TANGENT LINES AND OSCULATING CIRCLES

In these notes, we discuss some properties of tangent lines and osculating circles.

Proposition 1. *Let $P(s)$ be a curve parameterized by arclength, and let s_0 be fixed. The line $L(s)$ tangent to $P(s)$ at $P(s_0)$, parameterized by arclength, is the unique line such that*

$$\lim_{s \to s_0} \frac{|P(s) - L(s)|}{s - s_0} = 0\,. \tag{1}$$

The tangent line to $P(s)$ at $P(s_0)$ is parameterized by arclength by:

$$L(s) = P(s_0) + P'(s_0)(s - s_0)\,.$$

To prove Proposition 1, we first prove that $L(s)$ satisfies the condition (1).

Using the Taylor theorem for the curve $P(s)$, we get

$$P(s) = P(s_0) + P'(s_0)(s - s_0) + F(s)\,,$$

where $F(s)$ is a function such that:

$$\lim_{s \to s_0} \frac{F(s)}{s - s_0} = 0\,.$$

Therefore,

$$\lim_{s \to s_0} \frac{|P(s) - L(s)|}{s - s_0} = \lim_{s \to s_0} \frac{F(s)}{s - s_0} = 0$$

as claimed. It remains to prove uniqueness. For this purpose, we will suppose that $\widehat{L}$ is another line satisfying the condition of Equation (1), and we will prove that necessarily $\widehat{L} = L$.

Hence, let us suppose

$$\lim_{s \to s_0} \frac{|P(s) - \hat{L}(s)|}{s - s_0} = 0\,. \tag{2}$$

Therefore we have

$$\frac{|\widehat{L}(s) - L(s)|}{|s - s_0|} \leq \frac{|P(s) - L(s)|}{|s - s_0|} + \frac{|P(s) - \widehat{L}(s)|}{|s - s_0|} \to 0 \tag{3}$$

as $s \to s_0$. Now, since (2) holds, necessarily

$$\lim_{s \to s_0} L(s) = P(s_0)$$

and therefore $\widehat{L}(s_0) = P(s_0)$. Hence $\widehat{L}$ must have the form

$$\widehat{L}(s) = P(s_0) + v(s - s_0)\,,$$

where v is some vector. By computing $\widehat{L} - L$, we get:

$$\widehat{L}(s) - L(s) = (P(s_0) + v(s - s_0)) - (P(s_0) + P'(s_0)(s - s_0)) = (v - P'(s_0))(s - s_0)\,,$$

and therefore:

$$\frac{|\widehat{L}(s) - L(s)|}{|s - s_0|} = \frac{|v - P'(s_0)||s - s_0|}{|s - s_0|} = |v - P'(s_0)|\,.$$

This last quantity tends to 0 only if $|v - P'(s_0)| = 0$. Since by Equation (3), that quantity tends to 0 under our assumption, we get $v = P'(s_0)$, which means that $\widehat{L}(s) = L(s)$. This concludes the uniqueness part.

Remark 2. If $P''(s_0) = 0$, then from the Taylor expansion we get:

$$P(s) = P(s_0) + P'(s_0)(s - s_0) + F(s)\,,$$

where in this case $F(s)$ satisfies:

$$\lim_{s \to s_0} \frac{F(s)}{(s - s_0)^2} = 0\,. \tag{4}$$

In general, if $P''(s_0) \neq 0$, the condition (4) is not satisfied by the tangent line $L(s)$.

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Therefore, to obtain a second-order approximation of $P(s)$, one must use more complex approximating curves, which must be defined using the second-order properties of the curve $P(s)$, such as curvature and normal vector. This is the case of the osculating circle.

Proposition 3. *Let $P(s)$ be a curve parameterized by arclength, and let s_0 be fixed. The osculating circle $C(s)$ tangent to $P(s)$ at $P(s_0)$, parameterized by arclength, is the unique circle such that*

$$\lim_{s\to s_0} \frac{|P(s)-C(s)|}{(s-s_0)^2} = 0\,. \tag{5}$$

If $P''(s_0)=0$, then the curvature of P at s_0 is 0, and thus the osculating circle degenerates to the tangent line. As observed in Remark 2, in this case the tangent line satisfies the condition (5).

Hence we will assume $P''(s_0)\neq 0$. In this case, the curvature is defined as $c(s_0)=|P''(s_0)|$, the normal vector is $n=P''(s_0)/|P''(s_0)|$, and the osculating circle is the circle centered at $P(s_0)+(1/c)n$ of radius $1/c$.

To simplify the computation, we can apply:

- A translation of $\mathbb{R}^3$, sending $P(s_0)$ to the origin;
- An orthogonal transformation of $\mathbb{R}^3$, sending the orthonormal frame $\{-n,t,b\}$ to the standard frame;
- A change of parameterization (by adding a constant $s\mapsto s-s_0$), so that $s_0=0$.

Observe that the three operations above do not change any of the geometric quantities involves (distance, speed, curvature). For this reason, it suffices to prove Proposition 3 under these assumptions.

Therefore, denoting by $c:=c(0)$ the curvature of P at the reference point, by $R:=R(0)=1/c(0)$ the radius of curvature, the center of the osculating circle is $P(s_0)+cn=0+R(-1,0,0)=(-R,0,0)$. Moreover, the plane of the osculating circle, which is the plane spanned by t and n, is the xy-plane. The osculating circle is:

$$C(s)=\begin{pmatrix}-1/c\\0\\0\end{pmatrix}+(1/c)\begin{pmatrix}\cos(cs)\\\sin(cs)\\0\end{pmatrix}.$$

Indeed, $C(s)$ has curvature $1/R$, is tangent to $t(0)=(0,1,0)$, and $C(0)=0$.

Let us use again the Taylor expansion of $P(s)$, now up to the second order:

$$P(s)=P(0)+P'(0)s+\frac{1}{2}P''(s)s^2+F(s)=\begin{pmatrix}-(c/2)s^2\\s\\0\end{pmatrix}+F(s)\,,$$

where

$$\lim_{s\to s_0}\frac{F(s)}{s^2}=0\,.$$

Let us now compute:

$$\frac{P(s)-C(s)}{s^2}=\frac{1}{s^2}\begin{pmatrix}(1/c)(1-\cos(cs))-(c/2)s^2\\s-(1/c)\sin(cs)\\0\end{pmatrix}+\frac{F(s)}{s^2}\,.$$

We want to show that this last quantity tends to 0 as $s\to 0$. First, by the property of the Taylor theorem, the last term goes to 0 as just observed. Let us study the two non-zero entries of the first term:

$$\frac{1}{c}\frac{1-\cos(cs)}{s^2}-\frac{c}{2}=c\frac{1-\cos(cs)}{c^2s^2}-\frac{c}{2}\,.$$

Now, recalling the limit

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}\,,$$

which can be derived also by the Taylor expansion

$$\cos x\sim 1-(1/2)x^2+\text{lower order terms}\,.$$

Hence

$$\lim_{s\to 0}\frac{1-\cos(cs)}{c^2s^2}=\frac{1}{2}$$

and this concludes that the first entry of $(P(s)-C(s))/s^2$ tends to 0. For the second entry, we get

$$\frac{1}{s}-\frac{\sin(cs)}{cs^2}=c\frac{cs-\sin(cs)}{c^2s^2}\,.$$

As before, we can use the limit

$$\lim_{x\to 0} \frac{\sin x - x}{x^2} = 0\,,$$

which is a consequence of the Taylor expansion

$$\sin x \sim x - (1/6)x^3 + \text{lower order terms}\,.$$

This concludes the proof of the condition (5).

To prove uniqueness, as for tangent lines, let us suppose $\widehat{C}(s)$ is another circle with $\widehat{C}(0) = P(0)$ and satisfying

$$\lim_{s\to 0} \frac{|P(s) - \widehat{C}(s)|}{s^2} = 0\,.$$

Hence in particular $P(s) - \widehat{C}(s) \to 0$, which shows that $C(0) = P(0) = 0$, and moreover $|P(s) - \widehat{C}(s)|/|s| \to 0$. Hencem using Proposition 1, $\widehat{C}(s)$ must have the same tangent line as $P(s)$ (and thus also as $C(s)$) at $s = 0$. This shows that $\widehat{C}(s)$ has the form

$$\widehat{C}(s) = \begin{pmatrix} -1/d \\ 0 \\ 0 \end{pmatrix} + (1/d) \begin{pmatrix} \cos(ds) \\ \sin(ds) \\ 0 \end{pmatrix}\,.$$

Here $|d|$ is the curvature of $\widehat{C}$. (In principle, d might be a negative number, but this does not make any difference in the following argument.) As in the case of tangent lines,

$$\frac{|\widehat{C}(s) - C(s)|}{s^2} \le \frac{|P(s) - C(s)|}{s^2} + \frac{|P(s) - \widehat{C}(s)|}{s^2} \to 0$$

by the hypotheses. Now, the first entry of $(\widehat{C}(s) - C(s))/s^2$ is:

$$c\frac{1-\cos(cs)}{c^2s^2} - d\frac{1-\cos(ds)}{d^2s^2} \to \frac{c}{2} - \frac{d}{2}\,.$$

This quantity tends to 0 only if $c = d$. This shows that necessarily $\widehat{C} = C$.