Horn Clauses and SLD Resolution

Marco Piastra

Back to Propositional Logic

Horn Clauses (in L_P)

- Definition

A ***Horn Clause*** is a wff in CF
that contains at most one literal in positive form

- Three types of *Horn Clauses*:

Rule: two or more literals, one positive

Examples: $\{B, \neg D, \neg A, \neg C\}, \{A, \neg B\}$ (equivalent to: $(D \wedge A \wedge C) \rightarrow B,\ B \rightarrow A$)

Facts: just one positive literal

Examples: $\{B\}, \{A\}$

Goal: one or more literals, all negative

Examples: $\{\neg B\}, \{\neg A, \neg B\}$

More terminology:

Rules and facts are also called *definite clauses*

Goals are allo called *negative clauses*

Lost in Translation…

Many wffs can be translated into Horn clauses:

$(A \wedge B) \rightarrow C$

$\neg(A \wedge B) \vee C$ (rewriting $\rightarrow$)

$\neg A \vee \neg B \vee C$ (De Morgan - CF – it is a rule)

$A \rightarrow (B \wedge C)$

$\neg A \vee (B \wedge C)$ (rewriting $\rightarrow$)

$(\neg A \vee B) \wedge (\neg A \vee C)$ (distributing $\vee$)

$(\neg A \vee B), (\neg A \vee C)$ (CF – two rules)

$(A \vee B) \rightarrow C$

$\neg(A \vee B) \vee C$ (rewriting $\rightarrow$)

$(\neg A \wedge \neg B) \vee C$ (De Morgan)

$(\neg A \vee C) \wedge (\neg B \vee C)$ (distributing $\vee$)

$(\neg A \vee C), (\neg B \vee C)$ (CF – two rules)

But not all of them:

$(A \wedge \neg B) \rightarrow C$

$\neg(A \wedge \neg B) \vee C$ (rewriting $\rightarrow$)

$\neg A \vee B \vee C$ (De Morgan)

$A \rightarrow (B \vee C)$

$\neg A \vee B \vee C$ (rewriting $\rightarrow$)

SLD Resolution

Linear resolution with Selection function for Definite clauses

- Algorithm

  Starts from a set of *definite clauses* (also the *program*) + a *goal*

  1) At each step, the *selection function* identifies a *literal* in the *goal* (i.e. *subgoal*)
  2) All *definite clause* applicable to the *subgoal* are selected, in the given order
  3) The resolution rule is applied generating the *resolvent*

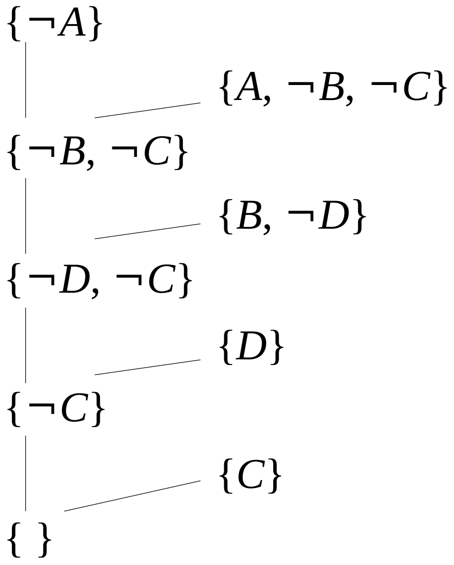
  Termination: either the empty clause { } is obtained or step 2) fails.

Example:

Selection function: *leftmost subgoal first*

Definite clauses: $\{C\}$, $\{D\}$, $\{B, \neg D\}$, $\{A, \neg B, \neg C\}$

Goal: $\{\neg A\}$



$\{\neg A\}$

$\{A, \neg B, \neg C\}$

$\{\neg B, \neg C\}$

$\{B, \neg D\}$

$\{\neg D, \neg C\}$

$\{D\}$

$\{\neg C\}$

$\{C\}$

$\{\ \}$

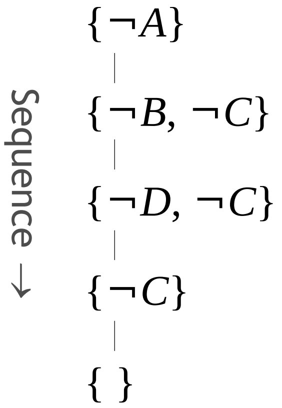
SLD trees

SLD derivations

Example: $\{C\}, \{D\}, \{B, \neg D\}, \{A, \neg B, \neg C\}$ goal $\{\neg A\}$

In this example each subgoal can be resolved in one mode only

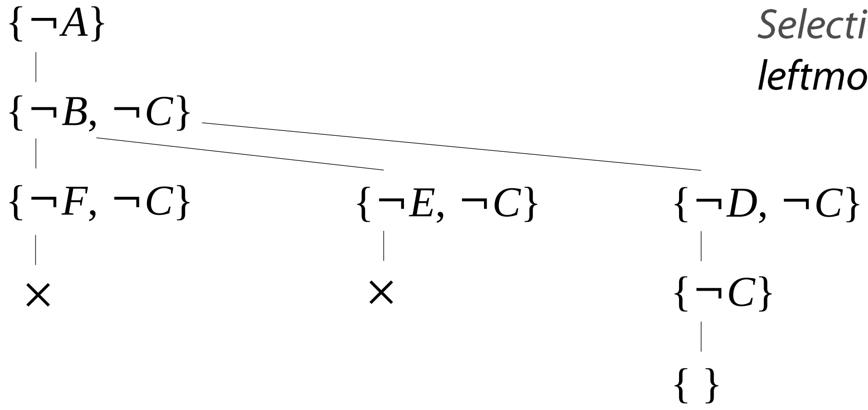
This is not true in general



- SLD trees (= trace of all SLD derivations from a goal)

Example: $\{C\}, \{D\}, \{B, \neg F\}, \{B, \neg E\}, \{B, \neg D\}, \{A, \neg B, \neg C\}$ goal $\{\neg A\}$

A few new rules have been added: there are now different possibilities



Selection function:
leftmost subgoal first

Each branch correspond to a possible resolution for a *subgoal*

SLD Resolution

- A resolution method <u>for Horn clauses</u> in L_P

    It always terminates

    It is *correct*: $\Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi$

    It is *complete*: $\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$

- Computationally efficient

  It has polynomial time complexity (w.r.t the # of propositional symbols occurring in Γ and φ)

SLD resolution
in First-Order Logic

Horn Clauses in L_{FO}

The definition is very similar to the propositional case

- Horn Clauses (of the skolemization of a set *sentences*)

  Each clause contains at most one literal in positive form

  Facts, rules and goals

  Fact: a clause with just an individual *atom*

  $\{Greek(socrates)\}, \{Pyramid(x)\}, \{Sister(sally, motherOf(paul))\}$

  Rule: a clause with at least two literals, exactly one in positive form

  $\{Human(x), \neg Greek(x)\},$

  $\forall x\ (Greek(x) \rightarrow Human(x))$

  $\{\neg Female(x), \neg Parent(k(x),x), \neg Parent(k(y),y), Sister(x,y)\}$

  $\forall x \forall y\ ((Female(x) \wedge \exists z\ (Parent(z,x) \wedge Parent(z,y))) \rightarrow Sister(x,y))$

  $\{\neg Above(x,y), On(x,k(x))\}, \{\neg Above(x,y), On(j(y),y)\}$

  $\forall x \forall y\ (Above(x,y) \rightarrow (\exists z\ On(x,z) \wedge \exists v\ On(v,y)))$

  Goal: a clause containing negative literals only

  $\{\neg Mortal(socrates)\}$

  $\{\neg Sister(sally,x), \neg Sister(x,paul)\}$

  Negation of $\exists x\ (Sister(sally,x) \wedge Sister(x,paul))$

SLD Resolution in L_{FO}

- Input: a program Π and a goal ϕ

Program Π (i.e. a set of *definite clauses:* rules + facts) in some predefined linear order:

$\gamma_1, \gamma_2, \dots, \gamma_n$ (each γ_i is a *definite clause*)

Goal ϕ (i.e. a conjunction of facts in negated form), which becomes the *current goal* ψ

Note: the *selection function* for the *current goal* and *subgoal* will be discussed in the next slide

Procedure:

1) Select a negative literal $\neg\alpha$ (i.e. the *subgoal*) in the *current goal* ψ
2) Scan the program (in the predefined order) to identify a clause candidate literal γ_i
3) Try unifying $\neg\alpha$ and $std(\gamma_i)$ (i.e. apply the standardization of variables to α')
4) If there is a *unifier* σ of $\neg\alpha$ and $std(\gamma_i)$, replace the current goal with the *resolvent* of $std(\gamma_i)[\sigma]$ and $\psi[\sigma]$ (i.e. first apply σ to both $std(\gamma_i)$ and ψ and then generate the resolvent)
5) Then, if the *resolvent* is the empty clause, terminate with <u>success</u>, otherwise select a new *current goal* and resume from step 1)
6) Else, if the unification fails , scan the program and select a new candidate literal γ_i and resume from step 3)
7) Else, if there are no further clauses in the program, select a new *current goal* and resume from step 1)
8) If all the goals in the tree have been fully explored, terminate with <u>failure</u>

SLD Resolution in L_{FO}

- *Two alternative selection functions:*

  Depth-first (which is the most common…)

  - Always select the *most recent goal*, i.e. the *resolvent* which has been generated last, as the *current goal* ϕ
  - Then, in the current goal ϕ, select the leftmost *subgoal* $\neg\alpha$ not selected yet
  - When the current goal ϕ is fully explored and no new *resolvent* has been generated, select the next *most recent* goal in the tree (*backtracking*)

  Breadth-first

  - Always select the *least recent* goal as the *current goal* ϕ
  - Then, in the current goal ϕ, select the leftmost *subgoal* $\neg\alpha$ not selected yet
  - When the current goal ϕ is fully explored select the next *least recent* goal in the tree

  Comparison

  Breadth-first is a *fair* selection function, in the sense that every possible resolution will be eventually attempted (i.e. 'it leaves nothing behind').

  Depth-first tends to save memory and be more efficient, but it is NOT *fair* (more to follow)

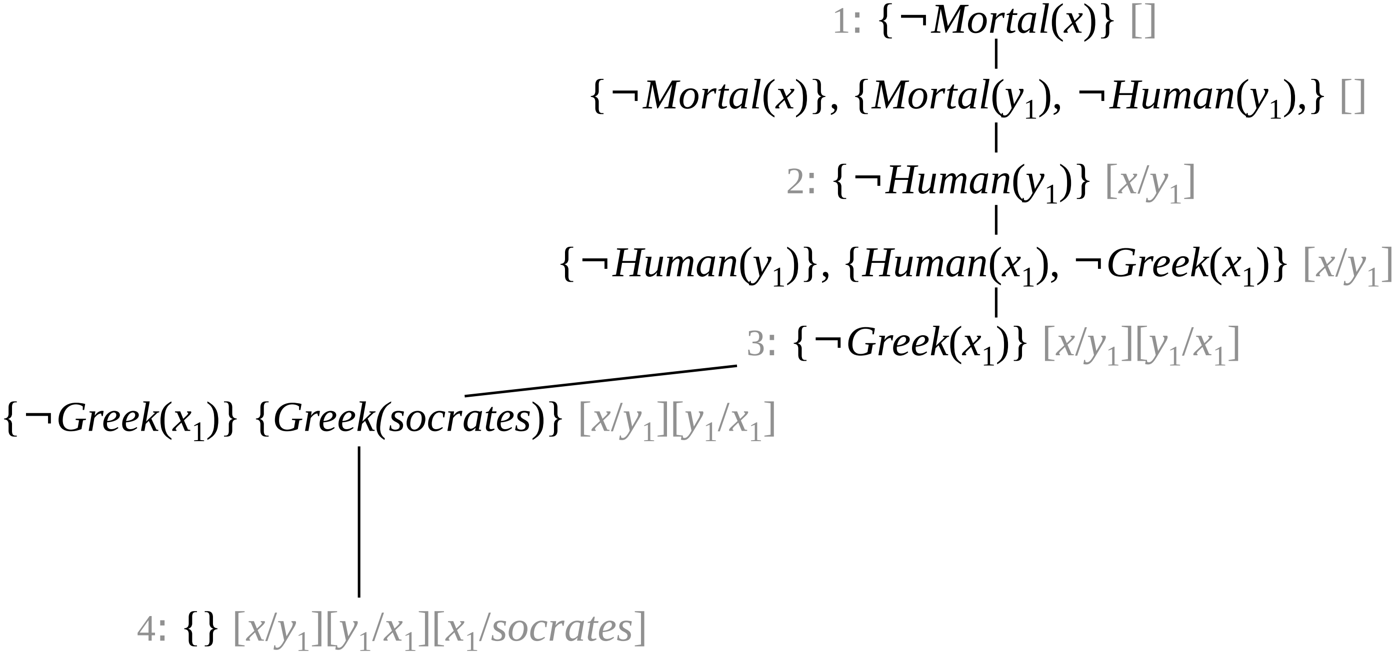
SLD Trees

- Example (depth-first selection function):

$$\Pi \equiv \{\{Human(x), \neg Greek(x)\}, \{Mortal(y), \neg Human(y)\},$$
$$\{Greek(socrates)\}, \{Greek(plato)\}, \{Greek(aristotle)\}\}$$

$$goal \equiv \{\neg Mortal(x)\}$$

"Is there anyone who is both human and mortal?"



1: $\{\neg Mortal(x)\}$ $[]$

$\{\neg Mortal(x)\}, \{Mortal(y_1), \neg Human(y_1),\}$ $[]$

2: $\{\neg Human(y_1)\}$ $[x/y_1]$

$\{\neg Human(y_1)\}, \{Human(x_1), \neg Greek(x_1)\}$ $[x/y_1]$

3: $\{\neg Greek(x_1)\}$ $[x/y_1][y_1/x_1]$

$\{\neg Greek(x_1)\}$ $\{Greek(socrates)\}$ $[x/y_1][y_1/x_1]$

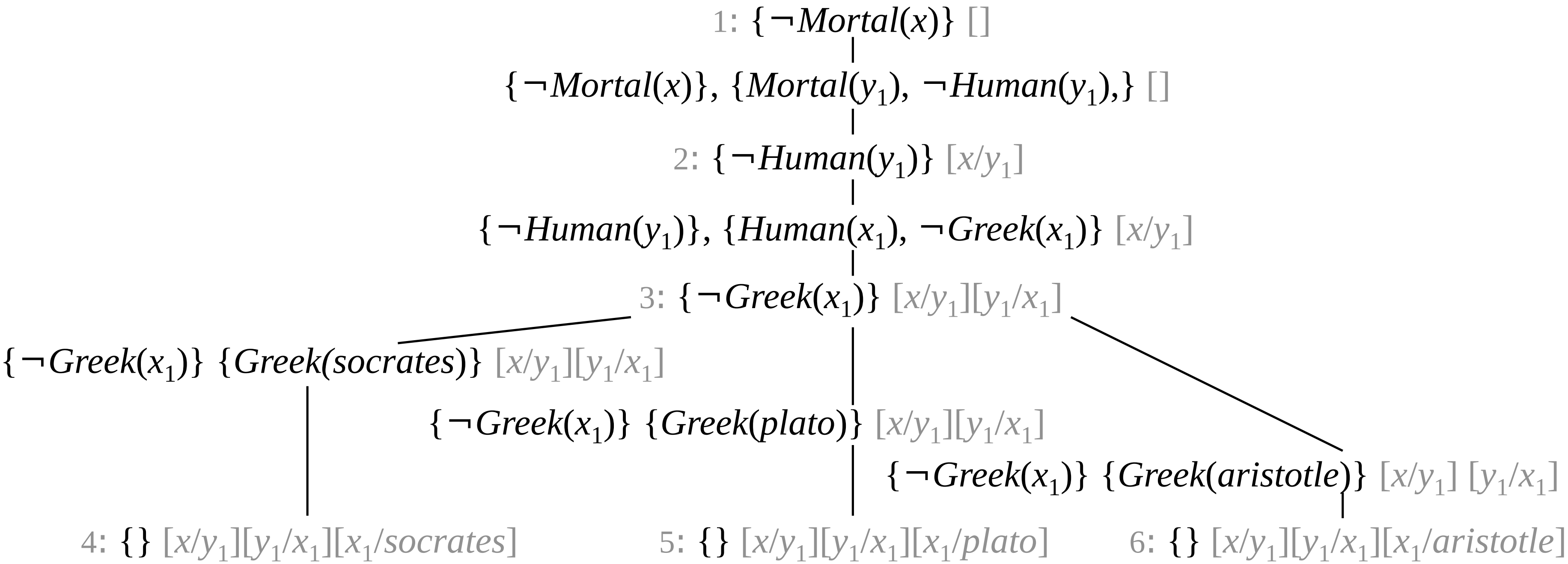
4: $\{\}$ $[x/y_1][y_1/x_1][x_1/socrates]$

SLD Trees

- Example (depth-first selection function, forcing full exploration of SLD tree):

$$\Pi \equiv \{\{Human(x), \neg Greek(x)\}, \{Mortal(y), \neg Human(y)\},$$
$$\{Greek(socrates)\}, \{Greek(plato)\}, \{Greek(aristotle)\}\}$$
$$goal \equiv \{\neg Mortal(x)\}$$

"Is there anyone who is both human and mortal?"



1: $\{\neg Mortal(x)\}$ $[]$

$\{\neg Mortal(x)\}, \{Mortal(y_1), \neg Human(y_1),\}$ $[]$

2: $\{\neg Human(y_1)\}$ $[x/y_1]$

$\{\neg Human(y_1)\}, \{Human(x_1), \neg Greek(x_1)\}$ $[x/y_1]$

3: $\{\neg Greek(x_1)\}$ $[x/y_1][y_1/x_1]$

$\{\neg Greek(x_1)\}$ $\{Greek(socrates)\}$ $[x/y_1][y_1/x_1]$

$\{\neg Greek(x_1)\}$ $\{Greek(plato)\}$ $[x/y_1][y_1/x_1]$

$\{\neg Greek(x_1)\}$ $\{Greek(aristotle)\}$ $[x/y_1]$ $[y_1/x_1]$

4: $\{\}$ $[x/y_1][y_1/x_1][x_1/socrates]$

5: $\{\}$ $[x/y_1][y_1/x_1][x_1/plato]$

6: $\{\}$ $[x/y_1][y_1/x_1][x_1/aristotle]$

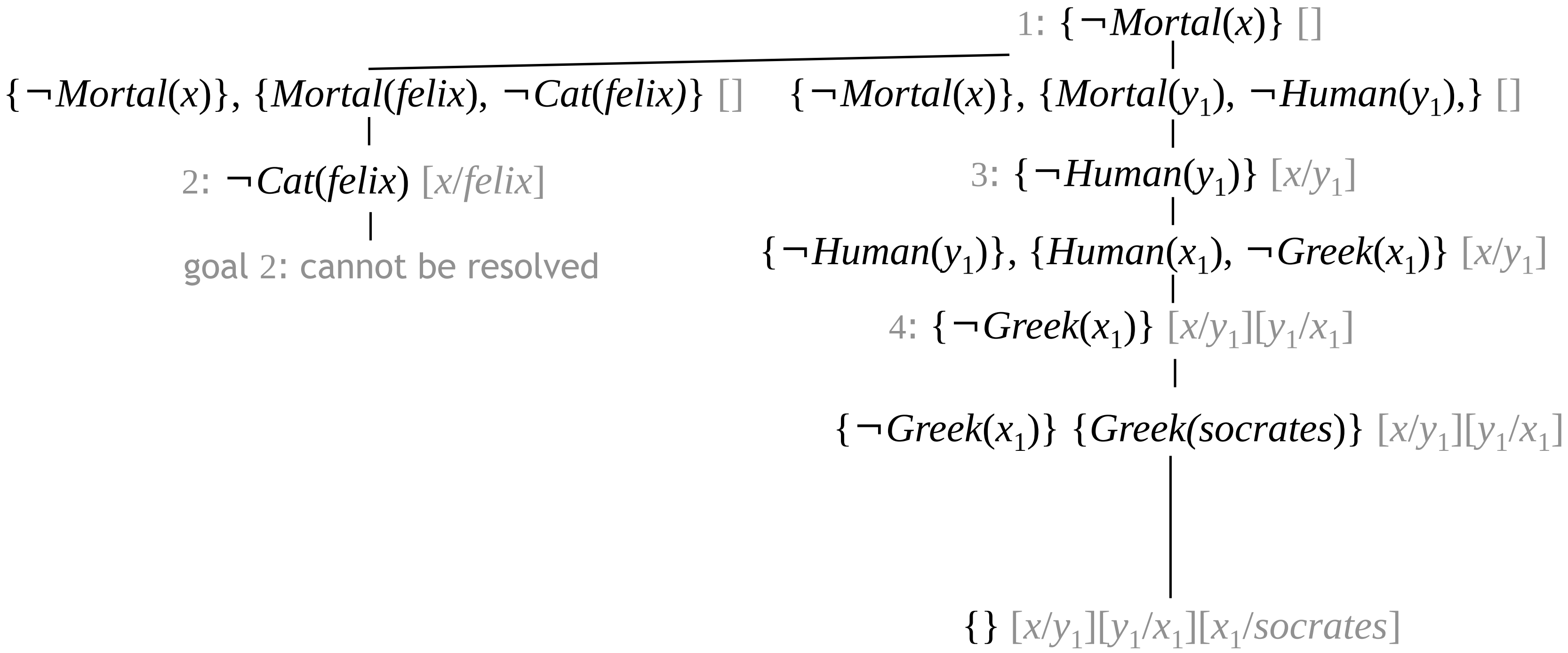
SLD Trees

- Another example (depth-first selection function):

$\Pi \equiv \{\{Mortal(felix), \neg Cat(felix)\}, \{Human(x), \neg Greek(x)\}, \{Mortal(y), \neg Human(y)\},$
$\{Greek(socrates)\}, \{Greek(plato)\}, \{Greek(aristotle)\}\}$

$goal \equiv \{\neg Mortal(x)\}$

"Is there anyone who is both human and mortal?"



1: $\{\neg Mortal(x)\}$ $[]$

$\{\neg Mortal(x)\}, \{Mortal(felix), \neg Cat(felix)\}$ $[]$

2: $\neg Cat(felix)$ $[x/felix]$

goal 2: cannot be resolved

$\{\neg Mortal(x)\}, \{Mortal(y_1), \neg Human(y_1),\}$ $[]$

3: $\{\neg Human(y_1)\}$ $[x/y_1]$

$\{\neg Human(y_1)\}, \{Human(x_1), \neg Greek(x_1)\}$ $[x/y_1]$

4: $\{\neg Greek(x_1)\}$ $[x/y_1][y_1/x_1]$

$\{\neg Greek(x_1)\}$ $\{Greek(socrates)\}$ $[x/y_1][y_1/x_1]$

$\{\}$ $[x/y_1][y_1/x_1][x_1/socrates]$

*The discreet charme of functions

- Representing data structures: *lists of items* $[a, b, c, \ldots]$

Symbols in Σ

$cons/2$
it's a function that associates items (e.g. a) to a list (e.g. $[b, c]$)
$cons(a, cons(b, cons(c, nil)))$ represents the list $[a, b, c]$

$Append/3$
it's a predicate: each pair of lists x and y is associated to their *concatenation* z

nil
it's a constant, represents the *empty list*.

Axioms ($\mathbf{AL}$)

$$\forall x\ Append(nil, x, x)$$
$$\forall x\ \forall y\ \forall z\ (Append(x, y, z) \rightarrow \forall s\ Append(cons(s, x), y, cons(s, z)))$$

Examples of entailment

$\{\mathbf{AL} + \exists z\ Append(cons(a, nil), cons(b, cons(c, nil), z)\ \}$
$\models Append(cons(a, nil), cons(b, cons(c, nil)), cons(a, cons(b, cons(\mathrm{c}, \mathrm{nil}))))$

$\{\mathbf{AL} + \exists x\ \exists y\ Append(x, y, cons(a, cons(b, nil)))\}$
$\models Append(cons(a, nil), cons(b, nil), cons(a, cons(b, nil)))$
$\models Append(nil, cons(a, cons(b, nil)), cons(a, cons(b, nil)))$
$\models Append(cons(a, cons(b, nil)),nil, cons(a, cons(b, nil)))$

The world of lists

- Lists of items $[a, b, c, \ldots]$

  $cons/2$
  it's a function that associates items (e.g. a) to a list (e.g. $[b, c]$)
  $cons(a,cons(b,cons(c,nil)))$ is the list $[a, b, c]$

  $Append/3$
  it's a predicate: each pair of lists x and y is associated to their *concatenation* z

  nil
  it's a constant, the *empty list*.

Shorthand notation (Prolog):

$[] \Leftrightarrow nil$
$[a] \Leftrightarrow cons(a,nil)$
$[a,b] \Leftrightarrow cons(a,cons(b,nil))$
$[a|[b,c]] \Leftrightarrow cons(a,[b,c])$

Axioms (**AL**)

$\forall x\ Append(nil,x,x)$
$\forall x \forall y \forall z\ (Append(x,y,z) \rightarrow \forall s\ Append([s,x],y,[s,z]))$

The world of lists

Problem: $\forall x\ Append(nil, x, x) \models \exists y\ \forall x\ Append(nil, cons(y, x), cons(a, x))$

1: $\forall x\ Append(nil, x, x),\ \neg\exists y\ \forall x\ Append(nil, cons(y, x), cons(a, x))$ (refutation)

2: $\forall x\ Append(nil, x, x),\ \forall y\ \exists x\ \neg Append(nil, cons(y, x), cons(a, x))$ (prenex normal form)

3: $\{Append(nil, x, x)\},\ \{\neg Append(nil, cons(y, k(y)), cons(a, k(y)))\}$

($k/1$ is a Skolem function, clausal form)

(N.B. there is no *skolemization* in Prolog : the programmer does it)

The pair of **literals**

$Append(nil, x, x),\ \neg Append(nil, cons(y, k(y)), cons(a, k(y))))$

... contains the same predicate $Append/3$ but the arguments are **<u>different</u>**

There is however an MGU $\sigma = [x/cons(a, k(a)), y/a]$ that yields

$\{Append(nil, cons(a,k(a)), cons(a,k(a)))\},\ \{\neg Append(nil, cons(a, k(a)), cons(a, k(a)))\}$

From this, the resolvent is the empty clause.

The world of lists in Prolog

```
% Identical to built-in predicate append/3, although it uses "cons"
% as a defined predicate, thus allowing trace-ability.

append(cons(S,X),Y,cons(S,Z)) :- append(X,Y,Z).
append(nil,X,X).

% WARNING: express your queries with cons. Examples:
% ?- append(cons(a,nil), cons(b,cons(c, nil)),cons(a,cons(b,cons(c, nil)))).
% ?- append(X,Y,cons(a,cons(b,cons(c, nil)))).
```

Infinite SLD Trees (*fairness of SLD*)

- An example:

$$\Pi \equiv \{\{S(a,b)\}, \{S(b,c)\}, \{S(x,z), \neg S(x,y), \neg S(y,z)\}\}$$

$$\neg\phi \equiv \{\neg S(a,x)\}$$

goal: $\neg S(a,x)$ []

|

$\{\neg S(a,x)\}, \{S(a,b)\}$ []

|

$\{\}$ $[x/b]$

Easy…

Infinite SLD Trees (*fairness of SLD*)

- An example:

$\Pi \equiv \{\{S(a,b)\}, \{S(b,c)\}, \{S(x,z), \neg S(x,y), \neg S(y,z)\}\}$

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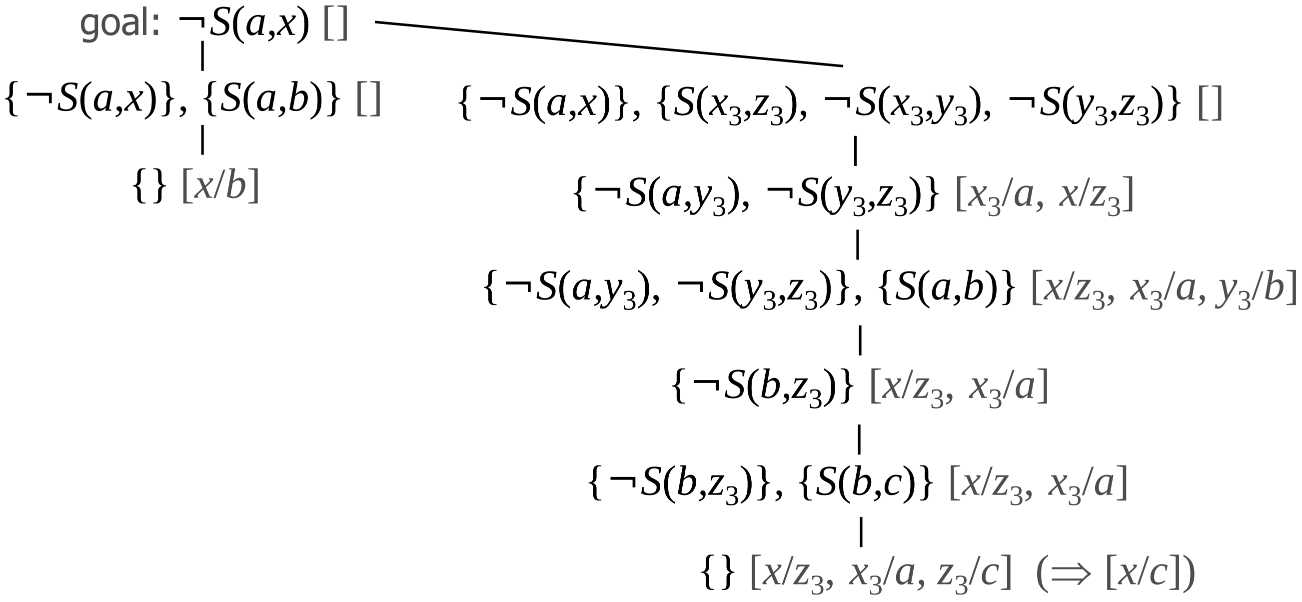


goal: $\neg S(a,x)$ []

$\{\neg S(a,x)\}, \{S(a,b)\}$ []

$\{\}$ $[x/b]$

$\{\neg S(a,x)\}, \{S(x_3,z_3), \neg S(x_3,y_3), \neg S(y_3,z_3)\}$ []

$\{\neg S(a,y_3), \neg S(y_3,z_3)\}$ $[x_3/a, x/z_3]$

$\{\neg S(a,y_3), \neg S(y_3,z_3)\}, \{S(a,b)\}$ $[x/z_3, x_3/a, y_3/b]$

$\{\neg S(b,z_3)\}$ $[x/z_3, x_3/a]$

$\{\neg S(b,z_3)\}, \{S(b,c)\}$ $[x/z_3, x_3/a]$

$\{\}$ $[x/z_3, x_3/a, z_3/c]$ $(\Rightarrow [x/c])$

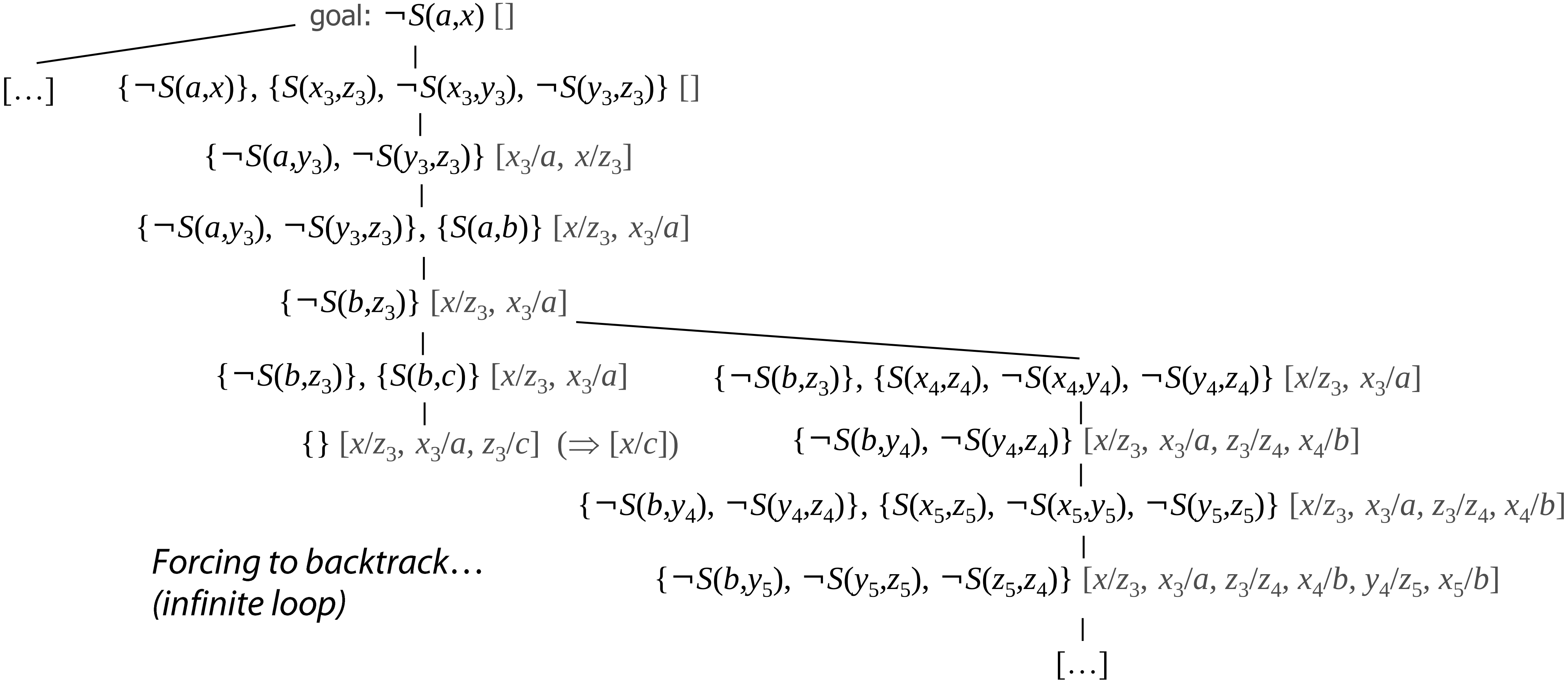
Forcing to backtrack…
(easy again)

Infinite SLD Trees (*fairness of SLD*)

- An example:

$\Pi \equiv \{\{S(a,b)\}, \{S(b,c)\}, \{S(x,z), \neg S(x,y), \neg S(y,z)\}\}$

$\neg\phi \equiv \{\neg S(a,x)\}$



goal: $\neg S(a,x)$ []

[…]

$\{\neg S(a,x)\}, \{S(x_3,z_3), \neg S(x_3,y_3), \neg S(y_3,z_3)\}$ []

$\{\neg S(a,y_3), \neg S(y_3,z_3)\}$ $[x_3/a,\ x/z_3]$

$\{\neg S(a,y_3), \neg S(y_3,z_3)\}, \{S(a,b)\}$ $[x/z_3,\ x_3/a]$

$\{\neg S(b,z_3)\}$ $[x/z_3,\ x_3/a]$

$\{\neg S(b,z_3)\}, \{S(b,c)\}$ $[x/z_3,\ x_3/a]$

$\{\}$ $[x/z_3,\ x_3/a,\ z_3/c]$ $(\Rightarrow [x/c])$

$\{\neg S(b,z_3)\}, \{S(x_4,z_4), \neg S(x_4,y_4), \neg S(y_4,z_4)\}$ $[x/z_3,\ x_3/a]$

$\{\neg S(b,y_4), \neg S(y_4,z_4)\}$ $[x/z_3,\ x_3/a,\ z_3/z_4,\ x_4/b]$

$\{\neg S(b,y_4), \neg S(y_4,z_4)\}, \{S(x_5,z_5), \neg S(x_5,y_5), \neg S(y_5,z_5)\}$ $[x/z_3,\ x_3/a,\ z_3/z_4,\ x_4/b]$

$\{\neg S(b,y_5), \neg S(y_5,z_5), \neg S(z_5,z_4)\}$ $[x/z_3,\ x_3/a,\ z_3/z_4,\ x_4/b,\ y_4/z_5,\ x_5/b]$

[…]

Forcing to backtrack…
(infinite loop)

Infinite SLD Trees (*fairness of SLD*)

- A second example:

$$\Pi \equiv \{\{S(x,z), \neg S(x,y), \neg S(y,z)\},\{S(a,b)\}, \{S(b,c)\}\}$$

$$\neg\phi \equiv \{\neg S(a,x)\}$$

Notice the change in clause ordering…..

goal: $\neg S(a,x)$ []

|

$\{\neg S(a,x)\}, \{S(x_1,z_1), \neg S(x_1,y_1), \neg S(y_1,z_1)\}$ []

|

$\{\neg S(a,y_1), \neg S(y_1,z_1)\}$ $[x_1/a, x/z_1]$

|

$\{\neg S(a,y_1), \neg S(y_1,z_1)\}, \{S(x_2,z_2), \neg S(x_2,y_2), \neg S(y_2,z_2)\}$ $[x_1/a, x/z_1]$

|

$\{\neg S(z_2,z_1), \neg S(x_2,y_2), \neg S(y_2,z_2)\}$ $[x_1/a, x/z_1, x_2/a, y_1/z_2]$

|

[…]

The *infinite loop* occurs immediately …

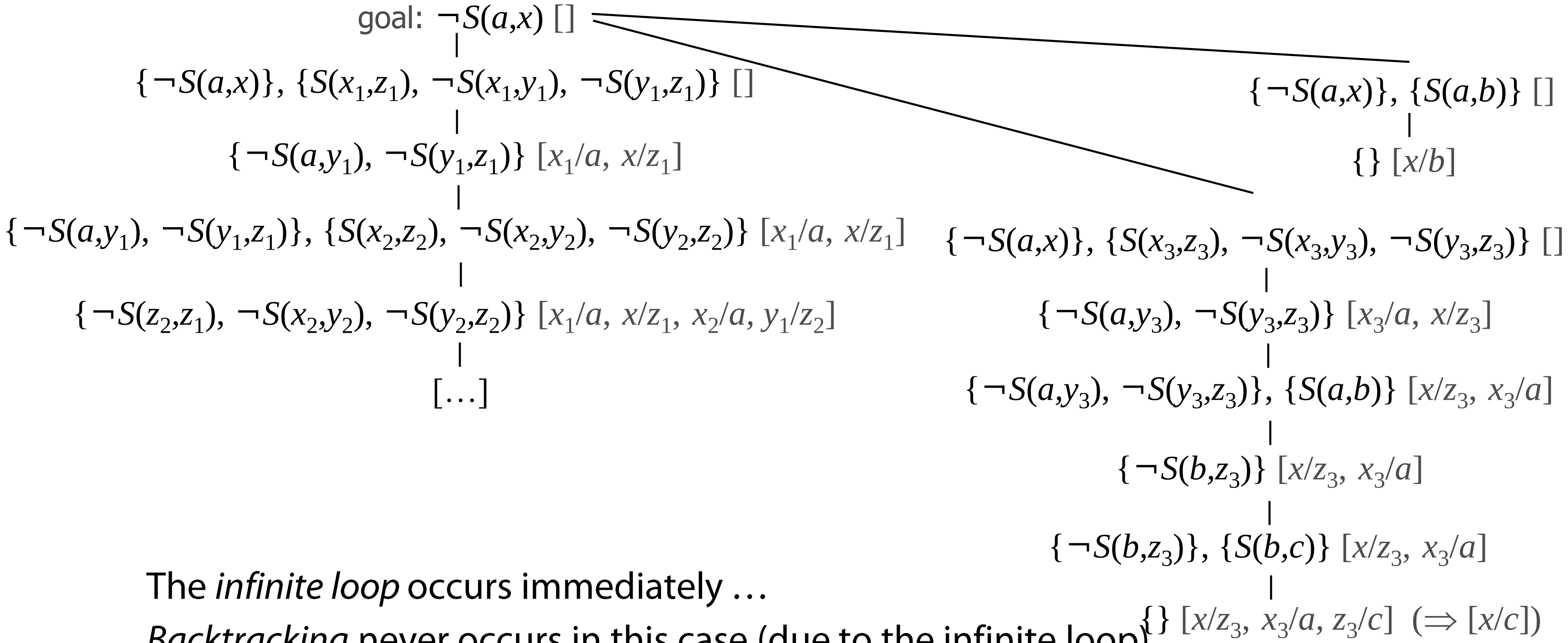
Infinite SLD Trees (*fairness of SLD*)

- A second example:

$$\Pi \equiv \{\{S(x,z), \neg S(x,y), \neg S(y,z)\},\{S(a,b)\}, \{S(b,c)\}\}$$

$$\neg\phi \equiv \{\neg S(a,x)\}$$

Notice the change in clause ordering…..



goal: $\neg S(a,x)$ []

$\{\neg S(a,x)\}, \{S(x_1,z_1), \neg S(x_1,y_1), \neg S(y_1,z_1)\}$ []

$\{\neg S(a,y_1), \neg S(y_1,z_1)\}$ $[x_1/a,\ x/z_1]$

$\{\neg S(a,y_1), \neg S(y_1,z_1)\}, \{S(x_2,z_2), \neg S(x_2,y_2), \neg S(y_2,z_2)\}$ $[x_1/a,\ x/z_1]$

$\{\neg S(z_2,z_1), \neg S(x_2,y_2), \neg S(y_2,z_2)\}$ $[x_1/a,\ x/z_1,\ x_2/a,\ y_1/z_2]$

[…]

$\{\neg S(a,x)\}, \{S(x_3,z_3), \neg S(x_3,y_3), \neg S(y_3,z_3)\}$ []

$\{\neg S(a,y_3), \neg S(y_3,z_3)\}$ $[x_3/a,\ x/z_3]$

$\{\neg S(a,y_3), \neg S(y_3,z_3)\}, \{S(a,b)\}$ $[x/z_3,\ x_3/a]$

$\{\neg S(b,z_3)\}$ $[x/z_3,\ x_3/a]$

$\{\neg S(b,z_3)\}, \{S(b,c)\}$ $[x/z_3,\ x_3/a]$

$\{\}$ $[x/z_3,\ x_3/a,\ z_3/c]$ $(\Rightarrow [x/c])$

$\{\neg S(a,x)\}, \{S(a,b)\}$ []

$\{\}$ $[x/b]$

The *infinite loop* occurs immediately …

Backtracking never occurs in this case (due to the infinite loop), yet, if it occurred it would have produced the two correct results

Infinite SLD Trees (*fairness of SLD*)

- Moral
  - In both previous examples the infinite loop (i.e. *divergence*) is unavoidable
  - Yet in the first one, the method first produces the right results and then diverges
  - So in the first case the result is *complete* (i.e. all entailed formulae are derived) while in the second case the method is not

A ***fair*** selection function is such that no possible resolution will be postponed indefinitely: that is, <u>any</u> possible resolution will be performed, eventually.

In the two previous examples, we used a *depth-first* exploration method of the SLD tree: which is <u>not</u> complete (in the above sense)

A *breadth-first* exploration method is ***fair*** hence it is complete (in the above sense)

In actual programming systems (e.g. Prolog) the depth-first is preferred for memory efficiency since the breadth-first method forces to keep (most of) the whole SLD tree in memory