# SPHERICAL CURVES PARAMETERIZED BY ARCLENGTH

The spherical coordinates on a unit sphere in $\mathbb{R}^3$ are:

$$(\phi, \varphi) \mapsto (\cos\varphi\cos\phi, \cos\varphi\sin\phi, \sin\varphi)\,.$$

where $\varphi \in (-\pi/2, \pi/2)$ and $\phi \in (-\pi, \pi]$. In fact, if we suppose the Earth is perfectly round, we choose our unit of measurement so that the Equator has length $2\pi$, we assume the $z$-axis is the axis of rotation and the North pole is $(0, 0, 1)$, then:

- The curves $\{\varphi = \text{cost.}\}$ are usually called *parallels*;
- The curves $\{\phi = \text{cost.}\}$ are usually called *meridians.*

Moreover, assuming that the meridian through the point $(1, 0, 0)$ is the Greenwich meridian, then

- $\varphi$ is usually called *latitude*;
- $\phi$ is usually called *longitude.*

The purpose of this notes is finding curves parameterized by arclength for which $\phi(t) = t$. That is, we will consider curves parameterized by

$$P(t) = (\cos\varphi(t)\cos t, \cos\varphi(t)\sin t, \sin\varphi(t))\,. \tag{1}$$

This is equivalent to the following question:

*At what speed must a point on Earth's surface travel along a meridian, so that its speed (with respect to the external reference frame) is the same as the speed of a point standing still on the Equator?*

Let us first compute the tangent vector:

$$P'(t) = (-\sin\varphi(t)\cos t\varphi'(t) - \cos\varphi(t)\sin t, -\sin\varphi(t)\sin t\varphi'(t) + \cos\varphi(t)\cos t, \cos\varphi(t)\varphi'(t))\,.$$

Therefore by an explicit computation,

$$|P'(t)|^2 = (\varphi'(t))^2 + \cos^2\varphi(t)$$

We need to impose $|P'(t)| = 1$, and therefore:

$$(\varphi'(t))^2 + \cos^2\varphi(t) = 1\,,$$

which implies

$$(\varphi'(t))^2 = \sin^2\varphi(t)\,.$$

Thus we have obtained two ODE's:

$$\varphi'(t) = \sin\varphi(t) \qquad \text{and} \qquad \varphi'(t) = -\sin\varphi(t)\,.$$

Let us solve the second equation by definiteness, the other equation being analogous. The equation has separable variables, hence we get

$$\int \frac{\varphi'(t)}{\sin\varphi(t)}dt = -\int dt = -t + C\,.$$

On the other hand, with the changes of variables $x = \varphi(t)$,

$$\int \frac{\varphi'(t)}{\sin\varphi(t)}dt = \int \frac{dx}{\sin x} = \int \frac{\sin x dx}{\sin^2 x} = \int \frac{\sin x dx}{1 - \cos^2 x}\,.$$

With another change of variables $y = \cos x$ we get:

$$\int \frac{\varphi'(t)}{\sin\varphi(t)}dt = \int \frac{dy}{1 - \cos^2 y} = \operatorname{arctanh}(y) + C = \operatorname{arctanh}(\cos\varphi(t)) + C\,.$$

Therefore we obtained:

$$-\operatorname{arctanh}(\cos\varphi(t)) = -t + C\,,$$

hence

$$\cos\varphi(t) = \tanh(t - C)\,. \tag{2}$$

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It is easy to check that the solution of the other equation is

$$\cos\varphi(t) = -\tanh(t-C)\,. \tag{3}$$

In both cases, we have

$$\sin\varphi(t) = \sqrt{1-\cos^2\varphi(t)} = \sqrt{1-\tanh^2(t-C)} = \pm\frac{1}{\cosh(t-C)}\,. \tag{4}$$

Getting back to the original curve, by using (2) and (4) (with positive sign) inside (1), it now has the form:

$$P(t) = \left(\tanh(t-C)\cos t, \tanh(t-C)\sin t, \frac{1}{\cosh(t-C)}\right)\,.$$

Changing the parameterization by $s = t - C$, we have

$$P(s) = \left(\tanh(s)\cos(s+C), \tanh(s)\sin(s+C), \frac{1}{\cosh(s)}\right)\,.$$

Choosing the different signs in (3) and (4), one obtains the other curves

$$P(s) = \left(\tanh(s)\cos(s+C), \tanh(s)\sin(s+C), -\frac{1}{\cosh(s)}\right)\,,$$

$$P(s) = \left(-\tanh(s)\cos(s+C), -\tanh(s)\sin(s+C), \frac{1}{\cosh(s)}\right)\,,$$

$$P(s) = \left(-\tanh(s)\cos(s+C), -\tanh(s)\sin(s+C), -\frac{1}{\cosh(s)}\right)\,,$$

which are obtained from the original one by reflections in the $xy$-plane by a rotation of angle $\pi$ around the $z$-axis. Moreover, by composing with a rotation around the $z$-axis of angle $C$, we can always assume $C = 0$. In conclusion, the curve

$$P(s) = \left(\tanh(s)\cos(s), \tanh(s)\sin(s), \frac{1}{\cosh(s)}\right)$$

is a curve which passes through the North pole for $s = 0$, and since

$$\lim_{s\to\pm\infty}\tanh(s) = \pm 1 \qquad \lim_{s\to\pm\infty}\frac{1}{\cosh(s)} = 0$$

the curve approaches (but never crosses) the Equator as $s \to \pm\infty$.

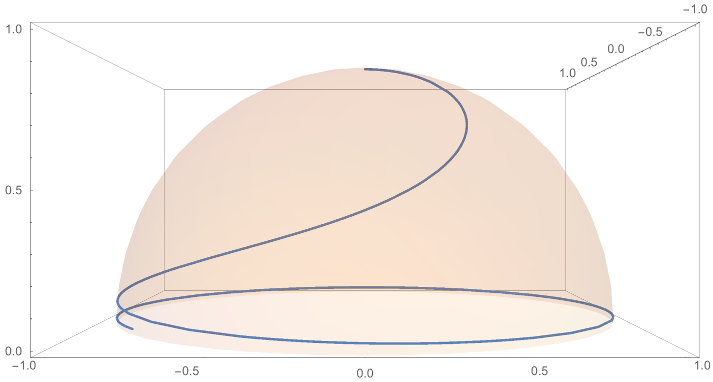


FIGURE 1. The curve $P(s)$.

To conclude, let us now compute the curvature of $P(s)$. The tangent vector is:

$$\begin{aligned} P'(s) &= \left(\frac{\cos s}{\cosh^2 s} - \sin s\tanh s, \frac{\sin s}{\cosh^2 s} + \cos s\tanh s, -\frac{\tanh s}{\cosh s}\right) \\ &= \frac{1}{\cosh^2 s}(\cos s, \sin s, 0) + \tanh s\left(-\sin s, \cos s, -\frac{1}{\cosh s}\right) \end{aligned}$$

Therefore

$$\begin{aligned}P''(s) &= -\frac{\sinh s}{2\cosh^3 s}(\cos s, \sin s, 0) + \frac{1}{\cosh^2 s}(-\sin s, \cos s, 0)\\ &+ \frac{1}{\cosh^2 s}\left(-\sin s, \cos s, -\frac{1}{\cosh s}\right) + \tanh s\left(-\cos s, -\sin s, \frac{\sinh s}{\cosh^2 s}\right)\\ &= \frac{2}{\cosh^2 s}(-\sin s, -\cos s, 0) - \tanh s\left(\frac{1}{2\cosh^2 s} + 1\right)(\cos s, \sin s, 0) + \left(0, 0, \frac{-1+\sinh^2 s}{\cosh^3 s}\right).\end{aligned}$$

Observe that in this last form, $P''(s)$ is written as the sum of three mutually orthogonal vectors. Hence $|P''(s)|^2$ is the sum of the squared norm of each of these three vectors, namely:

$$|P''(s)|^2 = \frac{4}{\cosh^4 s} + \tanh^2 s\left(\frac{1}{2\cosh^2 s} + 1\right)^2 + \frac{(-1+\sinh^2 s)^2}{\cosh^6 s}.$$

In particular, when $s = 0$ (i.e. at the North pole), the curvature is $c(0) = |P''(0)| = \sqrt{5}$. On the other hand, when $s \to \pm\infty$, recalling that $\tanh s \to \pm 1$ and $\sinh s \sim e^{\pm s}$, $\cosh s \sim e^s$, we find

$$\lim_{s\to\pm\infty} c(s) = 1\,,$$

as it was to be expected, since $P(s)$ is approaching the equatorial circle.