Artificial Intelligence

A course about foundations

Probabilistic Reasoning: Supervised Learning

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Machine Learning

Types of machine learning problems

Consider a number of observations (i.e. a dataset) made by an agent

$\{D^{(1)}, D^{(2)}, ..., D^{(N)}\}$

- ***Supervised learning***

  Learning form *complete* observations: each of the observations $\{D^{(1)}, D^{(2)}, ..., D^{(N)}\}$ include values for all the random variables in the model

  The objective is learning a *distribution* P

- ***Unsupervised learning***

  Learning form *incomplete* observations: observations $\{D^{(1)}, D^{(2)}, ..., D^{(N)}\}$ do not necessarily include values for all the random variables in the model

  The objective is learning a *distribution* P

- ***Reinforcement learning***

  The observations $\{D^{(1)}, D^{(2)}, ..., D^{(N)}\}$ are *states* o *situations*, at each state X_i the agent must perform an ***action*** a_i that produces a ***result*** r_i.

  The objective is learning a *distribution* π over possible actions in each state

  which describes a *policy* that the agent will follow

  Such policy should maximize the expected value of a *reward function* $v(< r_1, r_2, ..., r_n >)$ of the sequence of *results*

Observations and Independence

Each observation could be the outcome of an experiment or a test

The outcome of a particular experiment can be represented by a set of *random variables*

For example, if the model makes use of the two random variables $\{X, Y\}$, the N outcomes of the experiments are $D^{(1)} = (X^{(1)}, Y^{(1)}), \ldots, D^{(N)} = (X^{(N)}, Y^{(N)})$

That is, a *dataset*

$$D := \{(X^{(i)}, Y^{(i)})\}_{i=1}^{N}$$

- **Independent observations, same probability distribution**

Independent and Identically Distributed (IID) *random variables*

Definition

A sequence or a set of random variables $\{X_1, X_2, \ldots, X_n\}$ is *Independent and Identically Distributed* (IID) iff:

1) $\langle X_i \perp X_j \rangle, \;\; \forall\, i \neq j$ (*independence*)
2) $P(X_i = x) = P(X_j = x), \;\; \forall\, i \neq j, \forall\, x$ (*identical distribution*)

CAUTION: *Being* IID *is not an obvious property of observations*

e.g. different measurements on different patients may be IID,
but different measurements over time on the same patient are not IID

ML = Representation + Evaluation + Optimization

Assume that an I.I.D. dataset D is available

- **Representation**

  The objective is learning a specific distribution

  $$P(\{X_r\};\theta)$$

  where $\{X_r\}$ are all the random variables of interest and θ is a *set* of parameters

  Which kind of distribution (i.e. the *model* or also the *learner*) do we select?

  Example: assume we select the anti-spam filter (i.e. Naïve Bayesian Classifier) as the model
  the parameters in such case are the numerical probabilities in the CPTs

- **Evaluation**

  Given a dataset D, how well does a specific set of parameter values $\hat{\theta}$
  make the distribution P fit the dataset?

  An estimator, i.e. a scoring function of some sort, must be selected

- **Optimization**

  How can we find the optimal set of parameter values θ^*
  with respect to the *estimator* of choice?

  In general, this is an optimization problem

Maximum Likelihood Estimator
(MLE)

Likelihood

A probabilistic model $P(X)$, with parameters θ

θ is a set of values that characterizes $P(X)$ *completely*: once θ is defined, $P(X)$ is also defined.

A set of IID observations (data items) $D = \{D^{(1)}, \dots, D^{(N)}\}$

- *Likelihood function (or conditional probability)*

A function, or a conditional probability, derived from the model $P(X)$

$$L(\theta \mid D) = P(D \mid \theta) = P(D^{(1)}, \dots, D^{(N)} \mid \theta)$$

Note the 'trick':
likelihood of the dataset given the parameters

where $P(D \mid \theta)$ is the conditional probability that the parameter θ, considered as a random variables, could *generate* the observations D

When the observations $\{D^{(1)}, \dots, D^{(N)}\}$ are IID:

$$P(D \mid \theta) = P(D^{(1)}|\theta) \; \dots \; P(D^{(N)}|\theta) = \prod_m P(D^{(m)} \mid \theta)$$

Maximum Likelihood Estimator (MLE)

A probabilistic model $P(X)$, with parameters θ

θ is a set of values that characterizes $P(X)$ *completely*: once θ is defined, $P(X)$ is also defined.

A set of IID observations (data items) $D = \{D^{(1)}, \dots , D^{(N)}\}$

- *Maximum Likelihood Estimation*

$$\theta^*_{ML} := \text{argmax}_\theta L(\theta|D)$$

Since the observations are IID, using *log-Likelihood* could ease computations:

$$\ell(\theta \mid D) = \log\ L(\theta|D)\ =\ \log \prod_m P(D^{(m)}|\theta)\ =\ \sum_m \log\ P(D^{(m)}|\theta)$$

$$\theta^*_{ML} = \text{argmax}_\theta \ell(\theta|D)$$

true because log *is monotonically increasing*

Example: coin tossing (*Bernoulli Trials*)

Experiment: tossing a coin X, not necessarily *fair* ($X = 1$ head, $X = 0$ tail)

Parameters: $\theta := \{ \pi \} \quad \Leftrightarrow \quad P(X = 1) = \pi,\ P(X = 0) = 1 - \pi$

Observations: a sequence of experimental outcomes

$D = \{D_1 = \{X^{(1)} = x^{(1)}\},\ D_2 = \{X^{(2)} = x^{(2)}\}, \ldots, D_N = \{X^{(N)} = x^{(N)}\}\}$

- Binomial distribution

$\binom{N}{k} := \frac{N!}{k!\,(N-k)!}$ *binomial coefficient*

$$P(D|\theta) = \binom{N}{N_{X=1}} \prod_i P(X^{(i)}|\theta) = \binom{N}{N_{X=1}} P(X=1|\theta)^{N_{X=1}}\ P(X=0|\theta)^{N_{X=0}}$$

$N_{X=1}$ is the number of $X = 1$ (i.e. heads) in a sequence of N trials

$$= \binom{N}{N_{X=1}} \pi^{N_{X=1}} (1-\pi)^{N_{X=0}}$$

It is the probability of obtaining $N_{X=1}$ times 'head' in a sequence of N trials

In this case, it is assumed to be the likelihood of $\{D^{(1)}, \ldots, D^{(N)}\}$ *given the parameters* θ

MLE as optimization

Example: coin tossing (*Bernoulli Trials*)

- *(Log-)Likelihood Function*

$$\ell(\theta|D) = \log P(D|\theta) = \log P(\{X^{(i)}\}|\theta) = \log \binom{N}{N_{X=1}} \prod_i P(X^{(i)}|\theta) = \log \binom{N}{N_{X=1}} + \sum_i \log P(X^{(i)}|\theta)$$

Rewrite $P(X \mid \theta)$ as:

$$P(X \mid \theta) = \pi^{[X=1]}(1-\pi)^{[X=0]} \quad \text{where:} \quad [X^{(i)} = v] := \begin{cases} 1 & \textit{if} \quad X^{(i)} = v \\ 0 & \textit{if} \quad X^{(i)} \neq v \end{cases}$$

Also called *indicator function*

$$\ell(\theta \mid D) = \log \binom{N}{N_{X=1}} + \sum_i \log \left(\pi^{[X^{(i)}=1]} (1 - \pi)^{[X^{(i)}=0]} \right) =$$

$$= \log \binom{N}{N_{X=1}} + \log \pi \sum_i [X^{(i)} = 1] + \log(1-\pi) \sum_i [X^{(i)} = 0]$$

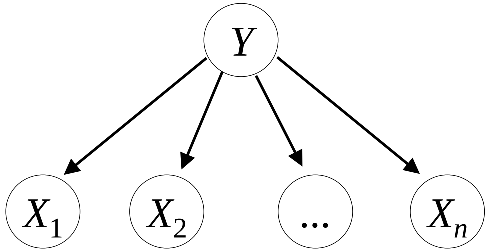
$$= \log \binom{N}{N_{X=1}} + N_{X=1} \log \pi + N_{X=0} \log(1-\pi)$$

- *Maximum Likelihood Estimation*

$$\frac{\partial \ell}{\partial \theta} = \frac{\partial \ell}{\partial \pi} = \frac{N_{X=1}}{\pi} - \frac{N_{X=0}}{(1-\pi)} \qquad \frac{\partial \ell}{\partial \theta} = 0 \Rightarrow \theta^*_{ML} = \frac{N_{X=1}}{N_{X=1} + N_{X=0}} = \frac{N_{X=1}}{N}$$

Naïve Bayesian Classifier

$$P(Y, X_1, \ldots, X_n) = P(Y) \prod_{i=1}^{n} P(X_i \mid Y)$$



Parameters: the *conditional probability tables* in the graphical model

$$\theta := \{\pi_k,\ \pi_{ijk}\} \quad , \quad P(Y = k) \ =:\ \pi_k \quad P(X_i = j \mid Y = k) \ =:\ \pi_{ijk}$$

Observations: a set of messages *with classification*

$$D = \{D^{(1)} = \{Y^{(1)} = 1, X_1^{(1)} = 1, \ldots, X_n^{(1)} = 0\},$$

$$\ldots,$$

$$D^{(N)} = \{Y_2^{(N)} = y^{(N)}, X_1^{(N)} = x_1^{(N)}, \ldots, X_n^{(N)} = x_n^{(N)}\}\}$$

- *Likelihood Function*

$$L(\theta|D) \ = \ P(D|\theta) \ = \ P(\{D^{(m)}\}|\{\pi_k, \pi_{ijk}\}) \ = \ \prod_m P(D^{(m)}|\{\pi_k, \pi_{ijk}\})$$

(data items are IID)

$$= \ \prod_m P(\{Y^{(m)} = y^{(m)}, X_i^{(m)} = x_i^{(m)}\}|\{\pi_k, \pi_{ijk}\})$$

(factorization)

$$= \ \prod_m P(Y^{(m)} = y^{(m)}|\{\pi_k, \pi_{ijk}\}) \ P(\{X_i^{(m)} = x_i^{(m)}\}|Y^{(m)} = y^{(m)}, \{\pi_k, \pi_{ijk}\})$$

(cond. independence)

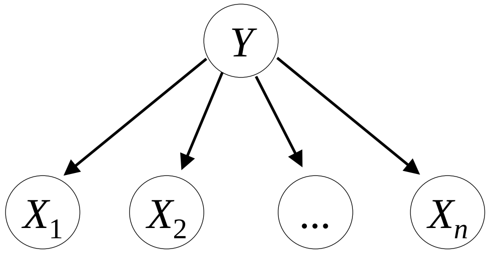
$$= \ \prod_m P(Y^{(m)} = y^{(m)}|\{\pi_k\}) \ P(\{X_i^{(m)} = x_i^{(m)}\}|Y^{(m)} = y^{(m)}, \{\pi_{ijk}\})$$

$(\langle X_i \perp X_j \mid Y \rangle)$

$$= \ \prod_m P(Y^{(m)} = y^{(m)}|\{\pi_k\}) \ \prod_i P(X_i^{(m)} = x_i^{(m)}|Y^{(m)} = y^{(m)}, \{\pi_{ijk}\})$$

Naïve Bayesian Classifier

$$P(Y, X_1, \ldots, X_n) = P(Y) \prod_{i=1}^{n} P(X_i \mid Y)$$



- *Log-Likelihood Function*

$$\ell(\{\pi_k, \pi_{ijk}\}|D) \;=\; \sum_m \log P(Y^{(m)} = y^{(m)}|\{\pi_k\}) + \sum_m \sum_i \log P({X_i}^{(m)} = {x_i}^{(m)}|Y^{(m)} = y^{(m)}, \{\pi_{ijk}\})$$

Alternative form for P: (i.e. rewritten using indicator functions)

$$P(Y = k|\{\pi_k\}) \;=\; \prod_k {\pi_k}^{[Y=k]}$$

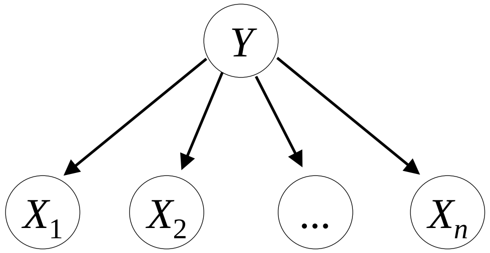
$$P(X_i = j|Y = k, \{\pi_{i\,j\,k}\}) \;=\; \prod_j \prod_k {\pi_{i,j,k}}^{[X_i=j][Y=k]}$$

$$\ell(\{\pi_k, \pi_{ijk}\}|D) \;=\; \sum_m \sum_k [Y^{(m)} = k] \;\log \pi_k + \sum_m \sum_i \sum_j \sum_k [{X_i}^{(m)} = j][Y^{(m)} = k] \;\log\; \pi_{ijk}$$

Being both positive and depending on different variables, the two terms above can be optimized separately

Naïve Bayesian Classifier

$$P(Y, X_1, \ldots, X_n) = P(Y) \prod_{i=1}^{n} P(X_i \mid Y)$$



- *Maximum Likelihood Estimation*

$$\ell(\{\pi_k, \pi_{ijk}\}|D) \;=\; \sum_m \sum_k [Y^{(m)} = k] \;\log \pi_k + \sum_m \sum_i \sum_j \sum_k [X_i{}^{(m)} = j][Y^{(m)} = k] \;\log\; \pi_{ijk}$$

Optimizing first term:

Lagrange multiplier

$$\ell^*(\{\pi_k\}|D) \;=\; \sum_m \sum_k [Y^{(m)} = k] \;\log \pi_k + \;\lambda(1 \;-\; \sum_k \pi_k)$$

$$\frac{\partial \ell^*}{\partial \pi_k} \;=\; \frac{\sum_m [Y^{(m)} = k]}{\pi_k} \;-\; \lambda$$

number of messages in D classified as k

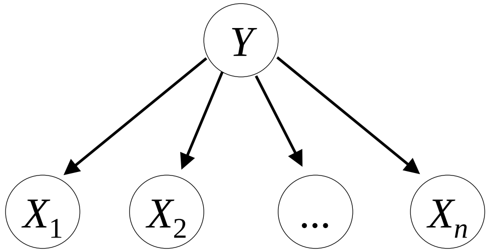
$$\frac{\partial \ell^*}{\partial \pi_k} \;=\; 0 \quad\Rightarrow\quad \pi_k \;=\; \frac{N_{Y=k}}{\lambda}$$

$$\sum_k \pi_k \;=\; 1 \quad\Rightarrow\quad \sum_k \frac{N_{Y=k}}{\lambda} \;=\; 1 \quad\Rightarrow\quad \lambda \;=\; \sum_k N_{Y=k} \;=\; N$$

$$\pi_k^* = \frac{N_{Y=k}}{N}$$ (*Maximum Likelihood Estimator* of π_k)

Naïve Bayesian Classifier

$$P(Y, X_1, \ldots, X_n) = P(Y) \prod_{i=1}^{n} P(X_i \mid Y)$$



- *Maximum Likelihood Estimation*

$$\ell(\{\pi_k, \pi_{ijk}\}|D) \;=\; \sum_m \sum_k [Y^{(m)} = k] \;\; \log\pi_k + \sum_m \sum_i \sum_j \sum_k [X_i{}^{(m)} = j][Y^{(m)} = k] \;\; \log \; \pi_{ijk}$$

Optimizing second term:

Lagrange multipliers

$$\ell^*(\{\pi_{ijk}\}|D) \;=\; \sum_m \sum_i \sum_j \sum_k [X_i{}^{(m)} = j][Y^{(m)} = k] \log \pi_{ijk} + \sum_i \sum_k \lambda_{ik}(1 \;-\; \sum_j \pi_{ijk})$$

$$\frac{\partial \ell^*}{\partial \pi_{ijk}} \;=\; \frac{\sum_m [X_i{}^{(m)} = j][Y^{\;(m)} = k]}{\pi_{ijk}} \;-\; \lambda_{ik}$$

$$\frac{\partial \ell^*}{\partial \pi_{ijk}} \;=\; 0 \quad \Rightarrow \quad \pi_{ijk} \;=\; \frac{N_{X_i=j,\; Y=k}}{\lambda_{ik}}$$

$$\sum_j \pi_{ijk} \;=\; 1 \quad \Rightarrow \quad \sum_j \frac{N_{X_i=j,\; Y=k}}{\lambda_{ik}} \;=\; 1 \quad \Rightarrow \quad \lambda \;=\; \sum_j N_{X_i=j,\; Y=k} \;=\; N_{Y=k}$$

$$\pi^*_{ijk} \;=\; \frac{N_{X_i=j,\; Y=k}}{N_{Y=k}}$$ (*Maximum Likelihood Estimator* of π_{ijk})

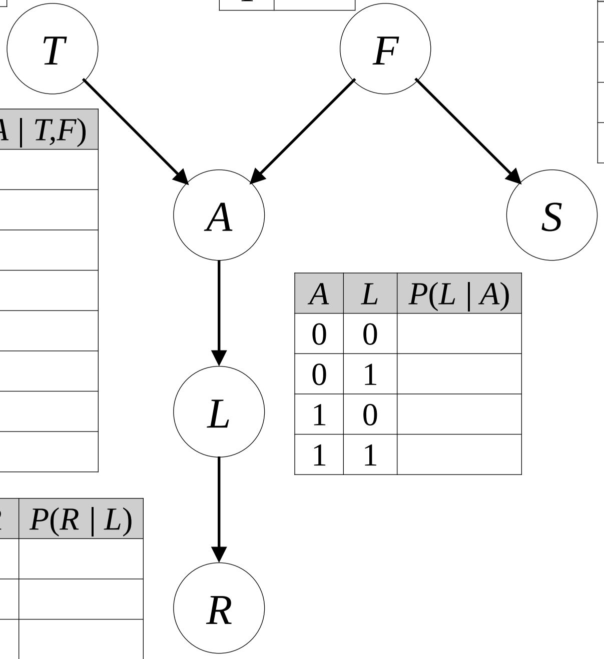
MLE for Graphical Models: A Practical Rule

Learning CPTs for a graphical model via MLE

Model: random variables plus the graph of dependencies

Observations: dataset of values, from *completely observed* outcomes

Parameters (to be determined): all conditional probabilities *(i.e. all CPTs)*



T	$P(T)$
0	
1	

F	$P(F)$
0	
1	

F	S	$P(S \mid F)$
0	0	
0	1	
1	0	
1	1	

T	F	A	$P(A \mid T,F)$
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

A	L	$P(L \mid A)$
0	0	
0	1	
1	0	
1	1	

L	R	$P(R \mid L)$
0	0	
0	1	
1	0	
1	1	

Learning CPTs for a graphical model via MLE

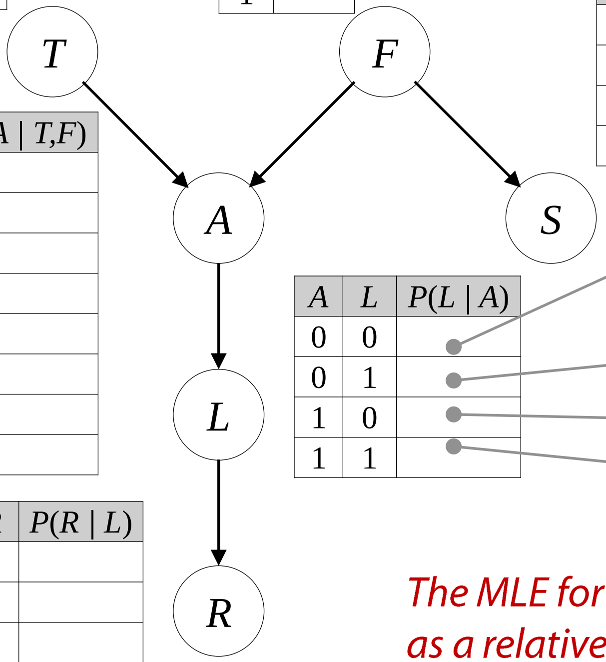
Model: random variables plus the graph of dependencies

Observations: dataset of values, from *completely observed* outcomes

Parameters (to be determined): all conditional probabilities *(i.e. all CPTs)*

T	$P(T)$
0	$\frac{N_{T=0}}{N}$
1	$\frac{N_{T=1}}{N}$

F	$P(F)$
0	$\frac{N_{F=0}}{N}$
1	$\frac{N_{F=1}}{N}$



F	S	$P(S \mid F)$
0	0	$\frac{N_{S=0,\ F=0}}{N_{F=0}}$
0	1	$\frac{N_{S=1,\ F=0}}{N_{F=0}}$
1	0	$\frac{N_{S=0,\ F=1}}{N_{F=1}}$
1	1	$\frac{N_{S=1,\ F=1}}{N_{F=1}}$

T	F	A	$P(A \mid T,F)$
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

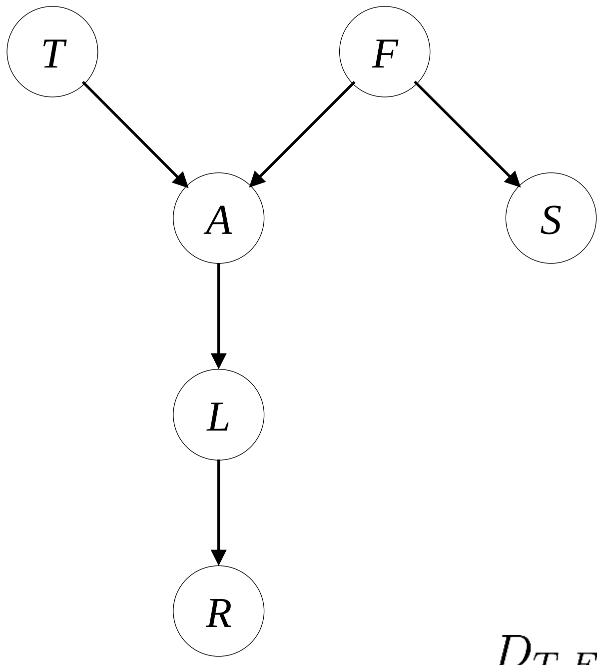
A	L	$P(L \mid A)$
0	0	$\frac{N_{L=0,\ A=0}}{N_{A=0}}$
0	1	$\frac{N_{L=1,\ A=0}}{N_{A=0}}$
1	0	$\frac{N_{L=0,\ A=1}}{N_{A=1}}$
1	1	$\frac{N_{L=1,\ A=1}}{N_{A=1}}$

L	R	$P(R \mid L)$
0	0	
0	1	
1	0	
1	1	

The MLE for each CPT is determined as a relative frequency of occurrence in the dataset

Learning CPTs for a graphical model via MLE

More in general:
The MLE of a (directed) graphical model is the MLE of each node
(in each corresponding observation subset)



$$\theta^*_{ML} := \text{argmax}_\theta \; P(D \mid \theta)$$

$$\theta = \{\pi_T, \pi_F, \pi_{S|F}, \pi_{A|S,F}, \pi_{L|A}, \pi_{R|L}\}$$

$$\pi^*_T := \text{argmax}_{\pi_T} \; P(D \mid \pi_T)$$
$$\pi^*_F := \text{argmax}_{\pi_F} \; P(D \mid \pi_F)$$
$$\pi^*_{S|F} := \text{argmax}_{\pi_{S|F}} \; P(D_F \mid \pi_{S|F})$$
$$\pi^*_{A|T,F} := \text{argmax}_{\pi_{A|T,F}} \; P(D_{T,F} \mid \pi_{A|T,F})$$
$$\pi^*_{L|A} := \text{argmax}_{\pi_{L|A}} \; P(D_A \mid \pi_{L|A})$$
$$\pi^*_{R|L} := \text{argmax}_{\pi_{R|L}} \; P(D_L \mid \pi_{R|L})$$

$D_{T,F}$ *denotes the subset of complete observation in which the random variables T, F have the corresponding values*

Bayesian Learning: Maximum a Posteriori (MAP) estimator

Bayesian learning

- *Maximum a Posteriori Estimation* (*MAP*)

  Instead of a *likelihood function*, the a posteriori probability is maximized

$$P(\theta|D) = \frac{P(D|\theta)\,P(\theta)}{P(D)} = \frac{P(D|\theta)\,P(\theta)}{\sum_{\theta} P(D|\theta)P(\theta)}$$

  Which is equivalent to optimize, w.r.t. θ:

$$P(D|\theta)\,P(\theta)$$

$$\theta^*_{MAP} := \text{argmax}_\theta\ P(D|\theta)\,P(\theta)$$

Advantages:

- Regularization: not all possible combinations of values might be present in D
- A formula for incremental learning:
  a priori terms could represent what was known *before* observations D

Problem:

- Which *prior* distribution? $P(\theta)$

Beta distribution

Gamma function (n integer > 0*)*

$$\Gamma(n) := (n-1)!$$

Beta <u>function</u> *(α and β integers* > 0*)*

$$\mathrm{B}(\alpha,\beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}$$

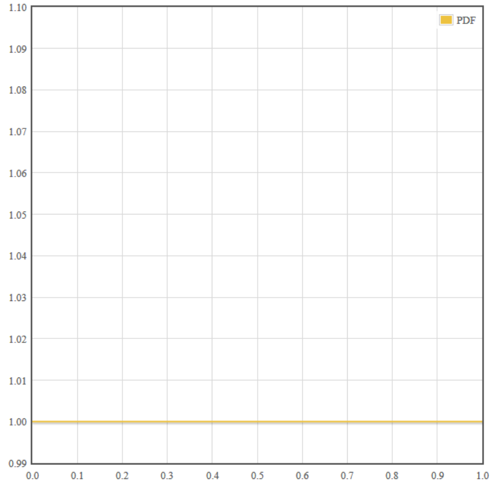
The definition is more complex when α and β are not integers (see Wikipedia)

- Beta probability density function (pdf) *(α and β integers* > 0*)*

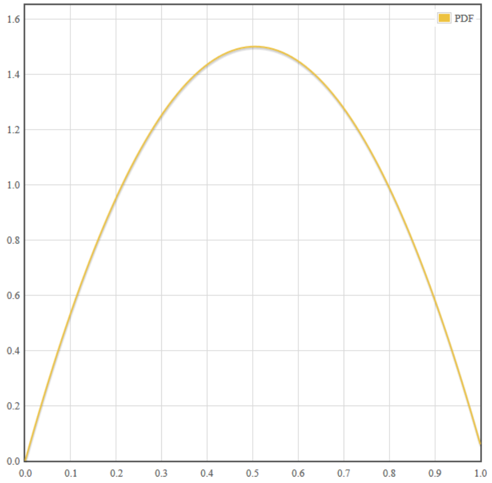
$$\mathrm{Beta}(\theta;\alpha,\beta) := \frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}$$

The maximum occurs at: $\theta = \dfrac{\alpha-1}{\alpha+\beta-2}$

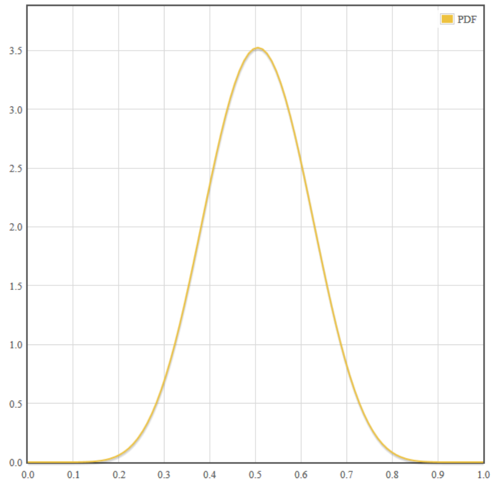
Beta(θ;1,1)



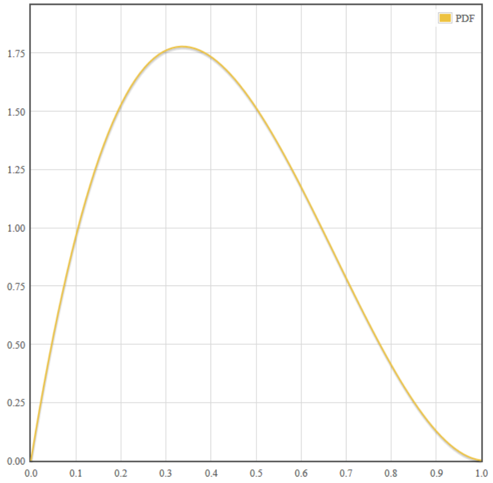
Beta(θ;2,2)



Beta(θ;10,10)



Beta(θ;2,3)



Conjugate prior distributions

Coin tossing (i.e. Binomial)

α_D and β_D are the result counts (i.e. heads and tails)

$$P(D|\theta) = \binom{\alpha_D + \beta_D}{\alpha_D} \prod_i P(X_i|\theta) = \binom{\alpha_D + \beta_D}{\alpha_D} \theta^{\alpha_D} (1-\theta)^{\beta_D}$$

A posteriori probability with Beta prior

*α_P and β_P are are the **hyperparameters** of the prior*

$$P(D|\theta)P(\theta) = \binom{\alpha_D + \beta_D}{\alpha_D} \theta^{\alpha_D} (1-\theta)^{\beta_D} \cdot \mathrm{Beta}(\theta; \alpha_P, \beta_P) = \binom{\alpha_D + \beta_D}{\alpha_D} \theta^{\alpha_D} (1-\theta)^{\beta_D} \cdot \frac{\theta^{\alpha_P - 1} (1-\theta)^{\beta_P - 1}}{\mathrm{B}(\alpha_P, \beta_P)}$$

$$= \binom{\alpha_D + \beta_D}{\alpha_D} \frac{\theta^{\alpha_D + \alpha_P - 1} (1-\theta)^{\beta_D + \beta_P - 1}}{\mathrm{B}(\alpha_P, \beta_P)} = \binom{\alpha_D + \beta_D}{\alpha_D} \frac{\mathrm{B}(\alpha_D + \alpha_P, \beta_D + \beta_P)}{\mathrm{B}(\alpha_P, \beta_P)} \mathrm{Beta}(\theta; \alpha_D + \alpha_P, \beta_D + \beta_P)$$

this factor is a positive constant (for θ)

Conjugate prior distributions

Coin tossing (i.e. Binomial)

α_D and β_D are the result counts (i.e. heads and tails)

$$P(D|\theta) = \binom{\alpha_D + \beta_D}{\alpha_D} \prod_i P(X_i|\theta) = \binom{\alpha_D + \beta_D}{\alpha_D} \theta^{\alpha_D} (1-\theta)^{\beta_D}$$

A posteriori probability with Beta prior

$$P(D|\theta)P(\theta) = \binom{\alpha_D + \beta_D}{\alpha_D} \frac{\mathrm{B}(\alpha_D + \alpha_P, \beta_D + \beta_P)}{\mathrm{B}(\alpha_P, \beta_P)} \mathrm{Beta}(\theta; \alpha_D + \alpha_P, \beta_D + \beta_P)$$

"is proportional to"

$$P(D|\theta)P(\theta) \propto \mathrm{Beta}(\theta; \alpha_D + \alpha_P, \beta_D + \beta_P)$$

Optimization:

$$\theta^*_{MAP} = \mathrm{argmax}_\theta \, \mathrm{Beta}(\theta; \alpha_D + \alpha_P, \beta_D + \beta_P) = \frac{\alpha_D + \alpha_P - 1}{\alpha_D + \alpha_P + \beta_D + \beta_P - 2}$$

which is the same as MLE but with the addition of $\alpha_P + \beta_P$ *pseudo-observations*

Being a ***conjugate prior*** $P(\theta)$ of a distribution $P(D|\theta)$ *in the above sense*
means that the posterior $P(D|\theta)P(\theta)$ is in *the same family* of $P(\theta)$

Conjugate prior distributions

Coin tossing (i.e. a specific observation i)

$$P(D_i|\theta) \;=\; \theta^{[X_i=1]}(1 \;-\; \theta)^{[X_i=0]}$$

Likelihood (of a dataset)

$$P(D|\theta) \;=\; \binom{N}{N_{X=1}} \prod_i P(D_i|\theta) \;=\; \binom{N}{N_{X=1}} \theta^{N_{X=1}}(1-\theta)^{N_{X=0}}$$

A posteriori probability with Beta prior

"is proportional to"

$$P(D|\theta)P(\theta) \;\propto\; \mathrm{Beta}(\theta,\; N_{X=1}+\alpha_P,\; N_{X=0}+\beta_P)$$

Therefore

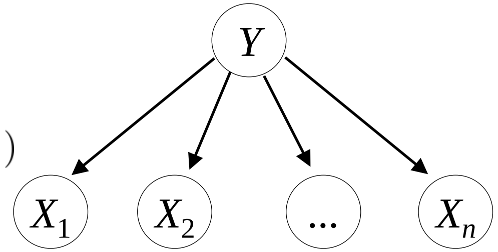
$$\theta^*_{MAP} \;=\; \mathrm{argmax}_\theta\; \mathrm{Beta}(\theta,\; N_{X=1}+\alpha_P,\; N_{X=0}+\beta_P) \;=\; \frac{N_{X=1}+\alpha_P-1}{N+\alpha_P+\beta_P-2}$$

which is the same as MLE but with the addition of $\alpha_P + \beta_P$ *pseudo-observations*

Being a ***conjugate prior*** $P(\theta)$ of a distribution $P(D|\theta)$ *in the above sense*
means that the posterior $P(D|\theta)P(\theta)$ is in *the same family* of $P(\theta)$

Anti-spam filter

$$P(Y, X_1, \ldots, X_n) = P(Y) \prod_{i=1}^{n} P(X_i \mid X_{i-1})$$



- *Maximum a Posteriori (MAP) Estimation*

  The adapted computations for:

$$\theta^*_{MAP} \; := \; \mathrm{argmax}_\theta \; P(D|\theta) \, P(\theta)$$

  yield:

$$\pi^*_k = \frac{\alpha_k + N_{Y=k} - 1}{\alpha_k + \beta_k + N - 2}$$

(*MAP Estimator* of π_k)

$$\pi^*_{ijk} = \frac{\alpha_{ijk} + N_{X_i=j,\, Y=k} - 1}{\alpha_{ijk} + \beta_{ijk} + N_{Y=k} - 2}$$

(*MAP Estimator* of π_{ijk})

  where the

$$\alpha_k, \beta_k, \alpha_{ijk}, \beta_{ijk}$$

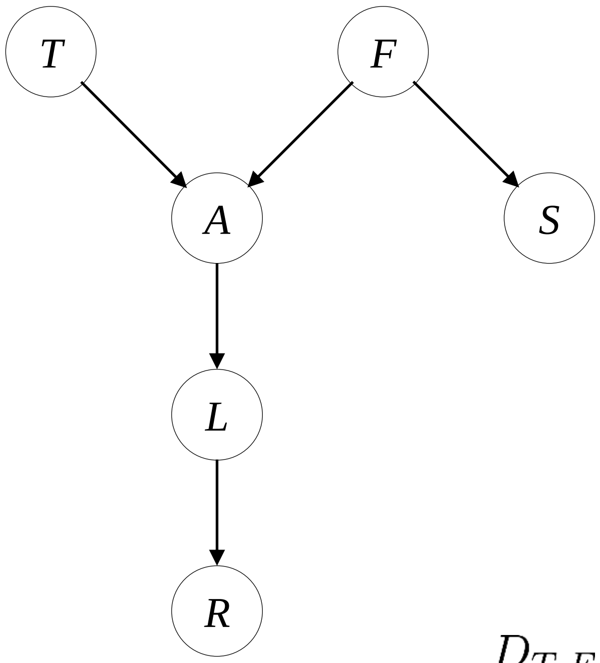
  are the *hyperparameters* of the prior distribution
  representing the *pseudo-observations*
  made *before* the arrival of new, actual observations D

Bayesian Learning:
MAP for Graphical Models

Learning CPTs for a graphical model

As *Maximum a Posteriori Estimation*

More in general:
The MAP of a (directed) graphical model is the MAP of each node
(in each corresponding observation subset)



$$\theta^*_{MAP} := \text{argmax}_\theta \; P(D \mid \theta) \; P(\theta)$$

$$\theta = \{\pi_T, \pi_F, \pi_{S|F}, \pi_{A|S,F}, \pi_{L|A}, \pi_{R|L}\}$$

$$\pi^*_T := \text{argmax}_{\pi_T} \; P(D \mid \pi_T) \; P(\pi_T)$$
$$\pi^*_F := \text{argmax}_{\pi_F} \; P(D \mid \pi_F) \; P(\pi_F)$$
$$\pi^*_{S|F} := \text{argmax}_{\pi_{S|F}} \; P(D_F \mid \pi_{S|F}) \; P(\pi_{S|F})$$
$$\pi^*_{A|T,F} := \text{argmax}_{\pi_{A|T,F}} \; P(D_{T,F} \mid \pi_{A|T,F}) \; P(\pi_{A|T,F})$$
$$\pi^*_{L|A} := \text{argmax}_{\pi_{L|A}} \; P(D_A \mid \pi_{L|A}) \; P(\pi_{L|A})$$
$$\pi^*_{R|L} := \text{argmax}_{\pi_{R|L}} \; P(D_L \mid \pi_{R|L}) \; P(\pi_{R|L})$$

$D_{T,F}$ *denotes the subset of complete observation in which the random variables T, F have the corresponding value*