EXERCISES ON TENSOR ALGEBRA

Problem 1. Consider the tensor $\mathbf{F} = 3\mathbf{e}_1 \otimes \mathbf{e}_1 - 2\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_2 + 2\mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_1$.

(1) Compute the determinant $\det \mathbf{F}$ and the trace $\operatorname{tr} \mathbf{F}$.

(2) What is the image under $\mathbf{F}$ of the vector $\mathbf{u} := 3\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$?

(3) Decompose $\mathbf{F}$ into its symmetric part $\text{sym}\mathbf{F}$ and its skew-symmetric part $\text{skw}\mathbf{F}$.

(4) What is the axial vector of $\text{skw}\mathbf{F}$?

(5) Write the inverse tensor $\mathbf{F}^{-1}$ and the adjugate tensor $\mathbf{F}^*$.

(6) What is the area dilation factor of $\mathbf{F}$ for surfaces orthogonal to $\mathbf{e}_2$?

Solution:

(1) $\det\mathbf{F} = 4$ and $\text{tr}\mathbf{F} = 2$

(2) $\mathbf{F}\mathbf{u} = 11\mathbf{e}_1 + 5\mathbf{e}_2 - 3\mathbf{e}_3$

(3) $\text{sym}\mathbf{F} = 3\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2 - \frac{1}{2}\mathbf{e}_1 \otimes \mathbf{e}_3 - \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_3 - \frac{1}{2}\mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes \mathbf{e}_2$
and $\text{skw}\mathbf{F} = -\mathbf{e}_1 \otimes \mathbf{e}_2 + \frac{1}{2}\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_3 - \frac{1}{2}\mathbf{e}_3 \otimes \mathbf{e}_1 - \mathbf{e}_3 \otimes \mathbf{e}_2$

(4) $\mathbf{w}(\text{skw}\mathbf{F}) = -\mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 + \mathbf{e}_3$

(5) $\mathbf{F}^{-1} = -\mathbf{e}_1 \otimes \mathbf{e}_3 - \frac{1}{2}\mathbf{e}_2 \otimes \mathbf{e}_1 - \frac{3}{2}\mathbf{e}_2 \otimes \mathbf{e}_3 - \frac{1}{4}\mathbf{e}_3 \otimes \mathbf{e}_1 + \frac{1}{2}\mathbf{e}_3 \otimes \mathbf{e}_2 - \frac{3}{4}\mathbf{e}_3 \otimes \mathbf{e}_3$
and $\mathbf{F}^* = -4\mathbf{e}_3 \otimes \mathbf{e}_1 - 2\mathbf{e}_1 \otimes \mathbf{e}_2 - 6\mathbf{e}_3 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_3 + 2\mathbf{e}_2 \otimes \mathbf{e}_3 - 3\mathbf{e}_3 \otimes \mathbf{e}_3$

(6) $|\mathbf{F}^*\mathbf{e}_2| = 2\sqrt{10}$

Problem 2. Consider the tensor $\mathbf{F} = -\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_3 + 2\mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 + 2\mathbf{e}_3 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3$.

(1) Compute the determinant $\det \mathbf{F}$ and the trace $\operatorname{tr} \mathbf{F}$.

(2) What is the image under $\mathbf{F}$ of the vector $\mathbf{u} := \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$?

(3) Decompose $\mathbf{F}$ into its symmetric part $\text{sym}\mathbf{F}$ and its skew-symmetric part $\text{skw}\mathbf{F}$.

(4) What is the axial vector of $\text{skw}\mathbf{F}$?

(5) Write the inverse tensor $\mathbf{F}^{-1}$ and the adjugate tensor $\mathbf{F}^*$.

(6) What is the area dilation factor of $\mathbf{F}$ for surfaces orthogonal to $\mathbf{e}_3$?

Solution:

(1) $\det\mathbf{F} = 3$ and $\text{tr}\mathbf{F} = -3$

(2) $\mathbf{Fu} = 3\mathbf{e}_2 - 3\mathbf{e}_3$

(3) $\text{sym}\mathbf{F} = -\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_2 + \frac{1}{2}\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_3 + \frac{1}{2}\mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes \mathbf{e}_2 - \mathbf{e}_3 \otimes \mathbf{e}_3$
and $\text{skw}\mathbf{F} = -\mathbf{e}_1 \otimes \mathbf{e}_2 + \frac{1}{2}\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_3 - \frac{1}{2}\mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes \mathbf{e}_2$

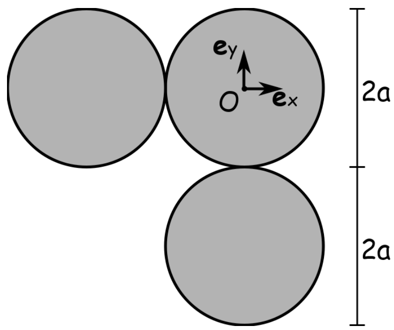
(4) $\mathbf{w}(\text{skw}\mathbf{F}) = \mathbf{e}_1 + \frac{1}{2}\mathbf{e}_2 + \mathbf{e}_3$

(5) $\mathbf{F}^{-1} = \frac{1}{3}\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{2}{3}\mathbf{e}_1 \otimes \mathbf{e}_2 + \frac{1}{3}\mathbf{e}_1 \otimes \mathbf{e}_3 + \frac{2}{3}\mathbf{e}_2 \otimes \mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{2}{3}\mathbf{e}_2 \otimes \mathbf{e}_3 + \frac{4}{3}\mathbf{e}_3 \otimes \mathbf{e}_1 + \frac{2}{3}\mathbf{e}_3 \otimes \mathbf{e}_2 + \frac{1}{3}\mathbf{e}_3 \otimes \mathbf{e}_3$
and $\mathbf{F}^* = \mathbf{e}_1 \otimes \mathbf{e}_1 + 2\mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes \mathbf{e}_1 + 2\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_2 + 2\mathbf{e}_3 \otimes \mathbf{e}_2 + 4\mathbf{e}_1 \otimes \mathbf{e}_3 + 2\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_3$

(6) $|\mathbf{F}^*\mathbf{e}_3| = \sqrt{21}$

EXERCISES ON INERTIA

Problem 3. Suppose the following grey body to be in $\mathbb{R}^3$ and to have uniform mass density ρ:



(1) Write the vector position $C - O$ of the center of mass of the body.

(2) Write a principal basis for the central tensor of inertia $\mathbf{I}_C$ (that is: a basis of $\mathbb{R}^3$, made of principal versors of the tensor of inertia of the body at C).

(3) Write the central tensor of inertia $\mathbf{I}_C$ of the body.

(4) Write the tensor of inertia $\mathbf{I}_O$ of the body.

Solution:

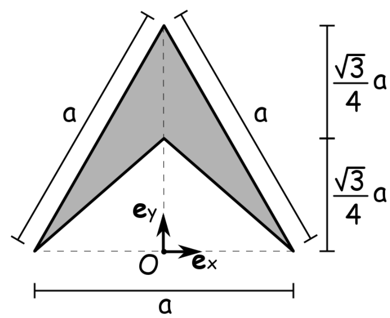
(1) $C - O = -\frac{2}{3}a(\mathbf{e}_x + \mathbf{e}_y)$

(2) A principal basis is $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, with $\mathbf{e}_1 := \frac{\sqrt{2}}{2}(\mathbf{e}_x + \mathbf{e}_y)$, $\mathbf{e}_2 := \frac{\sqrt{2}}{2}(-\mathbf{e}_x + \mathbf{e}_y)$ and $\mathbf{e}_3 := \mathbf{e}_x \times \mathbf{e}_y$

(3) $\mathbf{I}_C = \frac{19}{4}\pi\rho a^4 \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{25}{12}\pi\rho a^4 \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{41}{6}\pi\rho a^4 \mathbf{e}_3 \otimes \mathbf{e}_3$

(4) $\mathbf{I}_O = \frac{19}{4}\pi\rho a^4 \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{257}{108}\pi\rho a^4 \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{385}{54}\pi\rho a^4 \mathbf{e}_3 \otimes \mathbf{e}_3$

Problem 4. Suppose the following grey body to be in $\mathbb{R}^3$ and to have uniform mass density ρ:



(1) Write the vector position $C - O$ of the center of mass of the body.

(2) Write a principal basis for the central tensor of inertia $\mathbf{I}_C$ (that is: a basis of $\mathbb{R}^3$, made of principal versors of the tensor of inertia of the body at C).

(3) Write the central tensor of inertia $\mathbf{I}_C$ of the body.

(4) Write the tensor of inertia $\mathbf{I}_O$ of the body.

Solution:

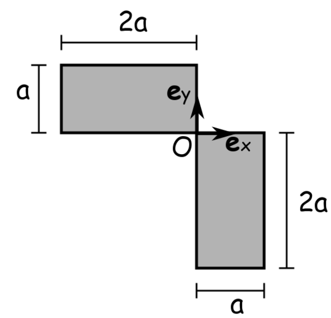
(1) $C - O = \frac{\sqrt{3}}{4} a \mathbf{e}_y$

(2) A principal basis is $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$, with $\mathbf{e}_z := \mathbf{e}_x \times \mathbf{e}_y$

(3) $\mathbf{I}_C = \frac{\sqrt{3}}{256}\rho a^4 \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{\sqrt{3}}{192}\rho a^4 \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{7\sqrt{3}}{768}\rho a^4 \mathbf{e}_3 \otimes \mathbf{e}_3$

(4) $\mathbf{I}_O = \frac{7\sqrt{3}}{256}\rho a^4 \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{\sqrt{3}}{192}\rho a^4 \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{25\sqrt{3}}{768}\rho a^4 \mathbf{e}_3 \otimes \mathbf{e}_3$

Problem 5. Suppose the following grey body to be in $\mathbb{R}^3$ and to have uniform mass density ρ:



(1) Write the vector position $C - O$ of the center of mass of the body.

(2) Write a principal basis for the central tensor of inertia $\mathbf{I}_C$ (that is: a basis of $\mathbb{R}^3$, made of principal versors of the tensor of inertia of the body at C).

(3) Write the central tensor of inertia $\mathbf{I}_C$ of the body.

(4) Write the tensor of inertia $\mathbf{I}_O$ of the body.

Solution:

(1) $C - O = -\frac{1}{4}a(\mathbf{e}_x + \mathbf{e}_y)$

(2) A principal basis is $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, with $\mathbf{e}_1 := \frac{\sqrt{2}}{2}(\mathbf{e}_x + \mathbf{e}_y)$, $\mathbf{e}_2 := \frac{\sqrt{2}}{2}(-\mathbf{e}_x + \mathbf{e}_y)$ and $\mathbf{e}_3 := \mathbf{e}_x \times \mathbf{e}_y$

(3) $\mathbf{I}_C = \frac{37}{12}\rho a^4(\mathbf{e}_x \otimes \mathbf{e}_x + \mathbf{e}_y \otimes \mathbf{e}_y + 2\mathbf{e}_z \otimes \mathbf{e}_z) + \frac{9}{4}\rho a^4(\mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x)$

(4) $\mathbf{I}_C = \frac{10}{3}\rho a^4(\mathbf{e}_x \otimes \mathbf{e}_x + \mathbf{e}_y \otimes \mathbf{e}_y + 2\mathbf{e}_z \otimes \mathbf{e}_z) + 2\rho a^4(\mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x)$

EXERCISES ON CURVES

Problem 6. Consider the curve defined as follows:

$$P(\vartheta) := \left(\frac{\sqrt{14}}{4}\vartheta, \frac{1}{6}(\sqrt{1+\vartheta})^3, \frac{1}{6}(\sqrt{1-\vartheta})^3 \right) \qquad \text{with } \vartheta \in \left[-\frac{1}{2}, \frac{1}{2} \right]$$

(1) Write the arc-length parameter s of the curve.

(2) Find the length L of the curve in the interval $\left[-\frac{1}{2}, \frac{1}{2} \right]$.

(3) Write the tangent vector $\mathbf{t}$, the normal vector $\mathbf{n}$ and the binormal vector $\mathbf{b}$ of the curve for $\vartheta \in \left[-\frac{1}{2}, \frac{1}{2} \right]$.

(4) Compute the curvature c and the torsion τ of the curve for $\vartheta \in \left[-\frac{1}{2}, \frac{1}{2} \right]$.

Solution:

(1) $s = \vartheta$

(2) $L = 1$

(3) $\mathbf{t}(\vartheta) = \left(\frac{\sqrt{14}}{4}, \frac{\sqrt{1+\vartheta}}{4}, -\frac{\sqrt{1-\vartheta}}{4} \right)$, $\mathbf{n}(\vartheta) = \left(0, \frac{\sqrt{1-\vartheta}}{\sqrt{2}}, \frac{\sqrt{1+\vartheta}}{\sqrt{2}} \right)$ and

$\mathbf{b}(\vartheta) = \left(\frac{\sqrt{2}}{4}, -\frac{\sqrt{7}\sqrt{1+\vartheta}}{4}, \frac{\sqrt{7}\sqrt{1-\vartheta}}{4} \right)$

(4) $c(\vartheta) = \frac{\sqrt{2}}{8\sqrt{1-\vartheta^2}}$ and $\tau(\vartheta) = -\frac{\sqrt{14}}{8\sqrt{1-\vartheta^2}}$

Problem 7. Consider the curve defined as follows:

$$P(\vartheta) := \left(\vartheta, -\frac{\sqrt{2}}{2}\vartheta^2, \frac{\sqrt{2}}{2}\vartheta^2\right) \qquad \text{with } \vartheta \in \mathbb{R}$$

(1) Write the tangent vector $\mathbf{t}$, the normal vector $\mathbf{n}$ and the binormal vector $\mathbf{b}$ of the curve for $\vartheta \in \mathbb{R}$.

(2) Compute the curvature c and the torsion τ of the curve for $\vartheta \in \mathbb{R}$.

Solution:

(1) $\mathbf{t}(\vartheta) = \dfrac{1}{\sqrt{4\vartheta^2+1}}\left(1, -\sqrt{2}\vartheta, \sqrt{2}\vartheta\right)$, $\mathbf{n}(\vartheta) = \dfrac{1}{\sqrt{4\vartheta^2+1}}\left(-2\vartheta, -\sqrt{2}, \sqrt{2}\right)$ and $\mathbf{b}(\vartheta) = \dfrac{\sqrt{2}}{2}(0, -1, -1)$

(2) $c(\vartheta) = \dfrac{2}{(\sqrt{4\vartheta^2+1})^3}$ and $\tau(\vartheta) = 0$

Problem 8. Consider the curve defined as follows:

$$P(\vartheta) := \left(\vartheta, \vartheta^2, \frac{2}{3}\vartheta^3\right) \qquad \text{with } \vartheta \in \mathbb{R}$$

(1) Find the length L of the curve in the interval $[-1, 3]$.

(2) Write the tangent vector $\mathbf{t}$, the normal vector $\mathbf{n}$ and the binormal vector $\mathbf{b}$ of the curve for $\vartheta \in \mathbb{R}$.

(3) Compute the curvature c and the torsion τ of the curve for $\vartheta \in \mathbb{R}$.

Solution:

(1) $L = \dfrac{68}{3}$

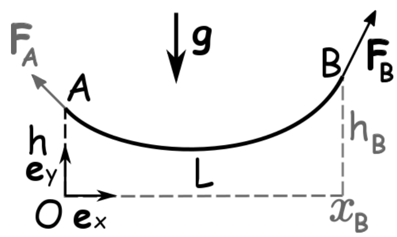
(2) $\mathbf{t}(\vartheta) = \dfrac{1}{2\vartheta^2+1}\left(1, 2\vartheta, 2\vartheta^2\right)$, $\mathbf{n}(\vartheta) = \dfrac{1}{2\vartheta^2+1}\left(-2\vartheta, 1-2\vartheta^2, 2\vartheta\right)$ and

$\mathbf{b}(\vartheta) = \dfrac{1}{2\vartheta^2+1}\left(2\vartheta^2, -2\vartheta, 1\right)$

(3) $c(\vartheta) = \dfrac{2}{(2\vartheta^2+1)^2}$ and $\tau(\vartheta) = -\dfrac{2}{(2\vartheta^2+1)^2}$

EXERCISES ON CABLES

Problem 9. A cable with length L and linear mass density λ is hung in A at height h. At B a force $\mathbf{F}_B = \mathbf{e}_x + \frac{2}{3}\lambda g L \mathbf{e}_y$ is exerted on the cable, as shown in the following picture:



Assume that the only distributed force acting on the cable is due to the gravitational acceleration $\mathbf{g} = -g\mathbf{e}_y$.

(1) Let $y(x)$ be the function that describes the shape of the cable. What is the condition for $y(0)$, at equilibrium?

(2) Let $T(x)$ be the function that describes the tension of the cable, and let $\mathbf{t}(x)$ be the tangent versor to the shape of the cable. What is the condition that involves both the tension and the tangent at the point B, at equilibrium?

(3) Write the function $y(x)$.

(4) What is the height h_B of the point B at equilibrium?

(5) For which value of x the tension is minimal?

(6) What is the reactive force $\mathbf{F}_A$ the cable is subjected to at A?

Solution:

(1) $y(0) = h$

(2) $T(x_B)\mathbf{t}(x_B) = \mathbf{e}_x + \frac{2}{3}\lambda g L \mathbf{e}_y$

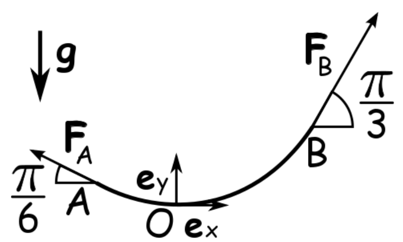
(3) $y(x) = \frac{1}{\lambda g}\cosh\left(\lambda g x - \operatorname{asinh}\left(\frac{\lambda g L}{3}\right)\right) + h - \frac{1}{\lambda g}\cosh\left(\operatorname{asinh}\left(\frac{\lambda g L}{3}\right)\right)$

(4) $h_B = \frac{1}{\lambda g}\left(\cosh\left(\operatorname{asinh}\left(\frac{2\lambda g L}{3}\right)\right) - \cosh\left(\operatorname{asinh}\left(\frac{\lambda g L}{3}\right)\right)\right) + h$

(5) $x = \frac{1}{\lambda g}\operatorname{asinh}\left(\frac{\lambda g L}{3}\right)$

(6) $\mathbf{F}_A = -\mathbf{e}_x + \frac{1}{3}\lambda g L \mathbf{e}_y$

Problem 10. A cable with linear mass density $\lambda = \frac{2p}{g}$ is subjected in A and B respectively to two forces $\mathbf{F}_A$ and $\mathbf{F}_B$. The force $\mathbf{F}_A$ makes an angle $\frac{\pi}{6}$ with the horizontal direction $\mathbf{e}_x$ and the force $\mathbf{F}_B$ makes an angle $\frac{\pi}{3}$ with the horizontal direction $\mathbf{e}_x$, as shown in the following picture:



The norm of the force $\mathbf{F}_B$ is $|\mathbf{F}_B| = 3pl$, and the only distributed force acting on the cable is due to the gravitational acceleration $\mathbf{g} = -g\mathbf{e}_y$. Consider a reference system centered at the point O of minimal tension (as in the picture: $x_O = 0$ and $y_O = 0$).

(1) Let $y(x)$ be the function that describes the shape of the cable. What are the conditions for $y(0)$ and $\dot{y}(0)$, at equilibrium?

(2) Let $T(x)$ be the function that describes the tension of the cable, and let $\mathbf{t}(x)$ be the tangent versor to the shape of the cable. What is the condition that involves both the tension and the tangent at the point B, at equilibrium?

(3) Write the function $y(x)$.

(4) Write the vector $A - O$ and $B - O$.

(5) What is the length L of the cable?

Solution:

(1) $y(0) = 0$ and $\dot{y}(0) = 0$

(2) $T(x_B)\mathbf{t}(x_B) = \mathbf{F}_B = \frac{3}{2}pl\mathbf{e}_x + \frac{3\sqrt{3}}{2}pl\mathbf{e}_y$

(3) $y(x) = \frac{3}{4}l\left(\cosh\left(\frac{4}{3l}x\right) - 1\right)$

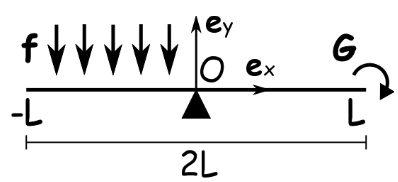
(4) $A - O = -\frac{3l}{4}\operatorname{asinh}\left(\frac{1}{\sqrt{3}}\right)\mathbf{e}_x + \frac{3l}{4}\left(\frac{2}{\sqrt{3}} - 1\right)\mathbf{e}_y$

and $B - O = \frac{3l}{4}\operatorname{asinh}\left(\sqrt{3}\right)\mathbf{e}_x + \frac{3l}{4}\mathbf{e}_y$

(5) $L = \sqrt{3}l$

EXERCISES ON BEAMS

Problem 11. A uniformly elastic beam with $EI_2 = B$ and with length $2L$ is simply supported at its middle point O. The first half of the beam ($s \in [-L, 0]$) is subjected to a distributed force with density $\mathbf{f} = -f\mathbf{e}_y$ ($f > 0$) and a couple force $\mathbf{G} = G\mathbf{e}_z$ is exerted on the second end ($s = L$) of the beam, as shown in the following picture:



(Recall: *the simple support exerts on the beam a reactive force* $\mathbf{F}_O$ *only, and* $\mathbf{F}_O$ *is orthogonal to the center line of the beam at* O).

(1) A equilibrium, what is the reactive force $\mathbf{F}_O$ exerted on the beam and what is the relation between G and f? (Hint: *use the total balance of forces acting on the beam and the total balance of torques at* $s = L$).

(2) Let $P(s) = (x(s), y(s))$ be the curve that describes the shape of the beam at equilibrium, $\boldsymbol{\varphi}(s)$ be the stress force at equilibrium and $\vartheta(s)$ be the angle between $\mathbf{e}_x$ and the tangent to $P(s)$ at equilibrium. What are the boundary conditions for $y(s)$, $\boldsymbol{\varphi}(s)$ and $\vartheta'(s)$ at the ends $s = -L$ and $s = L$?

(3) Write the conditions for $y(0)$ and $\vartheta(0)$.

(4) Write the function $y(x)$ which represents the shape of the beam at equilibrium, in case of small deflections.

(5) What is the maximum deflection $y_{\max} := \max\{|y(x)|\}$ of the beam, and for which values of x is this attained?

(6) What is the maximum slope $\vartheta_{\max} := \max\{|\vartheta(x)|\}$ of the beam, and for which values of x is this attained?

(7) What is the condition on the value of f that makes the small deflections approximation viable?

Solution:

(1) $\mathbf{F}_O = fL\mathbf{e}_y$ and $G = -\frac{1}{2}fL^2$

(2) $\vartheta'(-L) = 0$, $\vartheta'(L) = -\frac{fL^2}{2B}$, $\boldsymbol{\varphi}(-L) = \mathbf{0}$ and $\boldsymbol{\varphi}(L) = \mathbf{0}$

(3) $y(0) = 0$ and $\vartheta(0) = 0$

(4)

$$\begin{cases} y(x) = -\frac{f}{24B}x^2\left(x^2 + 4Lx + 6L^2\right) & x \in [-L, 0] \\ y(x) = -\frac{f}{4B}L^2x^2 & x \in [0, L] \end{cases}$$

(5) $y_{\max} = |y(L)| = \frac{fL^4}{4B}$

(6) $\vartheta_{\max} = |\vartheta(L)| = \frac{fL^3}{2B}$

(7) $f \ll \frac{2B}{L^3}$

Problem 12. A uniformly elastic beam with $EI_2 = B$ and with length L is clamped at one end and supported by a smooth disk of radius R at the other end, as shown in the following picture:

(Recall: *the disk exerts a reactive force that is directed radially outward and orthogonal to the beam at the contact point*).

(1) Let $P(s) = (x(s), y(s))$ be the curve that describes the shape of the beam at equilibrium and $\vartheta(s)$ be the angle between $\mathbf{e}_x$ and the tangent to $P(s)$ at equilibrium. What are the boundary conditions for $y(s)$, $\vartheta(s)$ and $\vartheta'(s)$ at the ends $s = 0$ and $s = L$?

(2) Write the balance equations for forces and torques at equilibrium at the generic point s.

(3) Write the function $y(x)$ which represents the shape of the beam at equilibrium, in case of small deflections.

(4) What is the maximum deflection $y_{\max} := \max\{|y(x)|\}$ of the beam, and for which values of x is this attained?

(5) What is the maximum slope $\vartheta_{\max} := \max\{|\vartheta(x)|\}$ of the beam, and for which values of x is this attained?

(6) What is the condition on the radius R that makes the small deflections approximation viable?

Solution:

(1) $y(0) = 0$, $\vartheta(0) = 0$, $y(L) = R\cos(\vartheta(L))$ and $\vartheta'(L) = 0$

(2) $\boldsymbol{\varphi}'(s) = \mathbf{0}$ and $(B\vartheta''(s) + F\cos(\vartheta(L))\cos(\vartheta(s)) + F\sin(\vartheta(L))\sin(\vartheta(s)))\mathbf{e}_z = \mathbf{0}$, with $F = |\mathbf{F}_L|$.

(3) $y(x) = \dfrac{R}{2L^3}x^2\,(3L - x)$

(4) $y_{\max} = y(L) = R$

(5) $\vartheta_{\max} = \vartheta(L) = \dfrac{3R}{2L}$

(6) $R \ll \dfrac{2}{3}L$