TEST - 15^{th} SEPTEMBER 2016 - LINEAR ALGEBRA

SURNAME:

NAME:

Instructions: Please read carefully the following instructions:

- remember to write your name;
- you must complete the test within 2 hours;
- neither calculators nor notes or books are allowed;
- if some request is unclear, please, ask;
- fill in the blanks with your answer and return both this solution sheet and the detailed computations you have worked out.

PROBLEMS

Problem 1. Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 5 \\ 3 & 7 & 2 \\ 3 & 2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix}$$

(1) Find the matrix $\mathbf{C}$ such that $\mathbf{A} + \mathbf{C} = \mathbf{B}$:

$$\mathbf{C} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

(2) Write the transpose $\mathbf{A}^\top$ of the matrix $\mathbf{A}$ and the matrix product $\mathbf{AB}$:

$$\mathbf{A}^\top = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \quad \mathbf{AB} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

(3) Is $\mathbf{AB} = \mathbf{BA}$?

(4) Compute the matrix $-3\mathbf{B}$ and the inverse matrix $\mathbf{B}^{-1}$ of the matrix $\mathbf{B}$:

$$-3\mathbf{B} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \quad \mathbf{B}^{-1} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix}$$

Problem 2. For each of the following pairs of vectors tell if they are linearly independent or orthogonal:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$$

LINEARLY INDEPENDENT: YES NO
ORTHOGONAL: YES NO

$$\mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

LINEARLY INDEPENDENT: YES NO
ORTHOGONAL: YES NO

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

LINEARLY INDEPENDENT: YES NO
ORTHOGONAL: YES NO

Problem 3. Consider the following three vectors:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

(1) Write their norms (=absolute values):

$|\mathbf{u}| =$ $\qquad$ $|\mathbf{v}| =$ $\qquad$ $|\mathbf{w}| =$

(2) Which of them are normal vectors?

(3) Is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ a set of linearly independent vectors?

(4) Write the vector

$$\mathbf{a} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$: $\quad \mathbf{a} =$

Problem 4. Consider the following sets of vectors:

$$\mathcal{U} = \left\{ \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} : u_1 + u_2 + u_3 = 0 \right\} \quad \text{and} \quad \mathcal{V} = \left\{ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} : v_1 v_2 v_3 = 0 \right\}$$

For each of them tell if it is a linear subspace of $\mathbb{R}^3$ or not. In the positive case, write a basis of the subspace. In the negative case, explain why the set is not a linear subspace.

	$\mathcal{U}$	$\mathcal{V}$
YES, and a basis is:		
NO, because:		

Problem 5. Consider the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix}$$

(1) Compute the determinant and the trace of $\mathbf{A}$:

$$\det \mathbf{A} = \qquad\qquad\qquad \operatorname{tr} \mathbf{A} =$$

(2) Is $\mathbf{A}$ invertible?

(3) What is the rank of $\mathbf{A}$?

$$\operatorname{rank} \mathbf{A} =$$

(4) Is the vector

$$\mathbf{u} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

in the image of $\mathbf{A}$?

(5) Is the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

in the null space of $\mathbf{A}$?

(6) What is the dimension of the null space of $\mathbf{A}$?

(7) Write a basis of the null space of $\mathbf{A}$:

Problem 6. Consider the matrix:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3h \\ 0 & 2h & h \end{bmatrix} \qquad \text{with } h \in \mathbb{R}.$$

(1) For which values of the parameter h the matrix $\mathbf{H}$ is diagonal?

(2) For $h = 1$ find the eigenvalues of $\mathbf{H}$ and their eigenspaces:

EIGENVALUES	BASIS OF THE EIGENSPACES

(3) For $h = 0$ find the eigenvalues of $\mathbf{H}$ and their eigenspaces:

EIGENVALUES	BASIS OF THE EIGENSPACES

Problem 7. Consider the matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & m \\ 1 & 1-m & 0 \\ m & 0 & 0 \end{bmatrix} \qquad \text{with } m \in \mathbb{R}.$$

(1) For which values of the parameter m the rows of the matrix $\mathbf{M}$ are linearly independent?

(2) For which values of the parameter m the matrix $\mathbf{M}$ is invertible?

(3) Discuss the rank of $\mathbf{M}$, depending on the parameter m:

(4) For $m = -1$, compute $\det(\mathbf{M}^8)$, $\det(-5\mathbf{M})$ and $\det(\mathbf{M}^{-1})$:

$\det(\mathbf{M}^8) =$ $\qquad\qquad$ $\det(-5\mathbf{M}) =$ $\qquad\qquad$ $\det(\mathbf{M}^{-1}) =$

Problem 8. Consider the two following bases of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{b}_3 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \right\}$$

and

$$\mathcal{C} = \left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(1) Which of them is an orthonormal basis?

(2) Write the change of basis matrices $\mathbf{M}_{\mathcal{B}}^{\mathcal{C}}$ (from the basis $\mathcal{B}$ to the basis $\mathcal{C}$) and $\mathbf{M}_{\mathcal{C}}^{\mathcal{B}}$ (from the basis $\mathcal{C}$ to the basis $\mathcal{B}$):

$$\mathbf{M}_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \qquad \mathbf{M}_{\mathcal{C}}^{\mathcal{B}} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

(3) Consider the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\begin{aligned} T(\mathbf{e}_1) &= \mathbf{e}_1 + 3\mathbf{e}_2 + 6\mathbf{e}_3 \\ T(\mathbf{e}_2) &= -3\mathbf{e}_1 - 5\mathbf{e}_2 - 6\mathbf{e}_3 \\ T(\mathbf{e}_3) &= 3\mathbf{e}_1 + 3\mathbf{e}_2 + 4\mathbf{e}_3 \end{aligned}$$

and write the matrix $\mathbf{T}_{\mathcal{C}}$ that represents T in the basis $\mathcal{C}$ and the matrix $\mathbf{T}_{\mathcal{B}}$ that represents T in the basis $\mathcal{B}$:

$$\mathbf{T}_{\mathcal{C}} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \qquad \mathbf{T}_{\mathcal{B}} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$