# Reinforcement learning

Marco Piastra

# Multi-Armed Bandit

[image from wikipedia]

A row of $N$ old-style slot machines

- Basic definitions

  $N$ arms or *bandits*

  Each arm $a$ yields a random reward $r$ with probability distribution $P(r \mid a)$

  *For simplicity, only Bernoullian rewards (i.e. either* $0$ *or* $1$*) will be considered here*

  Each time $t$ in a sequence, the player (i.e. the agent) selects the arm $\pi(t)$

  In other words, $\pi$ is the *policy* adopted by the agent

- Problem

  Find a policy $\pi$ that maximizes the <u>*total reward*</u> over time

  The policy will include random choices i.e. it will be *stochastic*

# Multi-Armed Bandit: strategies

- Informed (i.e. *optimal*) strategy

  At all times, select the bandit with higher probability of reward:

  $$\pi^*(t) = \text{argmax}_a P(r = 1 \,|\, a)$$

  Clearly, this strategy is optimal but requires knowing all distributions $P(r \mid a)$

  With enough data (*e.g. from other players*), these distributions can be learnt

- Random strategy

  At all times, select a bandit $a$ at random, with *uniform probability*

  *How does the Random strategy compare with the optimal, informed strategy?*

# Multi-Armed Bandit: basic definitions

- *Actions, Rewards*

  $a \in \mathcal{A}$ *in this case* $a \in \{1, \ldots, N\}$

  $r \in \mathcal{R}$ *in this case* $r \in \{0, 1\}$

- *Probability distribution (unknown)*

  $P(R \,|\, A)$ *the probability of reward* $R$ *for action* $A$ *(i.e. two random variables)*

- *Policy*

  $\pi : \mathbb{N}^+ \to \mathcal{A}$ *at each time, defines which action will be taken, it may be* *<u>stochastic</u>*

- *Q-value*

  *The* *<u>expected</u>* *reward of action* $a$

  $$Q(a) := \mathbb{E}[R \,|\, A = a] = \sum_r r \, P(r \,|\, A = a)$$

- *Optimal Value*

  *Maximum* *<u>expected</u>* *reward*

  $$V^* := Q(a^*) = \max_{a \in \mathcal{A}} Q(a)$$

# Multi-Armed Bandit: evaluating strategies

- *Total Expected Regret*

  *How far from optimality a policy is, considering the total reward over $T$ trials*

  For just <u>one</u> sequence of $T$ trials, the *Total Regret with <u>expected</u> rewards* is

  $$L(T) := TV^* - \sum_{t=1}^{T} Q(\pi(t))$$

  action taken at step $t$

  In a more general definition, the *Total <u>Expected</u> Regret* is

  $$\overline{L}(T) := TV^* - \sum_{a=1}^{N} \mathbb{E}[T_a(T)]Q(a) = \sum_{a=1}^{N} \mathbb{E}[T_a(T)]\Delta_a$$

  number of times action $a$ is taken in $T$ trials (i.e. *a random variable*)

  where

  $$\Delta_a := V^* - Q(a)$$

# Multi-Armed Bandit: evaluating strategies

- *Total Expected Regret*

$$\overline{L}(T) := TV^* - \sum_{a=1}^{N} \mathbb{E}[T_a(T)]Q(a) = \sum_{a=1}^{N} \mathbb{E}[T_a(T)]\Delta_a$$

number of times action $a$ is taken in $T$ trials (i.e. *a random variable*)

where

$$\Delta_a := V^* - Q(a)$$

With the optimal policy $\pi^*$ the total expected regret is $0$.

Whereas, with the *random policy* the total expected regret grows linearly over time:

$$\overline{L}(T) = \frac{T}{N} \sum_{a=1}^{N} \Delta_a$$

*…since, with a random strategy* $\mathbb{E}[T_a(T)] = \frac{T}{N}$

# Multi-Armed Bandit: *Online learning*

Adaptive policy: *exploration vs. exploitation*

***exploration***: make trials over the set of $N$ arms to improve on estimates $\hat{Q}(a)$

***exploitation***: make use of the current best estimates $\hat{Q}(a)$

- Greedy policy

Initialize all the estimates $\hat{Q}(a)$ at random

*Repeat*:

1) select the bandit with the current best estimated reward $a = \mathrm{argmax}_a \hat{Q}(a)$
2) update the current estimate about $a$ as

$$\hat{Q}(a) := \frac{\sum_{t=1}^{T_a} r_{a,t}}{T_a}$$

reward of arm *a* at trial *t*

Total number of times the arm *a* has been played

# Multi-Armed Bandit: *Online learning*

Adaptive policy: *exploration vs. exploitation*

***exploration***: make trials over the set of $N$ arms to improve on estimates $\hat{Q}(a)$

***exploitation***: make use of the current best estimates $\hat{Q}(a)$

- $\varepsilon$-greedy policy $(0 < \varepsilon < 1)$

  Initialize all the estimates $\hat{Q}(a)$ at random

  *Repeat*:

  1) with probability $(1 - \varepsilon)$ select the bandit $a = \text{argmax}_a \hat{Q}(a)$
     else (*i.e. with probability* $\varepsilon$) select one bandit at random
  2) update the current estimate about $a$

$$\hat{Q}(a) := \frac{\sum_{t=1}^{T_a} r_{a,t}}{T_a}$$

$r_{a,t}$: reward of arm $a$ at trial $t$

$T_a$: total number of times the arm $a$ has been played

# Multi-Armed Bandit: *Online learning*

Adaptive policy: *exploration vs. exploitation*

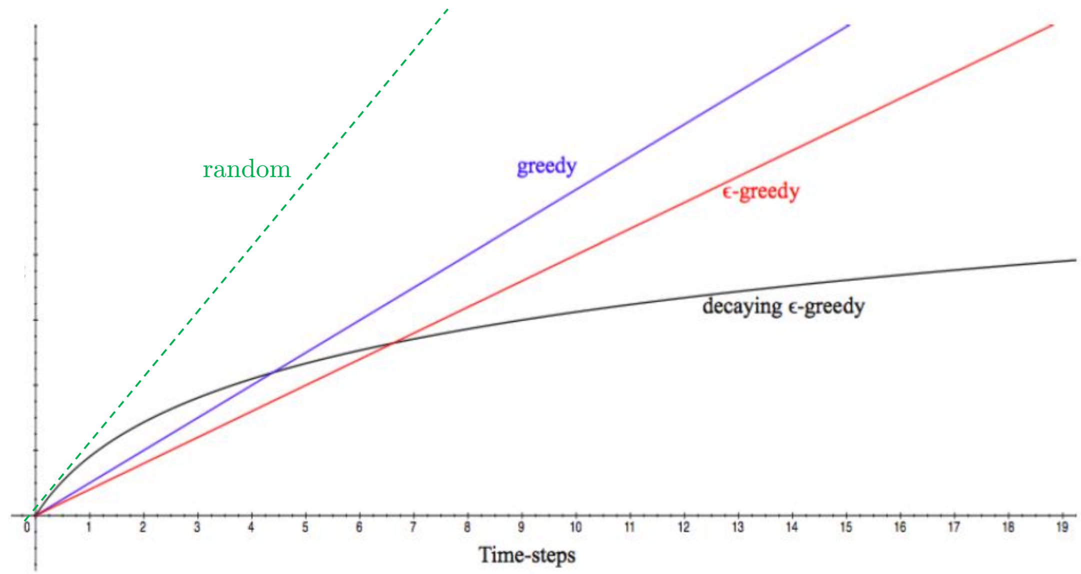
***exploration***: make trials over the set of $N$ arms to improve on estimates $\hat{Q}(a)$

***exploitation***: make use of the current best estimates $\hat{Q}(a)$

- Experimental comparison of different strategies (*Total Expected Regret*)

After a certain period of time, the *greedy* strategy stops exploring and exploits its estimates

whereas, the $\varepsilon$-greedy strategy keeps exploring and improving




Decaying $\varepsilon$-greedy strategy: $\varepsilon = \dfrac{\varepsilon_{initial}}{t}$

# Multi-Armed Bandit: evaluating strategies

- The two *greedy* strategies

  They are <u>*biased*</u>: they depend on the initial random estimates

  *Optimistic* variant: initially, set all $\hat{Q}(a) := 1$

  The average total regret grows <u>linearly</u>, in the long run

  In fact:

  - on the average, the *greedy* strategy will get stuck in a suboptimal choice
  - the $\varepsilon$-greedy strategy will continue to choose an arm at random (with probability $\varepsilon$)

  *Can we do any better?*

The decaying $\varepsilon$-greedy strategy does that…
Is there a minimum, i.e. a lower bound?

# Multi-Armed Bandit: Optimal *online learning*

- Lower bound theorem [Lai & Robbins 1985]

  Consider a generic, adaptive (i.e. learning) strategy for the multi-armed bandit problem with Bernoulli reward (i.e. $r \in \{0, 1\}$ )

$$\lim_{T \to \infty} \overline{L}(T) \geq \ln T \sum_{a | \Delta_a > 0} \frac{\Delta_a}{\mathrm{kl}(Q(a), V^*)} \qquad \Delta_a := V^* - Q(a)$$

  where

$$\mathrm{kl}(Q(a), V^*) := Q(a) \ln \frac{Q(a)}{V^*} + (1 - Q(a)) \ln \frac{(1 - Q(a))}{(1 - V^*)}$$

  the *Kullback-Leibler divergence*

  *In other words, we can achieve logarithmic growth for the total expected regret, but not better: on average, any adaptive strategy will choose suboptimal bandits a minimum number of times*

$$\lim_{T \to \infty} \mathbb{E}[T_a(T)] \geq \frac{\ln T}{\mathrm{kl}(Q(a), V^*)}$$

# Multi-Armed Bandit: UCB strategy

- Upper confidence bound (UCB) strategy [Auer, Cesa-Bianchi and Fisher 2002]

Initialize all the estimates of the expected reward $\hat{Q}(a) := 0$
Play each arm once (*to avoid zeroes in the formula below*)

*Repeat*:

1) select the bandit $a = \text{argmax}_k \left( \hat{Q}(a) + \sqrt{\frac{2 \ln T}{T_a}} \right)$

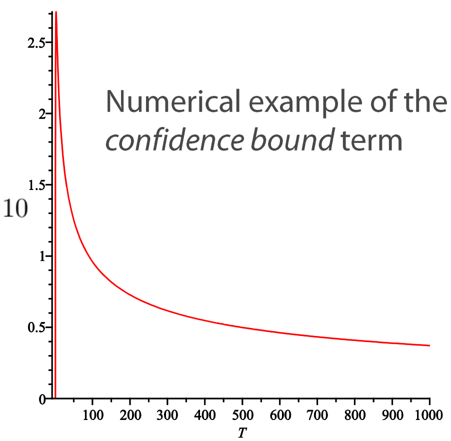
total number of trials

number of times the arm $k$ has been played

2) update the current estimate $\hat{Q}(a)$ as the *average* reward

Numerical example of the *confidence bound* term

$\sqrt{\frac{2 \ln T}{(T/N)}}, \; N = 10$




**Theorem**

With the UCB strategy, $\lim_{T \to \infty} \mathbb{E}[T_a(T)] \leq \frac{8 \ln T}{\Delta_a^2} + c$

i.e. a (small) constant

where it can be shown that $\frac{8}{\Delta_a^2} \geq \frac{1}{\text{kl}(Q(a), V^*)}$

(*i.e. there is a reasonably small gap between the two bounds – near optimality*)

# Multi-Armed Bandit: Thompson Sampling

- Thompson Sampling strategy (*also 'Bayesian Bandit'*) [Thompson, 1933]

  Initialize all the expected reward $\hat{Q}(a) :\sim \mathrm{Beta}(x; 1, 1)$

  i.e. assume this as a random variable with this distribution

  *Repeat*:

  1) <u>sample</u> each of the $N$ distributions to obtain an estimate $\hat{Q}(a)$
  2) select the bandit $a = \mathrm{argmax}_a \hat{Q}(a)$
  3) update the *posterior* distribution

     $\hat{Q}(a) :\sim \mathrm{Beta}(x; R_a + 1,\ T_a - R_a + 1)$

     total number of times the arm has been played

     total (*Bernoulli*) reward from this arm (*i.e. number of wins*)

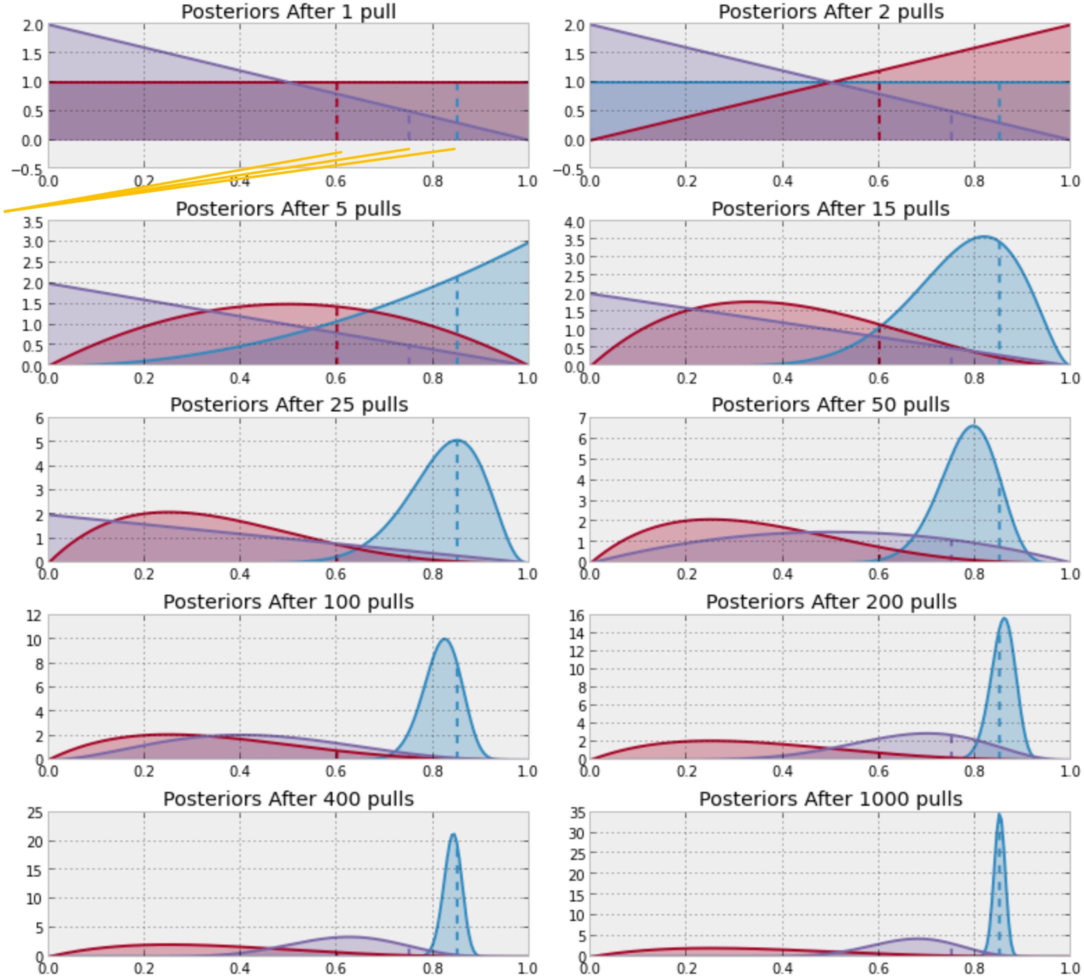
  **Theorem** [Kaufmann et al., 2012]

  The Thompson Sampling strategy has essentially the same theoretical bounds of the UCB strategy

- Thompson Sampling strategy (*also 'Bayesian Bandit'*) [Thompson, 1933]

*Example run with 3 arms: trace of the posterior probabilities for each* $\hat{Q}(a)$

*ground truth:* $Q(a)$

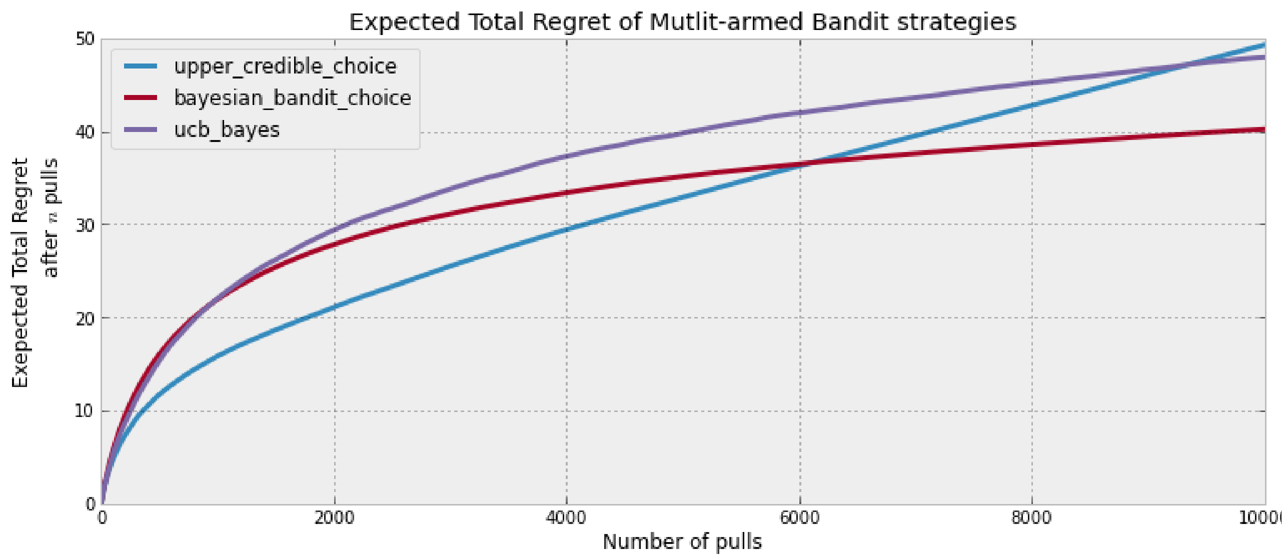



[image from: http://camdp.com/blogs/multi-armed-bandits]

# Multi-Armed Bandit: Thompson Sampling

- Thompson Sampling strategy (*also 'Bayesian Bandit'*) [Thompson, 1933]

  *In practical experiments, this strategy shows better performances in the long run*
  [Chapelle & Li, 2011]



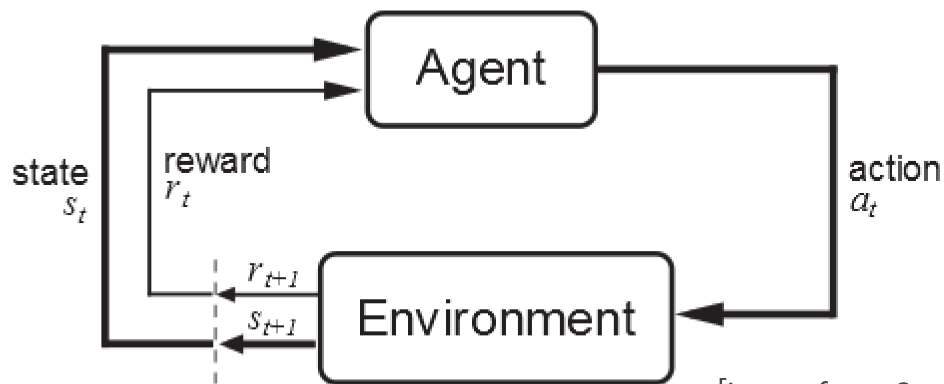

*Actually, Thompson Sampling is a preferred strategy at Google Inc.*
*(see https://support.google.com/analytics/answer/2846882?hl=en)*

[image from: http://camdp.com/blogs/multi-armed-bandits]

# Agent/Environment Interactions

*With multi-armed bandits, the context never changes*
*in the sense that the optimal choice does* ***not*** *depend on the current state*

What if the actions of the agent change the state of its interaction with the environment?




[image from: Sutton, Barto, *Reinforcement Learning.* 1998]

*Examples:*

- $a_t$ could be a *move in a game*, whereby the agent changes the state of the game
- $a_t$ could be a *movement*, whereby the agent changes its position in the environment

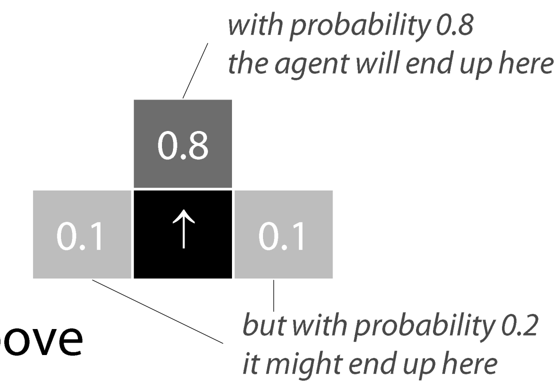
The agent could be wanting to learn an *optimal strategy* towards a given goal…

# An example: *gridworld*

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | -0.02 | -0.02 | -0.02 | 1 |
| 2 | -0.02 | | -0.02 | -1 |
| 3 | -0.02 | -0.02 | -0.02 | -0.02 |

The *state* of the agent is the position on the grid: e.g. (1,1), (3,4), (2,3)

At each time step, the agent can *move* one box in the directions ←↑↓→




*The effect of each move is somewhat stochastic, however*: for example, a move ↑ has a slight probability of producing a different (*and perhaps unwanted*) effect

Entering each state yields the *reward* shown in each box above

There are two *absorbing states*: entering either the green or the red box means exiting the *gridworld* and completing the game

- What is the best (*i.e. maximally rewarding*) movement policy?

# Markov Decision Process (MDP)

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | -0.02 | -0.02 | -0.02 | 1 |
| 2 | -0.02 | | -0.02 | -1 |
| 3 | -0.02 | -0.02 | -0.02 | -0.02 |

*Formalization and abstraction of the gridworld example*

**Markov Decision Process:** $< \mathcal{S}, \mathcal{A}, r, P, \gamma >$

A set of *states* : $\mathcal{S} = \{s_1, s_2, \dots\}$

A set of *actions* : $\mathcal{A} = \{a_1, a_2, \dots\}$

A *reward function* : $r : \mathcal{S} \to \mathbb{R}$

A *transition probability distribution* : $P(S_{t+1} \mid S_t, A_t)$ (also called a *model*)

***Markov property***: the transition probability depends only on the previous state and action

$$P(S_{t+1} \mid S_t, A_t) = P(S_{t+1} \mid S_t, A_t, S_{t-1}, A_{t-1}, S_{t-2}, A_{t-2}, \dots)$$

A *discount factor* : $0 \leq \gamma < 1$

# Markov Decision Process (MDP): policies and values

The agent is supposed to adopt a *deterministic* policy : $\pi : \mathcal{S} \to \mathcal{A}$

In other words, the agent always chooses its *action* depending on the *state* alone

Given a policy $\pi$, the ***state value function*** is defined, for each state $s$ as:

$$V^\pi(s) := \mathbb{E}[r(S_t) + \gamma r(S_{t+1}) + \gamma^2 r(S_{t+2}) + \ldots \mid \pi, S_t = s]$$

Note the role of the *discount factor*: a value $\gamma < 1$ means that that future rewards could be weighted less (by the agent) than immediate ones

Note also that all states $S_t$ must be described by *random variables* :
i.e. the policy is deterministic but the state transition is not

Note also that when the reward is *bounded*, i.e. $r(S) \leq r_{\max}$

$$\sum_{t=0}^{\infty} \gamma^t \, r(S_t) \;\leq\; r_{\max} \sum_{t=0}^{\infty} \gamma^t \;=\; r_{\max} \frac{1}{1-\gamma}$$

for $\gamma < 1$ this is the *geometric series*

# Markov Decision Process (MDP): policies and values

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Note also that all states $S_t$ must be described by *random variables* :
i.e. the policy is deterministic but the state transition is not

In the *gridworld* example:

- The set of states is finite
- The set of actions is finite
- For every policy, each entire story is finite

  *Sooner or later the agent will fall into one of the absorbing states*

# Bellman equations

By working on the definition of value function:

$$\begin{aligned} V^\pi(s) &:= \mathbb{E}[r(S_t) + \gamma r(S_{t+1}) + \gamma^2 r(S_{t+2}) + \ldots \mid \pi, S_t = s] \\ &= \mathbb{E}[r(S_t) + \gamma(r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots) \mid \pi, S_t = s] \\ &= r(s) + \gamma \mathbb{E}[r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots \mid \pi, S_t = s] \\ &= r(s) + \gamma \textstyle\sum_{s'} P(s' \mid s, \pi(s)) \cdot \mathbb{E}[r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots \mid \pi, S_{t+1} = s'] \\ &= r(s) + \gamma \textstyle\sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^\pi(S_{t+1}) \end{aligned}$$

This means that in a Markov Decision Process:

$$V^\pi(s) = r(s) + \gamma \textstyle\sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^\pi(S_{t+1})$$

This is true for any *state*, so there is one such equation for each of those

*If the set of states is finite, there are exactly $|S|$ (linear) Bellman equations for $|S|$ variables: in general, for any deterministic policy, $V^\pi$ can be computed analytically*

# Optimal policy – Optimal value function

- Basic definitions

$$V^*(s) := \max_\pi V^\pi(s), \ \forall s \in S$$
$$\pi^*(s) := \mathrm{argmax}_\pi V^\pi(s), \ \forall s \in S$$

**Property**: for every MDP, there exists such an optimal deterministic policy (*possibly non-unique*)

With Bellman Equations:

$$\max_\pi V^\pi(s) = r(s) + \gamma \max_\pi \left( \sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^\pi(S_{t+1}) \right)$$
$$V^*(s) = r(s) + \gamma \max_\pi \left( \sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^*(S_{t+1}) \right)$$
$$= r(s) + \gamma \max_a \left( \sum_{S_{t+1}} P(S_{t+1} \mid s, a) \cdot V^*(S_{t+1}) \right)$$

Therefore:

$$\pi^*(s) = \mathrm{argmax}_a \left( \sum_{S_{t+1}} P(S_{t+1} \mid s, a) V^*(S_{t+1}) \right)$$

*Computing* $V^*$ *directly from these equations is unfeasible, however*

*There are in fact* $|A|^{|S|}$ *possible strategies*

*However, once* $V^*$ *has been determined,* $\pi^*$ *can be determined as well*

# Optimal value function: value iteration

- Value iteration algorithm

  Initialize: $V(s) := r(s), \; \forall s \in S$

  *Repeat*:

  *Note that there is no policy: all actions must be explored*

  1) For every state, update: $V(s) := r(s) + \gamma \max_a \sum_{s'} P(s' \mid s, a) V(s')$

  **Theorem**: for every fair way (*i.e. giving an equal chance*) of visiting the states in $S$, this algorithm converges to $V^*$

# Value iteration and optimal policy

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | -0.02 | -0.02 | -0.02 | 1 |
| 2 | -0.02 | | -0.02 | -1 |
| 3 | -0.02 | -0.02 | -0.02 | -0.02 |

Initialize states
(e.g. using rewards as initial values)

Iterate and compute

$V^*$

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 0.86 | 0.90 | 0.93 | 1 |
| 2 | 0.82 | | 0.69 | -1 |
| 3 | 0.78 | 0.75 | 0.71 | 0.49 |

$V^*$

Define the optimal policy as:

$$\pi^*(s) := \text{argmax}_a \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, a) \cdot V^*(S_{t+1})\right)$$

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | → | → | → | 1 |
| 2 | ↑ | | ↑ | -1 |
| 3 | ↑ | ← | ← | ← |

$\pi^*$

# Optimal policy: policy iteration

- Policy iteration algorithm

Initialize $\pi(s), \forall s \in S$ at random
*Repeat*:

*This step is computationally expensive: either solve the equations or use value iteration (with fixed policy $\pi$)*

1) For each state, compute: $V(s) := V^{\pi}(s)$
2) For each state, define: $\pi(s) := \text{argmax}_a \sum_{s'} P(s' \mid s, a) V(s')$

**Theorem**: for every fair way (*i.e. giving an equal chance*) of visiting the states in $S$, this algorithm converges to $\pi^*$

*As with the value iteration algorithm, this algorithm uses partial estimates to compute new estimates.*
*It is also greedy, in the sense that it exploits its current estimate* $V^{\pi}(s)$

*Policy iteration* converges with very few number of iterations, but every iteration takes much longer time than that of *value iteration*

*The tradeoff with value iteration is the action space:*
*when action space is large and state space is small, policy iteration could be better*

# *Offline* vs. *Online* learning

- *Value iteration* and *policy iteration* are offline algorithms

  The <u>*model*</u> , i.e. the Markov Decision Process is known

  What needs to be learn is the optimal policy $\pi^*$

  In the algorithms, *visiting states* just means considering: there is no agent actually playing the game.

- Different conditions: *learning by doing …*

  Suppose the <u>*model*</u> (i.e. the MDP) is NOT known, or perhaps known only in part

  *Then the agent must learn by doing…*

# Action value function

*An analogous of the value function* $V^\pi$

Given a policy $\pi$, the ***action value function*** is defined, for each pair $(s, a)$ as:

$$
\begin{aligned}
Q^\pi(s,a) &:= \textstyle\sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot V^\pi(S_{t+1}) \\
&= \textstyle\sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot \mathbb{E}[r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots \mid \pi, S_{t+1}] \\
&= \textstyle\sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \mathbb{E}[\gamma r(S_{t+2}) + \ldots \mid \pi, S_{t+1}]] \\
&= \textstyle\sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \gamma Q^\pi(S_{t+1}, \pi(S_{t+1}))]
\end{aligned}
$$

In other words, $Q^\pi(s,a)$ is the expected value of the reward in $S_{t+1}$
by taking action $a$ in state $s$ and then following policy $\pi$ from that point on

Following a similar line of reasoning, the ***optimal*** *action value function* is

$$Q^*(s,a) = \textstyle\sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \gamma \max_{a'} Q^*(S_{t+1}, a')]$$

# Q-Learning

- Q-learning algorithm ($\varepsilon$-*greedy version*)

  Initialize $\hat{Q}(s,a)$ at random, put the agent is in a random state $s$
  *Repeat*:

  1) Select the action $\mathrm{argmax}_a \hat{Q}(s,a)$ with probability $(1-\varepsilon)$
     otherwise, select $a$ at random
  2) The agent is now in state $s'$ and has received the reward $r$
  3) Update $\hat{Q}(s,a)$ by

$$\Delta\hat{Q}(s,a) = \alpha[r + \gamma \max_{a'} \hat{Q}(s',a') - \hat{Q}(s,a)]$$

*Exponential Moving Average*
*(see later …)*

*Note that step 1) is closely similar to a **multi-armed bandit**:*
*in each state, the agent has to choose one among all actions in* $\mathcal{A}$
*and this will produce a random reward…*

# Q-Learning

- Q-learning algorithm

**Theorem** (Watkins, 1989): in the limit of that each action is played infinitely often and each state is visited infinitely often and $\alpha \to 0$ as experience progresses, then

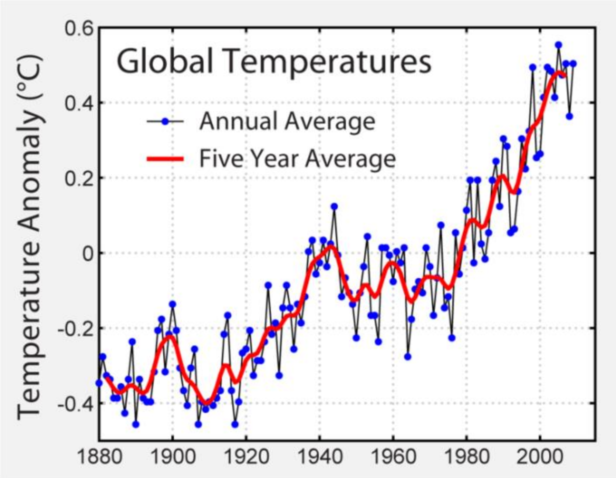
$$\hat{Q}(s,a) \to Q^*(s,a)$$

with probability $1$

*The Q-learning algorithm bypasses the MDP entirely, in the sense that the optimal strategy is learnt without learning the model* $P(S_{t+1} \mid S_t, A_t)$

# An aside: *moving averages*

*Following non-stationary phenomena*




[image from wikipedia]

- Average

  Definition: $$\overline{v}_T := \frac{1}{T}\sum_{k=1}^{T} v_k$$

  *Running implementation*:

$$\overline{v}_T = \frac{1}{T}(v_T + \sum_{k=1}^{T-1} v_k) = \frac{1}{T}(v_T + (T-1)\overline{v}_{T-1})$$

$$= \overline{v}_{T-1} + \frac{1}{T}(v_T - \overline{v}_{T-1}) = \frac{1}{T}\, v_T + (1 - \frac{1}{T})\,\overline{v}_{T-1}$$

*"the weight of newer observations diminishes with time"*

- Simple Moving Average (SMA)

$$\overline{v}_{T,n} := \frac{1}{n}\sum_{k=T-n}^{T} v_k$$

- Exponential Moving Average (EMA)

$$\overline{v}_{T,\alpha} := \alpha\, v_T + (1-\alpha)\,\overline{v}_{T-1,\alpha},\ \ \alpha \in [0,1]$$

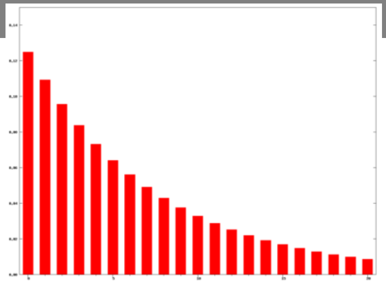
*"the weight of newer observations remains constant"*

# An aside: *moving averages*

- Exponential Moving Average (EMA)

$$\overline{v}_{T,\alpha} := \alpha\, v_T + (1-\alpha)\, \overline{v}_{T-1,\alpha},\ \ \alpha \in [0,1]$$

$(1-\alpha)^{\Delta_t}$
*"the weight of older observations diminishes with time"*



[image from wikipedia]

Expanding:

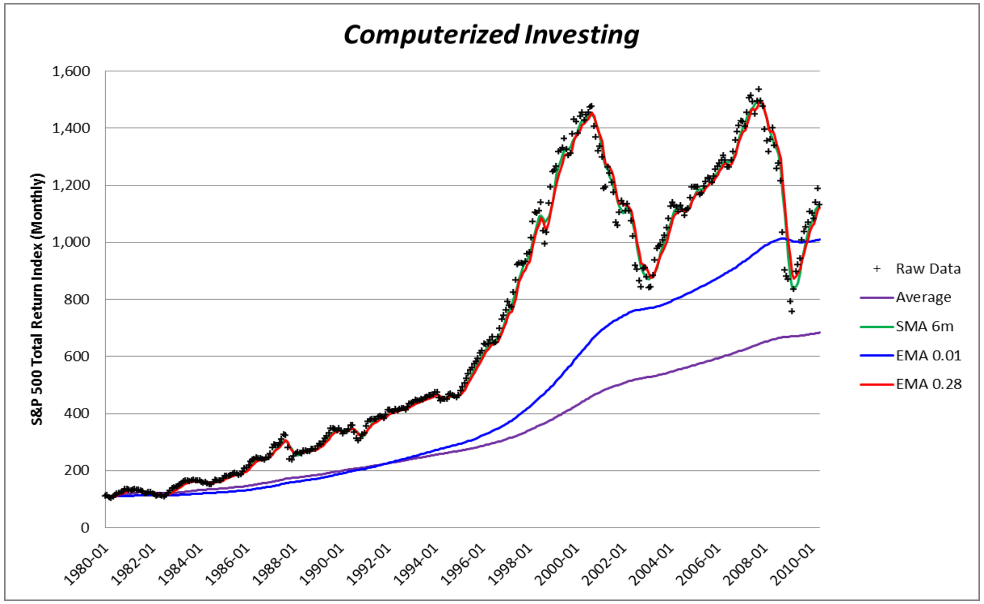
$$\begin{aligned}\overline{v}_{t,\alpha} &= \alpha\, v_t + (1-\alpha)\, \overline{v}_{t-1,\alpha} \\ &= \alpha\, v_t + (1-\alpha)(\alpha\, v_{t-1} + (1-\alpha)\overline{v}_{t-2,\alpha}) \\ &= \alpha\, v_t + (1-\alpha)(\alpha\, v_{t-1} + (1-\alpha)(\alpha\, v_{t-2} + (1-\alpha)\overline{v}_{t-3,\alpha})) \\ &= \alpha\,(v_t + (1-\alpha)\, v_{t-1} + (1-\alpha)^2\, v_{t-2}) + (1-\alpha)^3\, \overline{v}_{t-3,\alpha}\end{aligned}$$

*The weight of past contributions decays as*

$$(1-\alpha)^{\Delta_t}$$

*A SMA with $n$ previous values is approximately equal to an EMA with*

$$\alpha = \frac{2}{n+1}$$



# Q-Learning revisited

- Q-learning algorithm (*ε-greedy version*)

Initialize $\hat{Q}(s,a)$ at random, put the agent is in a random state $s$
*Repeat*:

1) Select the action $a = \text{argmax}_a \hat{Q}(s,a)$ with probability $(1-\varepsilon)$
otherwise, select $a$ at random
2) The agent is now in state $s'$ and has received the reward $r$
3) Update $\hat{Q}(s,a)$ by

$$\Delta\hat{Q}(s,a) = \alpha[r + \gamma \max_{a'} \hat{Q}(s',a') - \hat{Q}(s,a)]$$

*By rewriting step 3)*

$$\hat{Q}(s,a) = \hat{Q}(s,a) + \Delta\hat{Q}(s,a) = \hat{Q}(s,a) + \alpha[r + \gamma \max_{a'} \hat{Q}(s',a') - \hat{Q}(s,a)]$$

$$= \alpha[r + \gamma \max_{a'} \hat{Q}(s',a')] + (1-\alpha)\hat{Q}(s,a)$$

*Exponential Moving Average*

*compare with (see before)*:

$$Q^*(s,a) = \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \gamma \max_{a'} Q^*(S_{t+1},a')]$$

*Expectation*

# SARSA

- SARSA algorithm (*ε-greedy version*)

  Initialize $\hat{Q}(s, a)$ at random, put the agent is in a random state $s$
  *Repeat*:

  1) Select the action $a = \text{argmax}_a \hat{Q}(s, a)$ with probability $(1 - \varepsilon)$ otherwise, select $a$ at random
  2) The agent is now in state $s'$ and has received the reward $r$
  3) Select the action $a' = \text{argmax}_a \hat{Q}(s', a)$ with probability $(1 - \varepsilon)$ otherwise, select $a'$ at random
  4) Update $\hat{Q}(s, a)$ by

$$\Delta \hat{Q}(s, a) = \alpha[r + \gamma \hat{Q}(s', a') - \hat{Q}(s, a)]$$

*No more 'max' here*

Q-learning is a an *off-policy* algorithm: each update involves $\max_{a'} \hat{Q}(s', a')$
(i.e. *exploration* is not taken into account)
SARSA is a an *on-policy* algorithm: each update involves $\hat{Q}(s', a')$
(which involves the next policy action, *exploration* included)

# SARSA vs Q-Learning

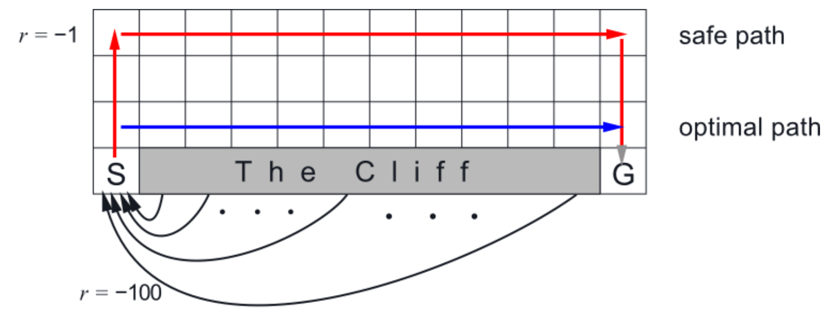
- Cliff World

  'S' is the start

  'G' is the goal

  Each white box has $r = -1$

  'The Cliff' region has $r = -100$ and entails going back to 'S'




- Experimental Results

  SARSA finds a sub-optimal but safer path since its learning takes into account the $\varepsilon$ risk of going off the cliff

  Q-learning finds the optimal path but, occasionally, it falls off the cliff during learning due to the $\varepsilon$-greedy strategy