial E_{ij} := \frac{\partial E(\boldsymbol{Y})}{\partial Y_{ij}}$$

i.e. the backpropagation component across Y_{ij}

$$\frac{\partial Y_{ij}}{\partial W_{lk}} = X_{(i+l)(j+k)}$$

$$\frac{\partial}{\partial W_{lk}} E(\boldsymbol{Y}) = \sum_{i=0}^{N-m} \sum_{j=0}^{N-m} \partial E_{ij}\, X_{(i+l)(j+k)}$$

Convolutional Layer Gradient

- **Gradient of *convolutional layer***

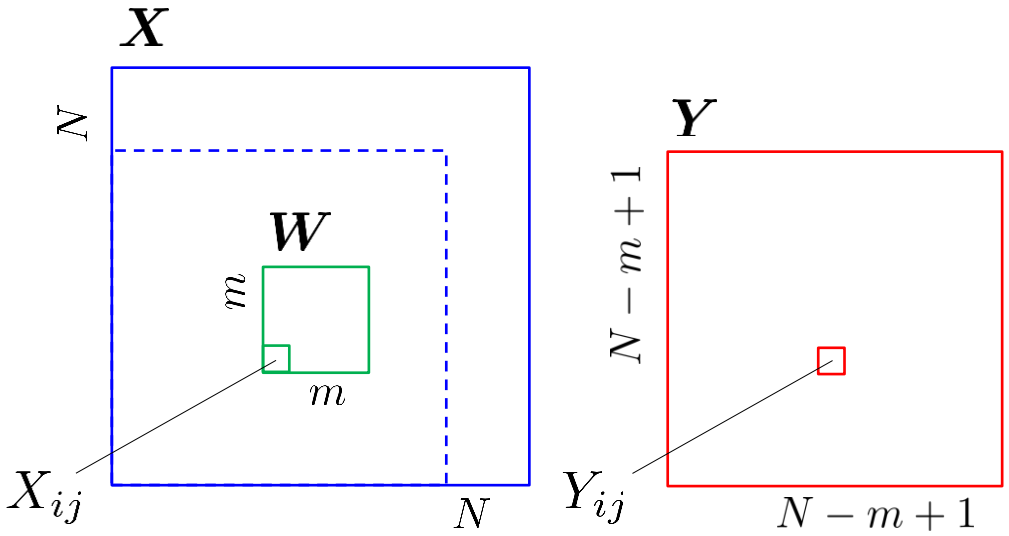
$$\boldsymbol{Y} = (\boldsymbol{W} * \boldsymbol{X})$$

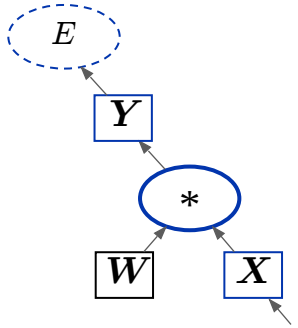
$$Y_{ij} = \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} W_{ab} X_{(i+a)(j+b)}$$

Case 2:

$$\frac{\partial}{\partial \boldsymbol{\vartheta}} E(\boldsymbol{Y})$$

$\boldsymbol{\vartheta} \neq \boldsymbol{W}$ is a generic parameter on which $\boldsymbol{X}$ depends





Convolutional Layer Gradient

- **Gradient of *convolutional layer***

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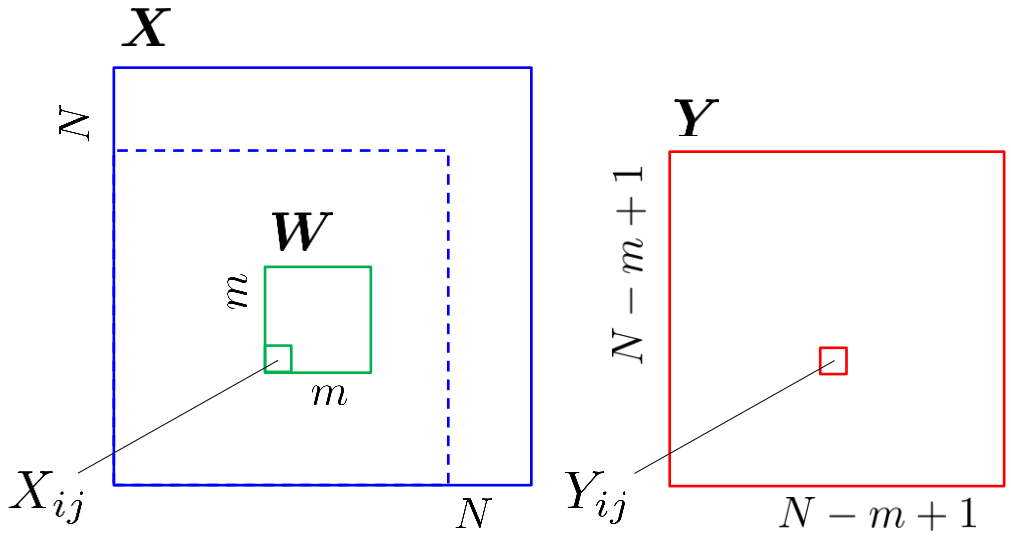

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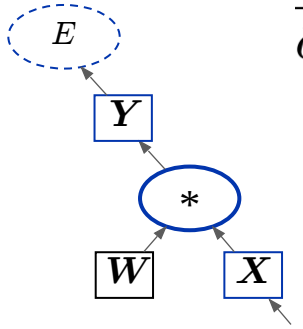

$$\frac{\partial}{\partial \vartheta} E(\boldsymbol{Y}) = \sum_{i=0}^{N-m} \sum_{j=0}^{N-m} \partial E_{ij} \frac{\partial Y_{ij}}{\partial \vartheta}$$

$$= \sum_{i=0}^{N-m} \sum_{j=0}^{N-m} \partial E_{ij} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} W_{ab} \frac{\partial}{\partial \vartheta} X_{(i+a)(j+b)}$$

This is inconvenient: the same X components appear multiple times - let's re-factor ….

Convolutional Layer Gradient

- **Gradient of *convolutional layer***

$$\boldsymbol{Y} = (\boldsymbol{W} * \boldsymbol{X})$$

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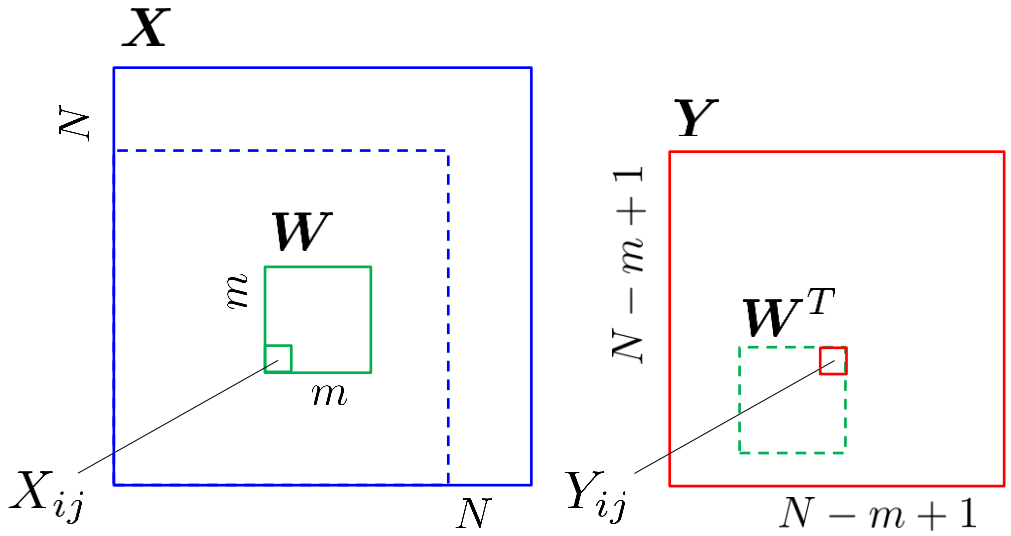

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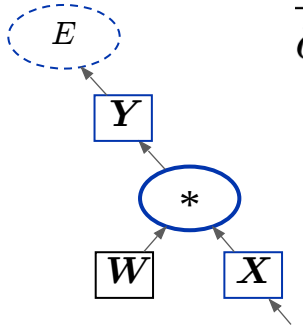

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$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \frac{\partial}{\partial \boldsymbol{\vartheta}} X_{ij} \sum_{a=0}^{m-1} \sum_{b=0}^{m-1} W_{ba} \partial E_{(i-a)(j-b)} \quad (i-a), (j-b) \geq 0$$

note the inversion of indexes

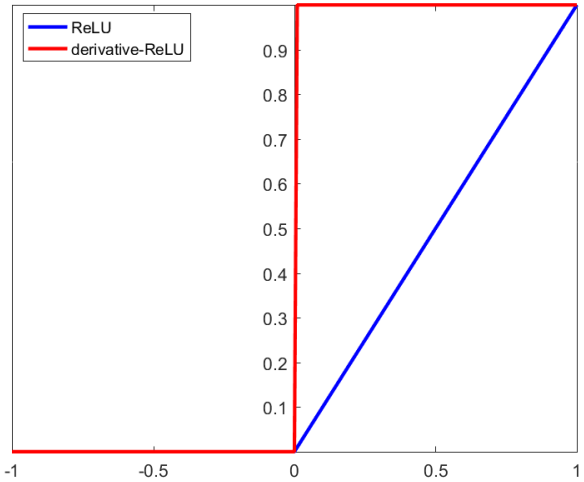
Convolutional Layer Gradient

- **Gradient of *ReLU*** *(see also Episode 1)*

$$\boldsymbol{Y} = \mathrm{ReLU}(\boldsymbol{X})$$

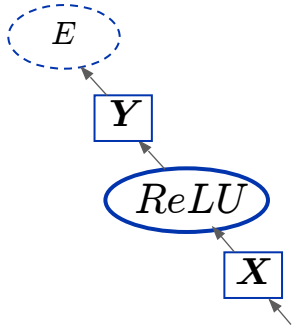
(ReLU has no parameters of its own)

$$\frac{\partial}{\partial x}\mathrm{ReLU}(x) = \frac{\partial}{\partial x}\max(x, 0) \approx \mathrm{step}(x)$$



So the gradient of ReLU acts like a 'switch'
When is it open?

Backpropagation alone 'does not know'



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- **Gradient of *ReLU*** *(see also Episode 1)*

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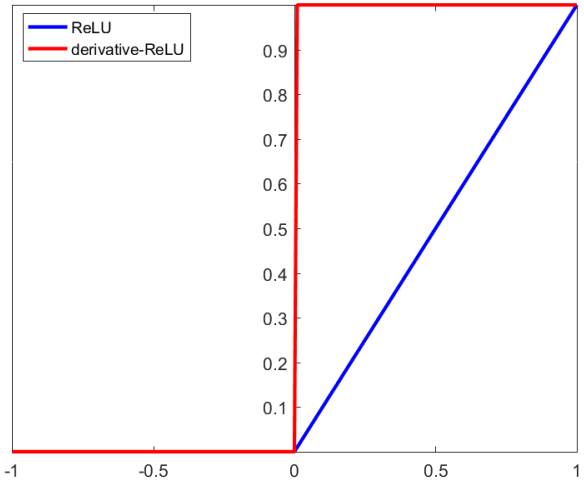

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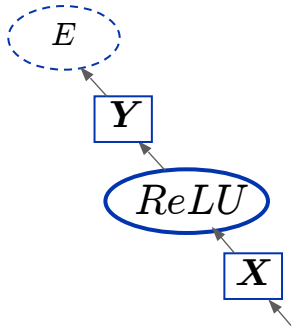


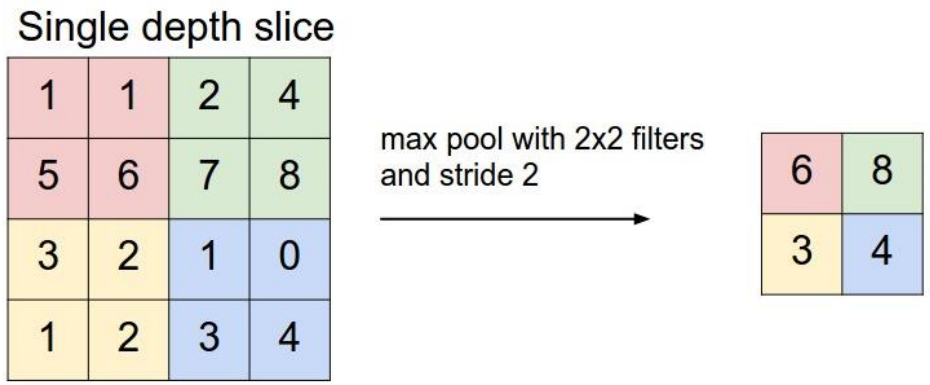
$\frac{\partial}{\partial \boldsymbol{\vartheta}} E(\boldsymbol{Y})$ *This is the gradient we want to compute*

as we have to apply it to each specific data item $\left(\frac{\partial}{\partial \boldsymbol{\vartheta}} E(\boldsymbol{Y})\right)(\boldsymbol{X}^{(i)})$

Moral: we need to perform one forward pass (i.e. activation)
to decide which component Y_{ij} is open (i.e. $/1$) and which is not (i.e. $/0$)

Max Pooling Gradient

- **Gradient of *Max Pooling***

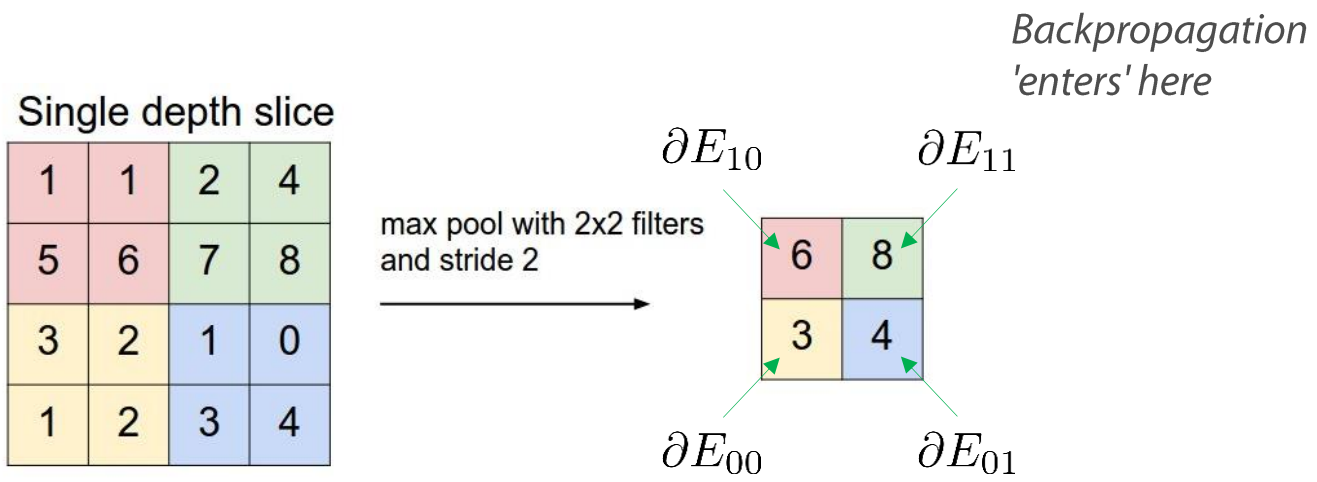


The gradient of max pooling acts as *multiplexer*

As with ReLU, one forward pass is required to determine which channel is selected

Max Pooling Gradient

- **Gradient of *Max Pooling***



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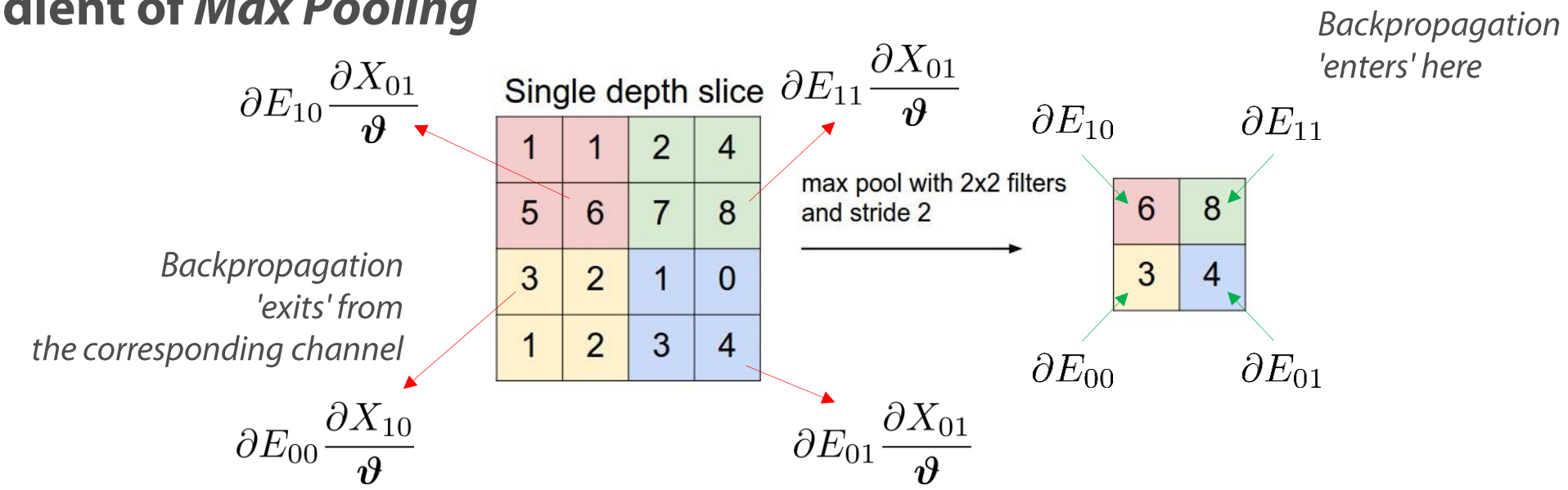
Max Pooling Gradient

- **Gradient of *Max Pooling***



The gradient of max pooling acts as *multiplexer*

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LRN Gradient

- **Gradient of *Local Response Normalization***

$$Y_{ijl} := \frac{X_{ijl}}{\left(a + \alpha \sum_{k \in \mathrm{Nbr}(i,j)} (X_{ijk})^2\right)^{\beta}}$$

where a, α, β are fixed hyperparameters

This formula is quite inconvenient: let's simplify….

LRN Gradient

- **Gradient of *Local Response Normalization***

$$Y_{ijl} := \frac{X_{ijl}}{\sum_{k=1}^{h} X_{ijk}}$$

i.e. plain, cross-map normalization

(simplified formula)

LRN Gradient

- **Gradient of *Local Response Normalization***

$$Y_{ijl} := \frac{X_{ijl}}{\sum_{k=1}^{h} X_{ijk}}$$

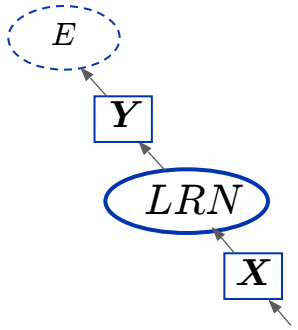
i.e. plain, cross-map normalization

i.e. the backpropagation component across Y_{ijl}

$$\frac{\partial}{\partial \boldsymbol{\vartheta}} E(\boldsymbol{Y}) = \sum_{i,j} \sum_{l} \partial E_{ijl} \frac{\partial Y_{ijl}}{\partial \boldsymbol{\vartheta}} \qquad \text{where} \quad \partial E_{ijl} := \frac{\partial E(\boldsymbol{Y})}{\partial Y_{ijl}}$$

$$= \sum_{i,j} \sum_{l} \partial E_{ijl} \frac{\partial}{\partial \boldsymbol{\vartheta}} \frac{X_{ijl}}{\sum_k X_{ijk}}$$

$$= \sum_{i,j} \sum_{l} \partial E_{ijl} \left(\frac{1}{\sum_k X_{ijk}} \frac{\partial X_{ijl}}{\partial \boldsymbol{\vartheta}} - \frac{X_{ijl}}{\left(\sum_k X_{ijk}\right)^2} \sum_k \frac{\partial X_{ijk}}{\partial \boldsymbol{\vartheta}} \right)$$



$$= \sum_{i,j} \sum_{l} \partial E_{ijl} \left(\frac{1}{c} \frac{\partial X_{ijl}}{\partial \boldsymbol{\vartheta}} - \frac{Y_{ijl}}{c} \sum_k \frac{\partial X_{ijk}}{\partial \boldsymbol{\vartheta}} \right)$$

where

$$c := \sum_k X_{ijk}$$

LRN Gradient

- **Gradient of *Local Response Normalization***

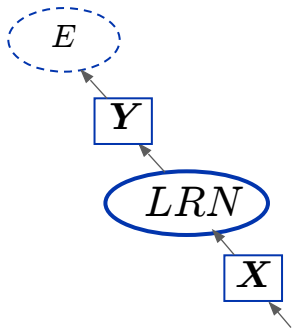
$$Y_{ijl} := \frac{X_{ijl}}{\sum_{k=1}^{h} X_{ijk}}$$

i.e. plain, cross-map normalization

$$\frac{\partial}{\partial \boldsymbol{\vartheta}} E(\boldsymbol{Y}) = \sum_{i,j} \sum_{l} \partial E_{ijl} \left(\frac{1}{c} \frac{\partial X_{ijl}}{\partial \boldsymbol{\vartheta}} - \frac{Y_{ijl}}{c} \sum_{k} \frac{\partial X_{ijk}}{\partial \boldsymbol{\vartheta}} \right)$$

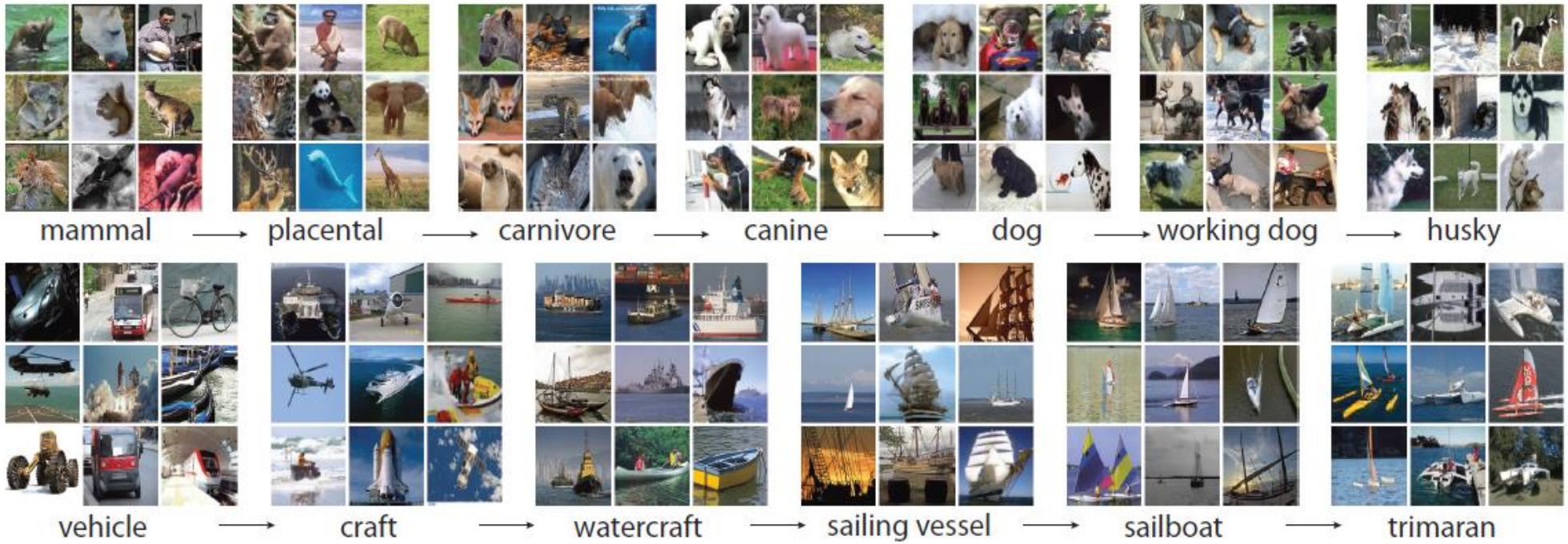
This is inconvenient: the same X components appear multiple times - let's re-factor ….

$$= \sum_{i,j} \sum_{l} \left(\frac{\partial E_{ijl}}{c} - \sum_{k} \left(Y_{ijk} \frac{\partial E_{ijk}}{c} \right) \right) \frac{\partial X_{ijl}}{\partial \boldsymbol{\vartheta}}$$



ImageNet Challenge

- The ImageNet Large Scale Visual Recognition Challenge



1,461,406 full resolution images

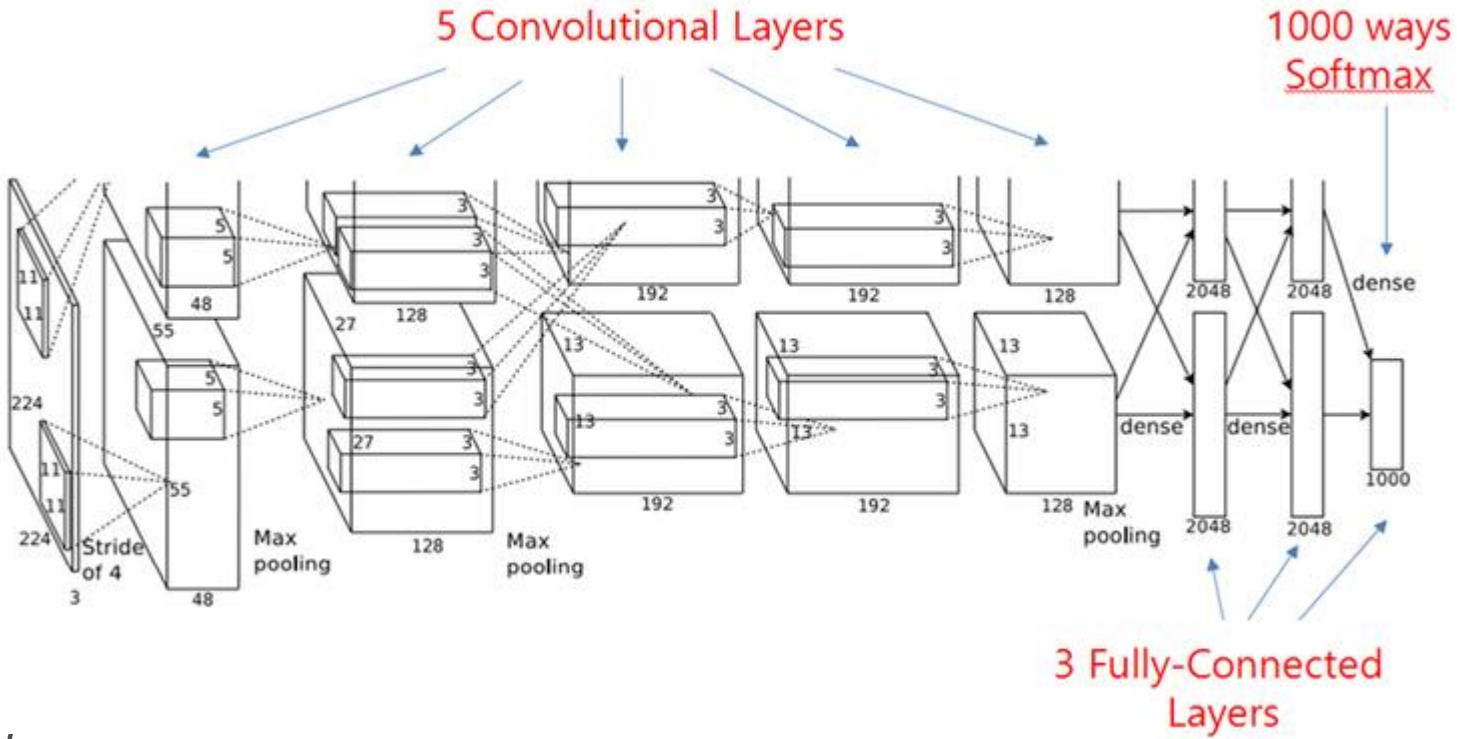
Complex and multiple textual annotation,
hierarchy of 1000 object classes along several dimensions

The image classification challenge is run annually since 2010

[figures from www.nvidia.com]

AlexNet (Krizhevsky, Sutskever & Hinton, 2012)

"The Mother of all DCNNs"



Trained with *batch gradient descent*

- the final supervised training set contained 15M images
- training was performed on two NVIDIA GTX 580 GPUs for six days

[image from https://world4jason.gitbooks.io/research-log/content/deepLearning/CNN/Model%20&%20ImgNet/alexnet/alexnet.html]

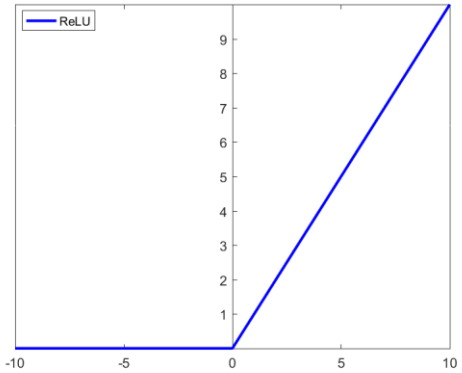
Deep Convolutional Neural Networks (DCNN)

- **AlexNet**

  Why ReLU and not another non-linearity?

  Because it is much faster to train.

$$y = \max(0, x)$$



ReLU

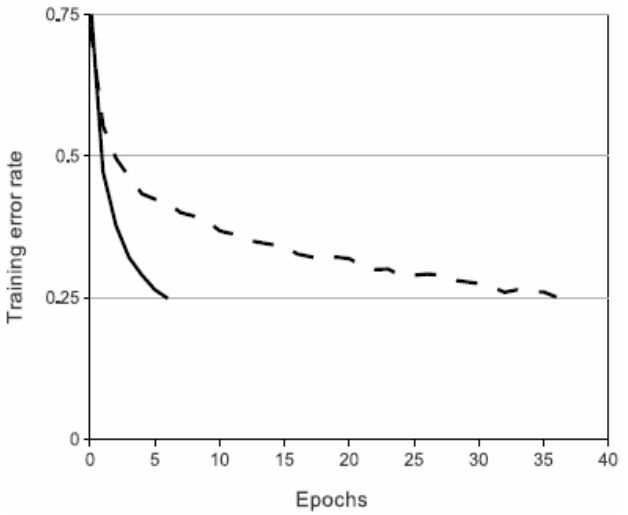


Figure 1: A four-layer convolutional neural network with ReLUs (**solid line**) reaches a 25% training error rate on CIFAR-10 six times faster than an equivalent network with tanh neurons (**dashed line**). The learning rates for each network were chosen independently to make training as fast as possible. No regularization of any kind was employed. The magnitude of the effect demonstrated here varies with network architecture, but networks with ReLUs consistently learn several times faster than equivalents with saturating neurons.

Image from [Krizhevsky, Sutskever & Hinton, 2012]