EXERCISES ON LINEAR ALGEBRA - SELF ASSESSMENT

Problem 1. For each of the following matrices compute the *trace*, the *determinant*, the *rank*, the *number of linearly independent rows*, the *number of linearly independent columns*, and tell if the matrix is *invertible* or not:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & 2 & 3 \\ -1 & 2 & 3 \\ -2 & 4 & 6 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Problem 2. For each of the matrices in Problem 1, find the real eigenvalues and a basis of the relative eigenspaces.

Problem 3. Consider the two following bases of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathcal{C} = \left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$\mathcal{D} = \left\{ \mathbf{d}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{d}_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \right\}$$

(1) Write the change of basis matrices $\mathbf{M}_{\mathcal{C}}^{\mathcal{B}}$ (from the basis $\mathcal{C}$ to the basis $\mathcal{B}$), $\mathbf{M}_{\mathcal{B}}^{\mathcal{C}}$ (from the basis $\mathcal{B}$ to the basis $\mathcal{C}$).

(2) Write the change of basis matrices $\mathbf{M}_{\mathcal{C}}^{\mathcal{D}}$ (from the basis $\mathcal{C}$ to the basis $\mathcal{D}$), $\mathbf{M}_{\mathcal{D}}^{\mathcal{C}}$ (from the basis $\mathcal{D}$ to the basis $\mathcal{C}$).

(3) Write the change of basis matrices $\mathbf{M}_{\mathcal{B}}^{\mathcal{D}}$ (from the basis $\mathcal{B}$ to the basis $\mathcal{D}$), $\mathbf{M}_{\mathcal{D}}^{\mathcal{B}}$ (from the basis $\mathcal{D}$ to the basis $\mathcal{B}$).

(4) Write the vectors $\mathbf{v} = 3\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$ and $\mathbf{u} = -\mathbf{e}_1 + 2\mathbf{e}_2 - 3\mathbf{e}_3$ as linear combinations of the elements of the basis $\mathcal{B}$ and as linear combinations of the elements of the basis $\mathcal{D}$.