Artificial Intelligence

A course about foundations

Entailment and Algorithms

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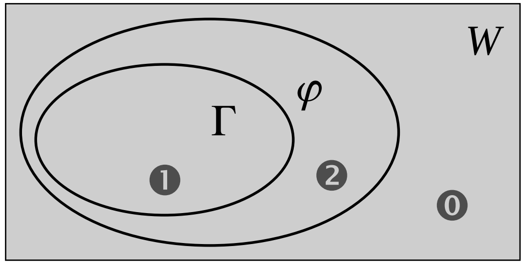
Entailment as *Satisfiability*

Transforming problems: entailment as satisfiability

- Step 1: the decision problem " $\Gamma \models \varphi$? "
  can be transformed into a *satisfiability* problem

  In fact, $\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is *not* satisfiable



($w(\Gamma)$ is the set of possible worlds that satisfy Γ)

$\Gamma \models \varphi \;\Rightarrow\; w(\Gamma) \subseteq w(\{\varphi\})$ ❶ ⊆ {❶, ❷}

$w(\{\neg\varphi\})$ = ⓿

$w(\Gamma \cup \{\neg\varphi\}) = w(\Gamma) \cap w(\{\neg\varphi\})$

$w(\Gamma \cup \{\neg\varphi\}) = \varnothing$ ❶ ∩ ⓿ = ∅

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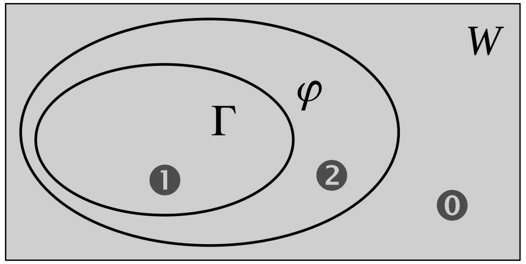

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- Step 2: the decision problem " is $\Gamma \cup \{\neg\varphi\}$ satisfiable? "
  can be transformed into a wff *satisfiability* problem

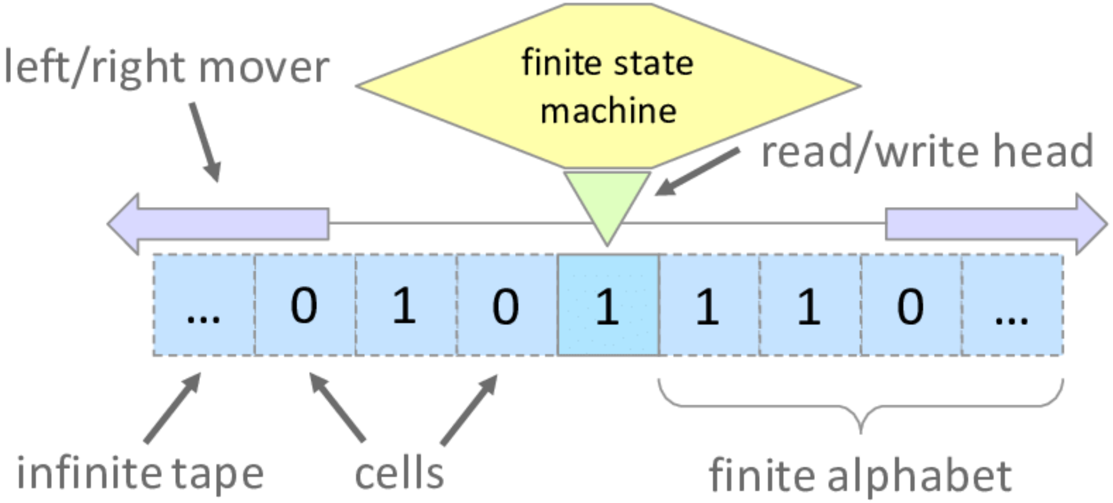
  Taking this one step further, we can transform $\Gamma \cup \{\neg\varphi\}$ into *just one formula*:

$$\bigwedge (\Gamma \cup \{\neg\varphi\})$$

This is the wff obtained by combing all the wffs in $\Gamma \cup \{\neg\varphi\}$ with $\wedge$,
it is called the *conjunctive closure* of the set $\Gamma \cup \{\neg\varphi\}$

"Algorithm"
(*Computational Complexity Theory in a Quick Ride*)

Turing Machine (A. Turing, 1937)



- A more precise definition
  - A non-empty and finite set of *states* S
    - At each instant the machine is in a state $s \in S$
  - A non-empty and finite alphabet of *symbols* Q
    - The alphabet Q includes a *blank*, default symbol b
    - Each cell in the tape contains a symbol $q \in Q$
  - A partial *transition* function

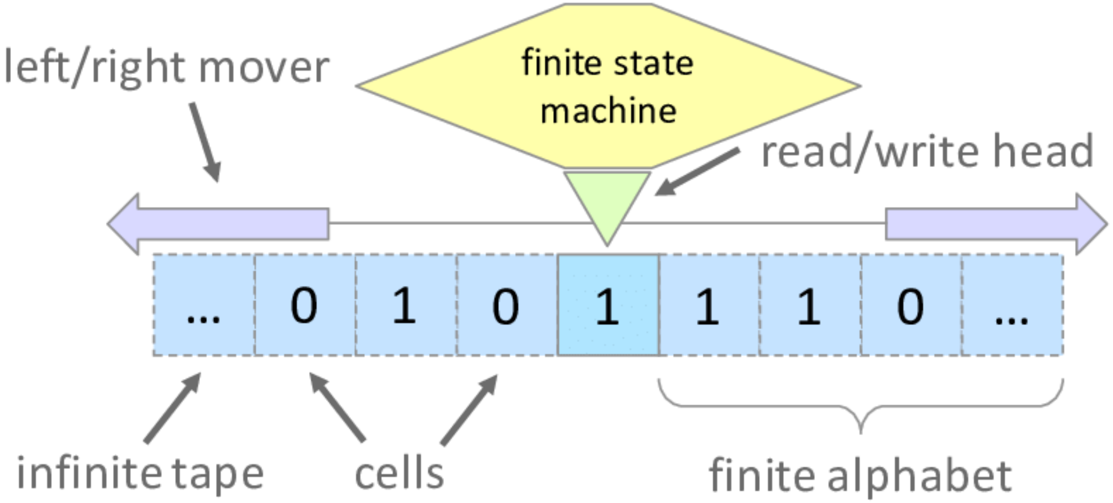
$$\tau : S \times Q \rightarrow S \times Q \times \{\text{Left}, \text{None}, \text{Right}\}$$

current state *input symbol* *next state* *output symbol* *head move*

It is partial in the sense it needs not be defined on any input tuple

  - A subset of *terminal* states $T \subseteq S$
  - An initial state $s_0 \in S$

Turing Machine (A. Turing, 1937)



- A ***busy beaver*** example (3 states)

$$S = \{A, B, C, \text{HALT}\}$$

$$s_0 = A \qquad T = \{\text{HALT}\}$$

$$Q = \{0, 1\} \qquad b = 0$$

$$\begin{aligned}
\tau = \\
& < A, 0 > \rightarrow < B, 1, \text{Right} > \\
& < A, 1 > \rightarrow < C, 1, \text{Left} > \\
& < B, 0 > \rightarrow < A, 1, \text{Left} > \\
& < B, 1 > \rightarrow < B, 1, \text{Right} > \\
& < C, 0 > \rightarrow < B, 1, \text{Left} > \\
& < C, 1 > \rightarrow < \text{HALT}, 1, \text{Right} >
\end{aligned}$$

Assume that the tape is infinite and plenty of blank symbols 0

What does this machine do?

An Aside: Church-Turing Thesis

- A possible formulation* (from Wikipedia):

> *“Every 'function which would naturally be regarded as computable‘*
> *can be computed by a Turing machine.”*

The vagueness in the above sentence gives raise to different interpretations.
One of these (though not entirely equivalent) is (from Wikipedia)*:*

> *“Every 'function that could be physically computed‘*
> *can be computed by a Turing machine.” ***

Searle: “... At present, obviously, the metaphor is the digital computer.”

** Caution: there is no such a thesis in the original writings of either author.*
Its formulation could be extrapolated from both, hence the attribution (made by others)

*** Quantum computation shatters complexity theory, but is (almost) innocuous to computability theory*

Decisions and decidability (automation)

- What is a *problem*?

  A *problem* is an association, a **relation** between *inputs* and *outputs* (= *solutions*)

  $$K = I \times S$$

- *Search problem*

  Typically, K associates *one* input to *many* solutions

  Optimization problems

  A *search problem* plus an *objective* or *cost* function

  $c : S \to \mathbb{R}$ (from S to the set of real numbers)

  In general, the task in a search problem is finding the solution(s) having maximal or minimal cost

- ***Decision problem***

  The solution space S is $\{0, 1\}$
  and K associates each input to a *unique* solution: $K : I \to \{0, 1\}$

  Example of decision problem: $\Gamma \models \varphi$?

  The input space I contains all possible combinations of set Γ of wffs with individual wffs φ
  The solution is uniquely defined for any instance of such problems in I

Decisions and decidability (automation)

- ***Decidable problem***

  A decision problem K for which there exists an algorithm, i.e a *Turing machine*,

  (there are other ways of defining an algorithm or an *effective procedure*: they are all equivalent)

  that ***always terminates*** and produces the right answer in ***finite time***.

Example of an *undecidable* problem: The *Halting Problem*

Given the formal description of a particular Turing machine and a specific input,
is it possible to tell if whether it will either halt, eventually, or run forever?

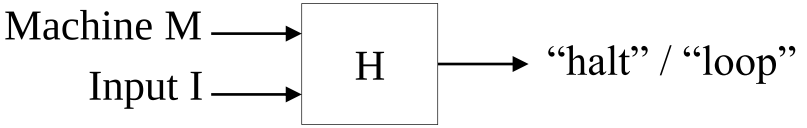
In other words, does it exist a Turing machine that, given in input the description of *another* Turing machine, will always produce the answer desired?

The answer is **no** (such a Turing machine *cannot* exist)

An aside: The *Halting Problem*

- Intuitive idea behind the proof (of the *undecidability* of this problem)

  Let's assume there exists a Turing machine H that, given the description of a Turing machine M with input I always terminates producing an output "halt" or "loop" depending on whether M with input I will terminate or not



(these are output *<u>symbols</u>)*

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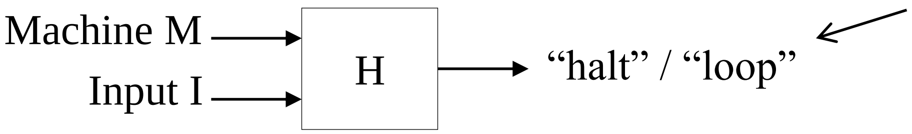


Machine M

Input I

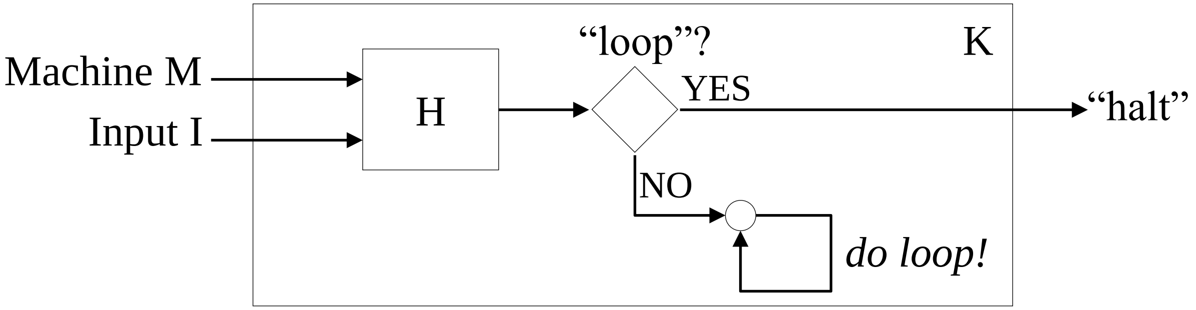
H

"halt" / "loop"

(these are output <u>symbols</u>)

<u>Assume H existed</u>

We could build another Turing machine K that enters an infinite loop whenever the output of H is "halt" and that terminates, with output "halt", when H outputs "loop"



Machine M

Input I

H

"loop"?

K

YES

"halt"

NO

do loop!

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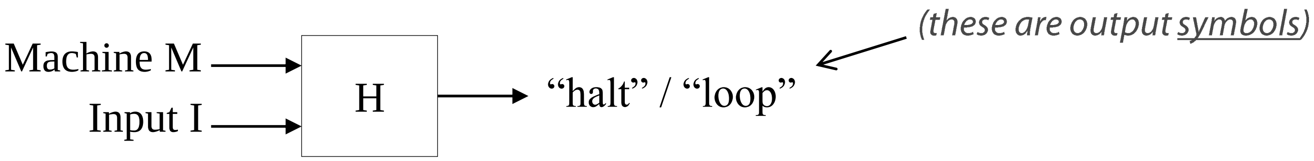

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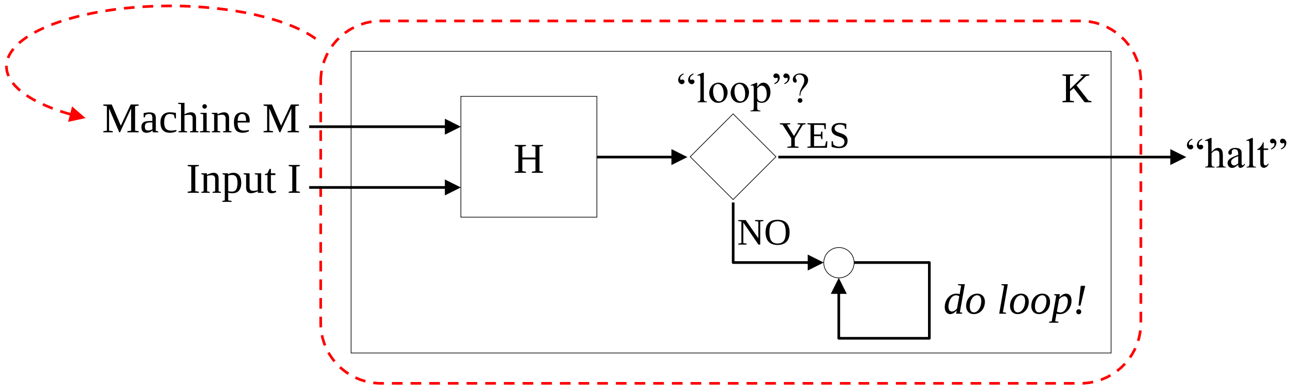


What will be the output of K when given K *itself* as the input?

K should *diverge* when K *terminates* and vice-versa: we have an absurdity

Computational complexity

CAUTION: These notions apply to *decidable problems* only

The benchmark is a (known) Turing machine that computes the correct answer in worst-case scenarios (= the least favorable inputs)

- *Time* complexity

  The number of *steps* that the Turing machine requires for computing the answer, as a function of some numerical dimension of the input (example: the number of atoms in a wff)

- *Memory* complexity

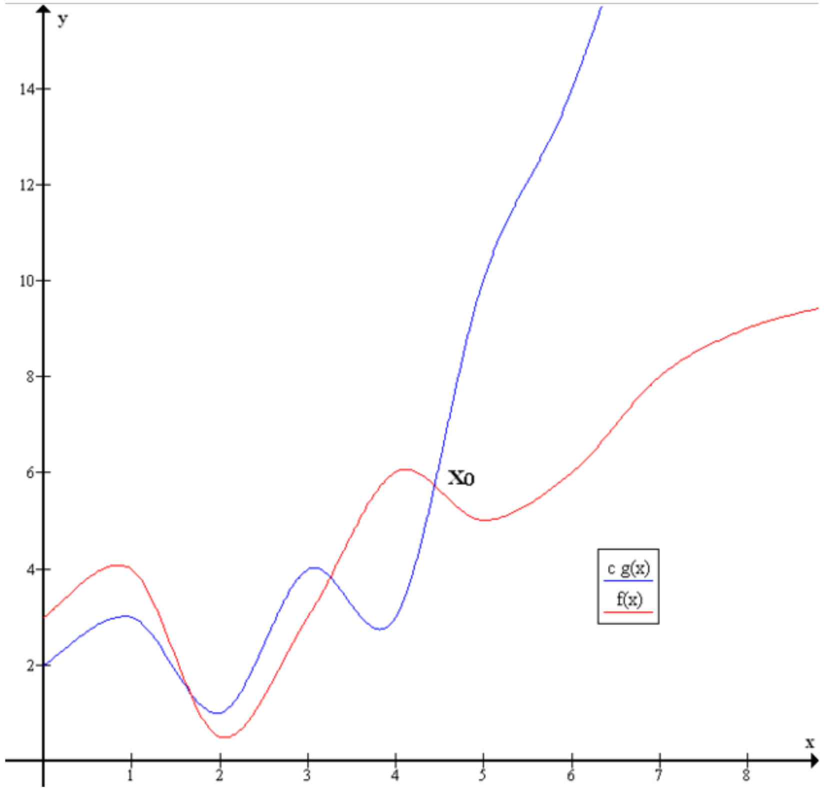
  The number of tape *cells* that the Turing machine requires for computing the answer, as a function of some numerical dimension of the input

- Big-O notation

$$f(x) = O(g(x))$$

means that

$$\exists M > 0,\ \exists x_0 > 0 \quad \text{such that} \quad |f(x)| \leq M|g(x)|,\ \ \forall x > x_0$$



Classes P, NP and NP-complete – The SAT problem

- Class P

  The class of problems for which there is a Turing machine that requires $O(\mathrm{P}(n))$ time

    where $\mathrm{P}(\ )$ is a polynomial of finite degree and n is the dimension of the (*worst-case*) input

- Class NP

  The class of all problems:

  a) A method for *enumerating* all possible answers (*recursive enumerability*)

  b) An algorithm in class P that *verifies* if a possible answer is also a *solution*

      It includes all problems in class P (that is, $\mathrm{P} \subseteq \mathrm{NP}$)

Classes P, NP and NP-complete – The SAT problem

- Class NP-complete

  It is a subclass of NP (NP-complete $\subseteq$ NP)

  A problem K is NP-complete if every problem in class NP is *reducible* to K

- Reducibility

  For class NP-complete

  Consider a problem K for which a decision algorithm $M(K)$ is known

  A problem J is *reducible* to K if there exist a decision algorithm $M(J)$ such that:

  a) algorithm $M(K)$ is called just once, as a “subroutine”, at the end of $M(J)$

  b) apart from $M(K)$, $M(J)$ has polynomial complexity

- The problem SAT

  Is NP-complete (*historically, it is the first one to be known*)

  Moral: if we had a polynomial decision algorithm for SAT, we would also have that

  $\mathrm{P} = \mathrm{NP}$

  This is not known for certain: it is commonly believed that $\mathrm{P} \neq \mathrm{NP}$

  (*and quite a lot will change in the digital world, if this belief turns to be false*)

Exhaustive (Tree) Search

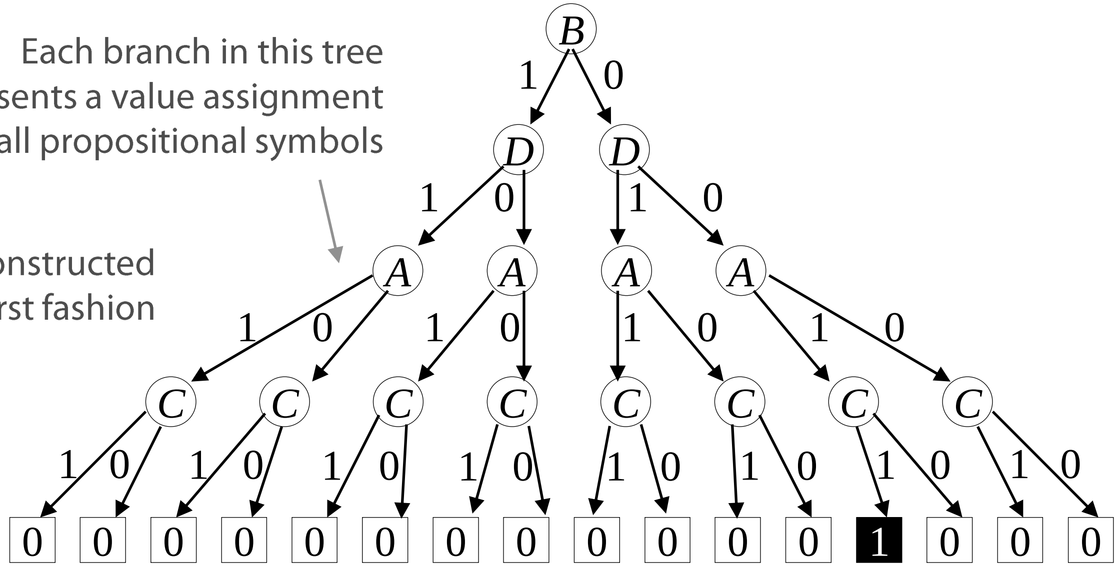
Satisfiability and decidability (in L_P)

- Is the decision problem " is the wff φ satisfiable? "?
  - It can be transformed into a *search* problem
    - that is, finding a possible world (in the set of all possible worlds) that satisfies φ
      - In the scientific literature, this problem is called "SAT"
    - *Intuition*: we can try every possible value assignment for the atoms in φ
    - *Hint:* the problem is NP-complete

Satisfiability and decidability (in L_P)

Example: is this wff *satisfiable*?

$\neg(B \vee D \vee \neg(A \wedge C))$



Each branch in this tree represents a value assignment to all propositional symbols

The tree can be constructed in a depth-first fashion

Each leaf in the tree is the value of the wff with the corresponding value assignments

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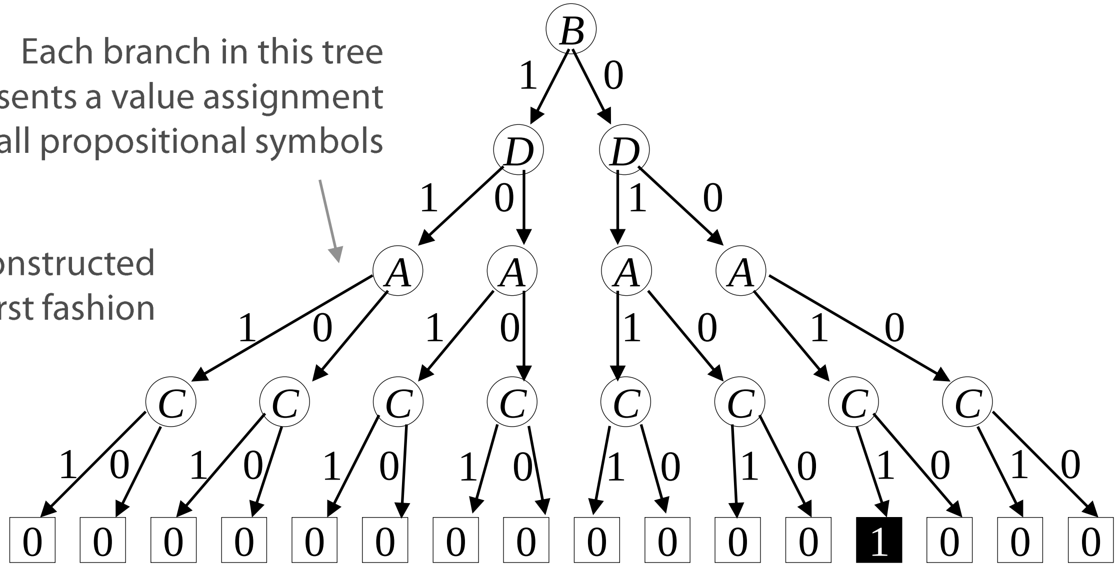

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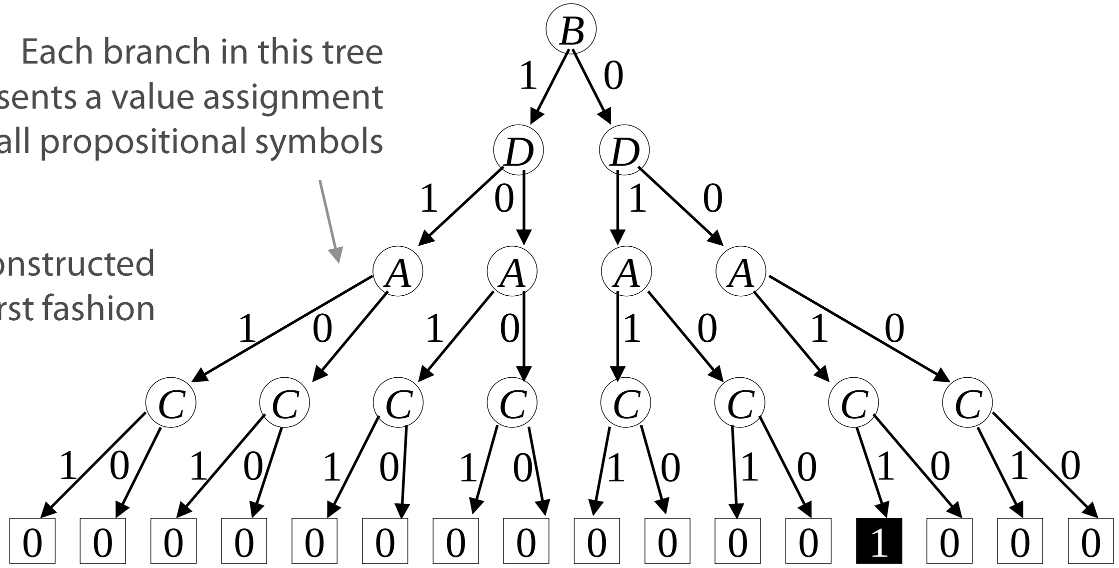
In this case, a depth-first algorithm stops here

But the algorithm is forced to try all possible assignments when ψ is *not* satisfiable, for example, with: $(\neg B \wedge \neg D \wedge \neg A \wedge \neg C)$

Satisfiability and decidability (in L_P)

Example: is this wff *satisfiable*?

$\neg(B \lor D \lor \neg(A \land C))$



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This method has $O(2^n)$ time complexity, where n is the number of propositional symbols