EXERCISES ON TENSOR ALGEBRA

Problem 1. Consider the tensor \( F = 3e_1 \otimes e_1 - 2e_1 \otimes e_2 - e_2 \otimes e_2 + 2e_2 \otimes e_3 - e_3 \otimes e_1 \).

(1) Compute the determinant \( \det F \) and the trace \( \text{tr} F \).

(2) What is the image under \( F \) of the vector \( u := 3e_1 - e_2 + 2e_3 \)?

(3) Decompose \( F \) into its symmetric part \( \text{sym} F \) and its skew-symmetric part \( \text{skw} F \).

(4) What is the axial vector of \( \text{skw} F \)?

(5) Write the inverse tensor \( F^{-1} \) and the adjugate tensor \( F^* \).

(6) What is the area dilation factor of \( F \) for surfaces orthogonal to \( e_2 \)?

Solution:

(1) \( \det F = 4 \) and \( \text{tr} F = 2 \)

(2) \( Fu = 11e_1 + 5e_2 - 3e_3 \)

(3) \( \text{sym} F = 3e_1 \otimes e_1 - e_1 \otimes e_2 - \frac{1}{2}e_1 \otimes e_3 - e_2 \otimes e_1 - e_2 \otimes e_2 + e_2 \otimes e_3 \)

\( -\frac{1}{2}e_3 \otimes e_1 + e_3 \otimes e_2 \)

and \( \text{skw} F = -e_1 \otimes e_2 + \frac{1}{2}e_1 \otimes e_3 + e_2 \otimes e_1 + e_2 \otimes e_3 - \frac{1}{2}e_3 \otimes e_1 - e_3 \otimes e_2 \)

(4) \( w(\text{skw} F) = -e_1 + \frac{1}{2}e_2 + e_3 \)

(5) \( F^{-1} = -e_1 \otimes e_3 - \frac{1}{2}e_2 \otimes e_1 - \frac{3}{2}e_2 \otimes e_2 - \frac{1}{4}e_3 \otimes e_3 - \frac{1}{2}e_3 \otimes e_1 + \frac{1}{2}e_3 \otimes e_2 - \frac{3}{4}e_3 \otimes e_3 \)

and \( F^* = -4e_3 \otimes e_1 - 2e_1 \otimes e_2 - 6e_3 \otimes e_2 - e_1 \otimes e_3 + 2e_2 \otimes e_3 - 3e_3 \otimes e_3 \)

(6) \( |F^*e_2| = 2\sqrt{10} \)
Problem 2. Consider the tensor $F = -e_1 \otimes e_1 + e_1 \otimes e_3 + 2e_2 \otimes e_1 - e_2 \otimes e_2 + 2e_3 \otimes e_2 - e_3 \otimes e_3$.

(1) Compute the determinant $\det F$ and the trace $\text{tr} \ F$.

(2) What is the image under $F$ of the vector $u := e_1 - e_2 + e_3$?

(3) Decompose $F$ into its symmetric part $\text{sym}\ F$ and its skew-symmetric part $\text{skw}\ F$.

(4) What is the axial vector of $\text{skw}\ F$?

(5) Write the inverse tensor $F^{-1}$ and the adjugate tensor $F^\ast$. 

(6) What is the area dilation factor of $F$ for surfaces orthogonal to $e_3$?

Solution:

(1) $\det F = 3$ and $\text{tr} \ F = -3$

(2) $Fu = 3e_2 - 3e_3$

(3) $\text{sym}\ F = -e_1 \otimes e_1 + e_1 \otimes e_3 + \frac{1}{2} e_2 \otimes e_1 + e_2 \otimes e_2 + e_2 \otimes e_3$

and $\text{skw}\ F = e_1 \otimes e_2 + \frac{1}{2} e_1 \otimes e_3 + e_2 \otimes e_1 - e_2 \otimes e_3 - \frac{1}{2} e_3 \otimes e_1 + e_3 \otimes e_2$

(4) $w(\text{skw}\ F) = e_1 + \frac{1}{2} e_2 + e_3$

(5) $F^{-1} = \frac{1}{3} e_1 \otimes e_1 + \frac{2}{3} e_1 \otimes e_2 + \frac{1}{3} e_1 \otimes e_3 + \frac{2}{3} e_2 \otimes e_1 + \frac{1}{3} e_2 \otimes e_2 + \frac{2}{3} e_2 \otimes e_3 + \frac{4}{3} e_3 \otimes e_1 + \frac{2}{3} e_3 \otimes e_2 + \frac{1}{3} e_3 \otimes e_3$

and $F^\ast = e_1 \otimes e_1 + 2e_2 \otimes e_1 + e_3 \otimes e_1 + 2e_1 \otimes e_2 + e_2 \otimes e_2 + 2e_3 \otimes e_2 + 4e_1 \otimes e_3 + 2e_2 \otimes e_3 + e_3 \otimes e_3$

(6) $|F^\ast e_3| = \sqrt{21}$
**EXERCISES ON INERTIA**

**Problem 3.** Suppose the following grey body to be in \( \mathbb{R}^3 \) and to have uniform mass density \( \rho \):

1. Write the vector position \( C - O \) of the center of mass of the body.

2. Write a principal basis for the central tensor of inertia \( I_C \) (that is: a basis of \( \mathbb{R}^3 \), made of principal versors of the tensor of inertia of the body at \( C \)).

3. Write the central tensor of inertia \( I_C \) of the body.

4. Write the tensor of inertia \( I_O \) of the body.

**Solution:**

1. \( C - O = -\frac{2}{3}a(e_x + e_y) \)

2. A principal basis is \((e_1, e_2, e_3)\), with \( e_1 := \frac{\sqrt{2}}{2}(e_x + e_y), \ e_2 := \frac{\sqrt{2}}{2}(-e_x + e_y) \) and \( e_3 := e_x \times e_y \)

3. \( I_C = \frac{19}{4} \pi \rho a^4 e_1 \otimes e_1 + \frac{25}{12} \pi \rho a^4 e_2 \otimes e_2 + \frac{41}{6} \pi \rho a^4 e_3 \otimes e_3 \)

4. \( I_O = \frac{19}{4} \pi \rho a^4 e_1 \otimes e_1 + \frac{257}{108} \pi \rho a^4 e_2 \otimes e_2 + \frac{385}{54} \pi \rho a^4 e_3 \otimes e_3 \)
Problem 4. Suppose the following grey body to be in $\mathbb{R}^3$ and to have uniform mass density $\rho$:

![Diagram of a grey body in $\mathbb{R}^3$ with dimensions and vectors labeled.]

1. Write the vector position $C - O$ of the center of mass of the body.

2. Write a principal basis for the central tensor of inertia $I_C$ (that is: a basis of $\mathbb{R}^3$, made of principal versors of the tensor of inertia of the body at $C$).

3. Write the central tensor of inertia $I_C$ of the body.

4. Write the tensor of inertia $I_O$ of the body.

Solution:

1. $C - O = \frac{\sqrt{3}}{4} a e_y$

2. A principal basis is $(e_x, e_y, e_z)$, with $e_z := e_x \times e_y$

3. $I_C = \frac{\sqrt{3}}{256} \rho a^4 e_1 \otimes e_1 + \frac{\sqrt{3}}{192} \rho a^4 e_2 \otimes e_2 + \frac{7 \sqrt{3}}{768} \rho a^4 e_3 \otimes e_3$

4. $I_O = \frac{7 \sqrt{3}}{256} \rho a^4 e_1 \otimes e_1 + \frac{\sqrt{3}}{192} \rho a^4 e_2 \otimes e_2 + \frac{25 \sqrt{3}}{768} \rho a^4 e_3 \otimes e_3$
**Problem 5.** Suppose the following grey body to be in $\mathbb{R}^3$ and to have uniform mass density $\rho$:

(1) Write the vector position $C - O$ of the center of mass of the body.

(2) Write a principal basis for the central tensor of inertia $I_C$ (that is: a basis of $\mathbb{R}^3$, made of principal versors of the tensor of inertia of the body at $C$).

(3) Write the central tensor of inertia $I_C$ of the body.

(4) Write the tensor of inertia $I_O$ of the body.

**Solution:**

(1) $C - O = -\frac{1}{4}a(e_x + e_y)$

(2) A principal basis is $(e_1, e_2, e_3)$, with $e_1 := \frac{\sqrt{2}}{2}(e_x + e_y)$, $e_2 := \frac{\sqrt{2}}{2}(-e_x + e_y)$ and $e_3 := e_x \times e_y$

(3) $I_C = \frac{37}{12} \rho a^4(e_x \otimes e_x + e_y \otimes e_y + 2e_z \otimes e_z) + \frac{9}{4} \rho a^4(e_x \otimes e_y + e_y \otimes e_x)$

(4) $I_C = \frac{10}{3} \rho a^4(e_x \otimes e_x + e_y \otimes e_y + 2e_z \otimes e_z) + 2\rho a^4(e_x \otimes e_y + e_y \otimes e_x)$
EXERCISES ON CURVES

Problem 6. Consider the curve defined as follows:

\[ P(\vartheta) := \left( \frac{\sqrt{14}}{4} \vartheta, \frac{1}{6}(\sqrt{1+\vartheta})^3, \frac{1}{6}(\sqrt{1-\vartheta})^3 \right) \text{ with } \vartheta \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \]

(1) Write the arc-length parameter \( s \) of the curve.

(2) Find the length \( L \) of the curve in the interval \( \left[ -\frac{1}{2}, \frac{1}{2} \right] \).

(3) Write the tangent vector \( t \), the normal vector \( n \) and the binormal vector \( b \) of the curve for \( \vartheta \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \).

(4) Compute the curvature \( c \) and the torsion \( \tau \) of the curve for \( \vartheta \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \).

Solution:

(1) \( s = \vartheta \)

(2) \( L = 1 \)

(3) \( t(\vartheta) = \left( \frac{\sqrt{14}}{4}, \frac{\sqrt{1+\vartheta}}{4}, \frac{\sqrt{1-\vartheta}}{4} \right) \), \( n(\vartheta) = \left( 0, \frac{\sqrt{1-\vartheta}}{\sqrt{2}}, \frac{\sqrt{1+\vartheta}}{\sqrt{2}} \right) \) and
\[
 b(\vartheta) = \left( \frac{\sqrt{2}}{4}, -\frac{\sqrt{14}\sqrt{1+\vartheta}}{4}, \frac{\sqrt{14}\sqrt{1-\vartheta}}{4} \right)
\]

(4) \( c(\vartheta) = \frac{\sqrt{2}}{8\sqrt{1-\vartheta^2}} \) and \( \tau(\vartheta) = -\frac{\sqrt{14}}{8\sqrt{1-\vartheta^2}} \).
Problem 7. Consider the curve defined as follows:

\[ P(\vartheta) := \left( \vartheta, -\frac{\sqrt{2}}{2} \vartheta^2, \frac{\sqrt{2}}{2} \vartheta^2 \right) \quad \text{with } \vartheta \in \mathbb{R} \]

(1) Write the tangent vector \( \mathbf{t} \), the normal vector \( \mathbf{n} \) and the binormal vector \( \mathbf{b} \) of the curve for \( \vartheta \in \mathbb{R} \).

(2) Compute the curvature \( c \) and the torsion \( \tau \) of the curve for \( \vartheta \in \mathbb{R} \).

Solution:

(1) \( \mathbf{t}(\vartheta) = \frac{1}{\sqrt{4\vartheta^2 + 1}} (1, -\sqrt{2} \vartheta, \sqrt{2} \vartheta) \), \( \mathbf{n}(\vartheta) = \frac{1}{\sqrt{4\vartheta^2 + 1}} (-2 \vartheta, -\sqrt{2}, \sqrt{2}) \) and 
\[ \mathbf{b}(\vartheta) = \frac{\sqrt{2}}{2} (0, -1, -1) \]

(2) \( c(\vartheta) = \frac{2}{(\sqrt{4\vartheta^2 + 1})^3} \) and \( \tau(\vartheta) = 0 \)
Problem 8. Consider the curve defined as follows:

\[ P(\vartheta) := \left( \vartheta, \vartheta^2, \frac{2}{3} \vartheta^3 \right) \quad \text{with} \ \vartheta \in \mathbb{R} \]

(1) Find the length \( L \) of the curve in the interval \([-1, 3]\).

(2) Write the tangent vector \( \mathbf{t} \), the normal vector \( \mathbf{n} \) and the binormal vector \( \mathbf{b} \) of the curve for \( \vartheta \in \mathbb{R} \).

(3) Compute the curvature \( c \) and the torsion \( \tau \) of the curve for \( \vartheta \in \mathbb{R} \).

Solution:

(1) \( L = \frac{68}{3} \)

(2) \( \mathbf{t}(\vartheta) = \frac{1}{2\vartheta^2 + 1} \left( 1, 2\vartheta, 2\vartheta^2 \right) \), \( \mathbf{n}(\vartheta) = \frac{1}{2\vartheta^2 + 1} \left( -2\vartheta, 1 - 2\vartheta^2, 2\vartheta \right) \) and \( \mathbf{b}(\vartheta) = \frac{1}{2\vartheta^2 + 1} \left( 2\vartheta^2, -2\vartheta, 1 \right) \)

(3) \( c(\vartheta) = \frac{2}{(2\vartheta^2 + 1)^2} \) and \( \tau(\vartheta) = -\frac{2}{(2\vartheta^2 + 1)^2} \)
EXERCISES ON CABLES

Problem 9. A cable with length $L$ and linear mass density $\lambda$ is hung in $A$ at height $h$. At $B$ a force $F_B = e_x + \frac{2}{3} \lambda g Le_y$ is exerted on the cable, as shown in the following picture:

Assume that the only distributed force acting on the cable is due to the gravitational acceleration $g = -ge_y$.

(1) Let $y(x)$ be the function that describes the shape of the cable. What is the condition for $y(0)$, at equilibrium?

(2) Let $T(x)$ be the function that describes the tension of the cable, and let $t(x)$ be the tangent versor to the shape of the cable. What is the condition that involves both the tension and the tangent at the point $B$, at equilibrium?

(3) Write the function $y(x)$.

(4) What is the height $h_B$ of the point $B$ at equilibrium?

(5) For which value of $x$ the tension is minimal?

(6) What is the reactive force $F_A$ the cable is subjected to at $A$?

Solution:

(1) $y(0) = h$

(2) $T(x_B)t(x_B) = e_x + \frac{2}{3} \lambda g Le_y$

(3) $y(x) = \frac{1}{\lambda g} \cosh \left( \lambda gx - \sinh \left( \frac{\lambda g L}{3} \right) \right) + h - \frac{1}{\lambda g} \cosh \left( \sinh \left( \frac{\lambda g L}{3} \right) \right)$

(4) $h_B = \frac{1}{\lambda g} \left( \cosh \left( \sinh \left( \frac{2\lambda g L}{3} \right) \right) - \cosh \left( \sinh \left( \frac{\lambda g L}{3} \right) \right) \right) + h$

(5) $x = \frac{1}{\lambda g} \sinh \left( \frac{\lambda g L}{3} \right)$

(6) $F_A = -e_x + \frac{1}{3} \lambda g Le_y$
Problem 10. A cable with linear mass density $\lambda = \frac{2p}{g}$ is subjected in $A$ and $B$ respectively to two forces $\mathbf{F}_A$ and $\mathbf{F}_B$. The force $\mathbf{F}_A$ makes an angle $\frac{\pi}{6}$ with the horizontal direction $\mathbf{e}_x$ and the force $\mathbf{F}_B$ makes an angle $\frac{\pi}{3}$ with the horizontal direction $\mathbf{e}_x$, as shown in the following picture:

![Cable diagram](image)

The norm of the force $\mathbf{F}_B$ is $|\mathbf{F}_B| = 3pl$, and the only distributed force acting on the cable is due to the gravitational acceleration $\mathbf{g} = -g\mathbf{e}_y$. Consider a reference system centered at the point $O$ of minimal tension (as in the picture: $x_O = 0$ and $y_O = 0$).

(1) Let $y(x)$ be the function that describes the shape of the cable. What are the conditions for $y(0)$ and $\dot{y}(0)$, at equilibrium?

(2) Let $T(x)$ be the function that describes the tension of the cable, and let $t(x)$ be the tangent versor to the shape of the cable. What is the condition that involves both the tension and the tangent at the point $B$, at equilibrium?

(3) Write the function $y(x)$.

(4) Write the vector $A - O$ and $B - O$.

(5) What is the length $L$ of the cable?

Solution:

(1) $y(0) = 0$ and $\dot{y}(0) = 0$

(2) $T(x_B)t(x_B) = \mathbf{F}_B = \frac{3}{2}pe_x + \frac{3\sqrt{3}}{2}pe_y$

(3) $y(x) = \frac{3}{4}l \left( \cosh \left( \frac{4}{3l}x \right) - 1 \right)$

(4) $A - O = -\frac{3l}{4} \text{ asinh} \left( \frac{1}{\sqrt{3}} \right) \mathbf{e}_x + \frac{3l}{4} \left( \frac{2}{\sqrt{3}} - 1 \right) \mathbf{e}_y$

and $B - O = \frac{3l}{4} \text{ asinh} \left( \sqrt{3} \right) \mathbf{e}_x + \frac{3l}{4} \mathbf{e}_y$

(5) $L = \sqrt{3}l$
EXERCISES ON BEAMS

Problem 11. A uniformly elastic beam with $EI_2 = B$ and with length $2L$ is simply supported at its middle point $O$. The first half of the beam ($s \in [-L, 0]$) is subjected to a distributed force with density $f = -f e_y$ ($f > 0$) and a couple force $G = Ge_z$ is exerted on the second end ($s = L$) of the beam, as shown in the following picture:

(Recall: the simple support exerts on the beam a reactive force $F_O$ only, and $F_O$ is orthogonal to the center line of the beam at $O$).

(1) At equilibrium, what is the reactive force $F_O$ exerted on the beam and what is the relation between $G$ and $f$? (Hint: use the total balance of forces acting on the beam and the total balance of torques at $s = L$).

(2) Let $P(s) = (x(s), y(s))$ be the curve that describes the shape of the beam at equilibrium, $\varphi(s)$ be the stress force at equilibrium and $\vartheta(s)$ be the angle between $e_x$ and the tangent to $P(s)$ at equilibrium. What are the boundary conditions for $y(s)$, $\varphi(s)$ and $\vartheta'(s)$ at the ends $s = -L$ and $s = L$?

(3) Write the conditions for $y(0)$ and $\vartheta(0)$.

(4) Write the function $y(x)$ which represents the shape of the beam at equilibrium, in case of small deflections.

(5) What is the maximum deflection $y_{\text{max}} := \max\{|y(x)|\}$ of the beam, and for which values of $x$ is this attained?

(6) What is the maximum slope $\vartheta_{\text{max}} := \max\{|\vartheta(x)|\}$ of the beam, and for which values of $x$ is this attained?

(7) What is the condition on the value of $f$ that makes the small deflections approximation viable?
Solution:

(1) $F_O = fLe_y$ and $G = -\frac{1}{2}fL^2$

(2) $\varphi'(-L) = 0$, $\varphi'(L) = -\frac{fL^2}{2B}$, $\varphi(-L) = 0$ and $\varphi(L) = 0$

(3) $y(0) = 0$ and $\vartheta(0) = 0$

(4)
$$
\begin{cases}
    y(x) = -\frac{f}{24B}x^2(x^2 + 4Lx + 6L^2) & x \in [-L, 0] \\
    y(x) = -\frac{f}{4B}L^2x^2 & x \in [0, L]
\end{cases}
$$

(5) $y_{\text{max}} = |y(L)| = \frac{fL^4}{4B}$

(6) $\vartheta_{\text{max}} = |\vartheta(L)| = \frac{fL^3}{2B}$

(7) $f \ll \frac{2B}{L^3}$
Problem 12. A uniformly elastic beam with $EI_2 = B$ and with length $L$ is clamped at one end and supported by a smooth disk of radius $R$ at the other end, as shown in the following picture:

(Recall: the disk exerts a reactive force that is directed radially outward and orthogonal to the beam at the contact point).

(1) Let $P(s) = (x(s), y(s))$ be the curve that describes the shape of the beam at equilibrium and $\theta(s)$ be the angle between $e_x$ and the tangent to $P(s)$ at equilibrium. What are the boundary conditions for $y(s)$, $\theta(s)$ and $\theta'(s)$ at the ends $s = 0$ and $s = L$?

(2) Write the balance equations for forces and torques at equilibrium at the generic point $s$.

(3) Write the function $y(x)$ which represents the shape of the beam at equilibrium, in case of small deflections.

(4) What is the maximum deflection $y_{\text{max}} := \max\{|y(x)|\}$ of the beam, and for which values of $x$ is this attained?

(5) What is the maximum slope $\vartheta_{\text{max}} := \max\{|\vartheta(x)|\}$ of the beam, and for which values of $x$ is this attained?

(6) What is the condition on the radius $R$ that makes the small deflections approximation viable?

Solution:
(1) $y(0) = 0$, $\theta(0) = 0$, $y(L) = R\cos(\theta(L))$ and $\theta'(L) = 0$

(2) $\varphi'(s) = 0$ and $(B\varphi''(s) + F\cos(\theta(L))\cos(\theta(s)) + F\sin(\theta(L))\sin(\theta(s)))e_z = 0$, with $F = |F_L|$.

(3) $y(x) = \frac{R}{2L^3}x^2(3L - x)$

(4) $y_{\text{max}} = y(L) = R$

(5) $\vartheta_{\text{max}} = \vartheta(L) = \frac{3R}{2L}$

(6) $R \ll \frac{2}{3}L$