

First-Order Logic

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Propositional possible worlds

Each possible world is a structure $\langle \{0,1\}, \mathbf{P}, \nu \rangle$

$\{0,1\}$ are the *truth values*

\mathbf{P} is the **signature** of the formal language: a set of propositional symbols

ν is a *function*: $\mathbf{P} \rightarrow \{0,1\}$ assigning truth values to the symbols in \mathbf{P}

Propositional symbols (*signature*)

Each symbol in \mathbf{P} stands for an actual *proposition* (in natural language)

In the simple convention, we use the symbols A, B, C, D, \dots

Caution: \mathbf{P} is not necessarily *finite*

Possible worlds

The class of structures contains all possible worlds:

$\langle \{0,1\}, \mathbf{P}, \nu \rangle$

$\langle \{0,1\}, \mathbf{P}, \nu' \rangle$

$\langle \{0,1\}, \mathbf{P}, \nu'' \rangle$

...

Each class of structure shares \mathbf{P} and $\{0,1\}$

The functions ν are different: the assignment of truth values varies, depending on the possible world

If \mathbf{P} is finite, there are only *finitely* many distinct possible worlds (actually $2^{|\mathbf{P}|}$)

An aside: tuples, relations and functions

▪ Tuple

Consider a generic set of objects \mathbf{U}

An example *set* of objects from \mathbf{U} is denoted as $\{u_1, u_2\}$, where $u_1, u_2 \in \mathbf{U}$

In a set, the order of elements is not relevant

An example of *tuple* of objects from \mathbf{U} is denoted as $\langle u_1, u_2 \rangle$, where $u_1, u_2 \in \mathbf{U}$

In a tuple, the order is relevant, i.e. $\langle u_1, u_2 \rangle \neq \langle u_2, u_1 \rangle$

▪ Cartesian product

The cartesian product $\mathbf{U} \times \mathbf{U} =: \mathbf{U}^2$ is the set of all tuples $\langle u_1, u_2 \rangle$, $u_1, u_2 \in \mathbf{U}$

Analogously, \mathbf{U}^3 is the set of all tuples $\langle u_1, u_2, u_3 \rangle$, $u_1, u_2, u_3 \in \mathbf{U}$

\mathbf{U}^4 is the set of all tuples $\langle u_1, u_2, u_3, u_4 \rangle$, $u_1, u_2, u_3, u_4 \in \mathbf{U}$ and so on ...

▪ Relation

arity is always an integer

A relation of *arity* n is a subset of \mathbf{U}^n

▪ Function

A function of type $\mathbf{U}^n \rightarrow \mathbf{U}$ is a relation of arity $n + 1$ such that each tuple is constructed by associating each tuple of \mathbf{U}^n with exactly one object from \mathbf{U}

First-order possible worlds

Possible worlds made of objects, functions and relations

Each possible world is a structure $\langle \mathbf{U}, \Sigma, \nu \rangle$

\mathbf{U} is a set of object, called **domain** (also *universe of discourse*)

Σ is a set of symbols, called **signature**

ν is a *function* that gives a *meaning* to the symbols in Σ with respect to \mathbf{U}

Signature Σ

- *individual constants*: a, b, c, d, \dots
- *function symbols (with arity)*: $f/n, g/p, h/q, \dots$
- *predicate symbols (with arity)*: $P/k, Q/l, R/m, \dots$

*Arity is an integer
that describes the expected number
of arguments*

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Term

A single *individual constant* is a **term**

If f/n is a *functional symbol* (with arity n) and t_1, \dots, t_n are **terms**, then $f(t_1, \dots, t_n)$ is a **term**

Atom

If P/n is a *predicate symbol* (with arity n) and t_1, \dots, t_n are **terms**, then $P(t_1, \dots, t_n)$ is an **atom** (i.e a first-order well-formed formula – wff)

First-order possible worlds

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Function ν (***interpretation***)

- ν assigns each *individual constant* to an *object* in \mathbf{U}
 $\nu(a) \in \mathbf{U}$ (a individual constant)
- ν assigns each *functional symbol* a *function* defined on \mathbf{U}
 $\nu(f/n) : \mathbf{U}^n \rightarrow \mathbf{U}$ (f/n functional symbol)
- ν assigns each *predicate symbol* a *relation* defined on \mathbf{U}
 $\nu(P/m) \subseteq \mathbf{U}^m$ (P/m predicate symbol)

First-order possible worlds

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If f/n is a *functional symbol* (with arity n) and t_1, \dots, t_n are **terms**, then $f(t_1, \dots, t_n)$ is a **term**

The semantics of a **term** $f(t_1, \dots, t_n)$ is

$$\nu(f/n) (\langle \nu(t_1), \dots, \nu(t_n) \rangle) \in \mathbf{U}$$

that is, the result of applying the function that ν associates to f/n to the tuple of objects in \mathbf{U} created from the semantics of t_1, \dots, t_n

It is yet an object in \mathbf{U}

First-order language (*without variables*)

■ Well-formed formulae (wff)

All symbols in the *signature* Σ (i.e. *constants, function and predicate symbols*)

Two (primary) **logical connectives**: \neg, \rightarrow

Three (derived) **logical connectives**: $\wedge, \vee, \leftrightarrow$

Parenthesis: $(,)$ (there are no *precedence rules* in this language)

The definition of *terms* and *atoms* (see before)

A set of syntactic rules

The set of all the **wff** of L_{FO} is denoted as $\text{wff}(L_{FO})$

φ is an atom $\Rightarrow \varphi \in \text{wff}(L_{FO})$

$\varphi \in \text{wff}(L_{FO}) \Rightarrow (\neg\varphi) \in \text{wff}(L_{FO})$

$\varphi, \psi \in \text{wff}(L_{FO}) \Rightarrow (\varphi \rightarrow \psi) \in \text{wff}(L_{FO})$

$\varphi, \psi \in \text{wff}(L_{FO}) \Rightarrow (\varphi \vee \psi) \in \text{wff}(L_{FO}), (\varphi \vee \psi) \Leftrightarrow ((\neg\varphi) \rightarrow \psi)$

$\varphi, \psi \in \text{wff}(L_{FO}) \Rightarrow (\varphi \wedge \psi) \in \text{wff}(L_{FO}), (\varphi \wedge \psi) \Leftrightarrow (\neg(\varphi \rightarrow (\neg\psi)))$

$\varphi, \psi \in \text{wff}(L_{FO}) \Rightarrow (\varphi \leftrightarrow \psi) \in \text{wff}(L_{FO}), (\varphi \leftrightarrow \psi) \Leftrightarrow ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$

Note that rules are identical to the propositional ones!

Satisfaction (*without variables*)

- Given a possible world $\langle \mathbf{U}, \Sigma, v \rangle$

If φ is an *atom* (i.e. φ has the form $P(t_1, \dots, t_n)$)

$$\langle \mathbf{U}, \Sigma, v \rangle \models \varphi \quad \text{iff} \quad \langle v(t_1), \dots, v(t_n) \rangle \in v(P/n)$$

If φ e ψ are wffs

$$\langle \mathbf{U}, \Sigma, v \rangle \models (\neg \varphi) \quad \text{iff} \quad \langle \mathbf{U}, \Sigma, v \rangle \not\models \varphi$$

$$\langle \mathbf{U}, \Sigma, v \rangle \models (\varphi \rightarrow \psi) \quad \text{iff} \quad \text{NOT } \langle \mathbf{U}, \Sigma, v \rangle \models \varphi \quad \text{OR} \quad \langle \mathbf{U}, \Sigma, v \rangle \models \psi$$

$$\langle \mathbf{U}, \Sigma, v \rangle \models (\varphi \wedge \psi) \quad \text{iff} \quad \langle \mathbf{U}, \Sigma, v \rangle \models \varphi \quad \text{AND} \quad \langle \mathbf{U}, \Sigma, v \rangle [s] \models \psi$$

$$\langle \mathbf{U}, \Sigma, v \rangle \models (\varphi \vee \psi) \quad \text{iff} \quad \langle \mathbf{U}, \Sigma, v \rangle \models \varphi \quad \text{OR} \quad \langle \mathbf{U}, \Sigma, v \rangle [s] \models \psi$$

What is *true*?

▪ A world of cats

Likes	Tom	Spot	Kitty	Felix
Tom	x			
Spot	x		x	
Kitty		x	x	
Felix			x	

translates into $\langle U, \Sigma, v \rangle$

▪ Universe

$U := \{\underline{tom}, \underline{spot}, \underline{kitty}, \underline{felix}\}$

← Could not put real cats in U :
underlined names represent *real objects*

▪ Signature

$\Sigma := \{tom, spot, kitty, felix, Likes/2\}$ i.e. four constants and one predicate symbol

▪ Interpretation

$v(tom) = \underline{tom}, \quad v(spot) = \underline{spot}, \quad v(kitty) = \underline{kitty}, \quad v(felix) = \underline{felix},$

$v(Likes/2) =$ a subset of $U \times U$

$\{\langle \underline{tom}, \underline{tom} \rangle, \langle \underline{spot}, \underline{tom} \rangle, \langle \underline{spot}, \underline{kitty} \rangle, \langle \underline{kitty}, \underline{spot} \rangle, \langle \underline{kitty}, \underline{kitty} \rangle, \langle \underline{felix}, \underline{kitty} \rangle\}$

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translates into $\langle \mathbf{U}, \Sigma, v \rangle$

▪ Sentences

$\langle \mathbf{U}, \Sigma, v \rangle \models \text{Likes}(\text{spot}, \text{kitty})$ because $\langle v(\text{spot}), v(\text{kitty}) \rangle \in v(\text{Likes}/2)$

$\langle \mathbf{U}, \Sigma, v \rangle \models \text{Likes}(\text{tom}, \text{tom})$ because $\langle v(\text{tom}), v(\text{tom}) \rangle \in v(\text{Likes}/2)$

$\langle \mathbf{U}, \Sigma, v \rangle \models \neg \text{Likes}(\text{kitty}, \text{felix})$ because $\langle v(\text{kitty}), v(\text{felix}) \rangle \notin v(\text{Likes}/2)$

$\langle \mathbf{U}, \Sigma, v \rangle \not\models \text{Likes}(\text{tom}, \text{kitty})$ because $\langle v(\text{tom}), v(\text{kitty}) \rangle \notin v(\text{Likes}/2)$

$\langle \mathbf{U}, \Sigma, v \rangle \not\models \neg \text{Likes}(\text{felix}, \text{kitty})$ because $\langle v(\text{felix}), v(\text{kitty}) \rangle \in v(\text{Likes}/2)$

What is *true*?

▪ A world of cats

Likes	Tom	Spot	Kitty	Felix
Tom	x			
Spot	x		x	
Kitty		x	x	
Felix			x	

translates into $\langle \mathbf{U}, \Sigma, \nu \rangle$

▪ Sentences

$$\langle \mathbf{U}, \Sigma, \nu \rangle \models (\text{Likes}(\text{spot}, \text{kitty}) \wedge \text{Likes}(\text{felix}, \text{kitty}))$$

$$\langle \mathbf{U}, \Sigma, \nu \rangle \models (\text{Likes}(\text{tom}, \text{kitty}) \vee \text{Likes}(\text{tom}, \text{tom}))$$

$$\langle \mathbf{U}, \Sigma, \nu \rangle \models (\text{Likes}(\text{spot}, \text{tom}) \vee \neg \text{Likes}(\text{spot}, \text{tom}))$$

is satisfied in this possible world but also in any possible world

First-order language

■ Well-formed formulae (wff)

All symbols in the *signature* Σ (i.e. *constants, function and predicate symbols*)

A set of **variables**: x, y, z

Two (primary) **logical connectives**: \neg, \rightarrow

Three (derived) **logical connectives**: $\wedge, \vee, \leftrightarrow$

Two **quantifiers**: \forall, \exists

Parentheses: $(,)$ (there are no *precedence rules* in this language)

An extended definition of *terms* and *atoms*

Term

A single *individual constant* or a **variable** is a **term**

If f/n is a *functional symbol* (with arity n) and t_1, \dots, t_n are **terms**, then $f(t_1, \dots, t_n)$ is a **term**

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If P/n is a *predicate symbol* (with arity n) and t_1, \dots, t_n are **terms**, then $P(t_1, \dots, t_n)$ is an **atom** (i.e a first-order well-formed formula – wff)

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An extended definition of *terms* and *atoms* (see before)

A set of syntactic rules

φ is an *atom* $\Rightarrow \varphi \in \text{wff}(L_{FO})$

$\varphi \in \text{wff}(L_{FO}) \Rightarrow (\neg\varphi) \in \text{wff}(L_{FO})$

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$\varphi, \psi \in \text{wff}(L_{FO}) \Rightarrow (\varphi \wedge \psi) \in \text{wff}(L_{FO}), (\varphi \wedge \psi) \Leftrightarrow (\neg(\varphi \rightarrow (\neg\psi)))$

$\varphi, \psi \in \text{wff}(L_{FO}) \Rightarrow (\varphi \leftrightarrow \psi) \in \text{wff}(L_{FO}), (\varphi \leftrightarrow \psi) \Leftrightarrow ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$

$\varphi \in \text{wff}(L_{FO}) \Rightarrow (\forall x \varphi) \in \text{wff}(L_{FO})$ $\leftarrow x$ can be any variable

$\varphi \in \text{wff}(L_{FO}) \Rightarrow (\exists x \varphi) \in \text{wff}(L_{FO})$

Satisfaction

- Given a possible world $\langle \mathbf{U}, \Sigma, v \rangle$ and a valuation s (on that world)

A valuation is a function $s : \text{Variables} \rightarrow \mathbf{U}$ \longleftarrow A valuation s transforms all variables into constants

If φ is an atom (i.e. φ has the form $P(t_1, \dots, t_n)$)

$$\langle \mathbf{U}, \Sigma, v \rangle [s] \models \varphi \text{ iff } \langle v(t_1) [s], \dots, v(t_n) [s] \rangle \in v(P) [s]$$

If φ e ψ are wffs

$$\langle \mathbf{U}, \Sigma, v \rangle [s] \models (\neg \varphi) \text{ iff}$$

$$\langle \mathbf{U}, \Sigma, v \rangle [s] \not\models \varphi$$

$$\langle \mathbf{U}, \Sigma, v \rangle [s] \models (\varphi \wedge \psi) \text{ iff}$$

$$\langle \mathbf{U}, \Sigma, v \rangle [s] \models \varphi \text{ AND } \langle \mathbf{U}, \Sigma, v \rangle [s] \models \psi$$

$$\langle \mathbf{U}, \Sigma, v \rangle [s] \models (\varphi \vee \psi) \text{ iff}$$

$$\langle \mathbf{U}, \Sigma, v \rangle [s] \models \varphi \text{ OR } \langle \mathbf{U}, \Sigma, v \rangle [s] \models \psi$$

$$\langle \mathbf{U}, \Sigma, v \rangle [s] \models (\varphi \rightarrow \psi) \text{ iff}$$

$$\text{NOT } \langle \mathbf{U}, \Sigma, v \rangle [s] \models \varphi \text{ OR } \langle \mathbf{U}, \Sigma, v \rangle [s] \models \psi$$

Quantified formulae

$$\langle \mathbf{U}, \Sigma, v \rangle [s] \models \forall x \varphi \text{ iff}$$

$$\text{FORALL } \underline{d} \in \mathbf{U} \text{ we have } \langle \mathbf{U}, \Sigma, v \rangle [s](x:\underline{d}) \models \varphi$$

$$\langle \mathbf{U}, \Sigma, v \rangle [s] \models \exists x \varphi \text{ iff}$$

$$\text{it EXISTS } \underline{d} \in \mathbf{U} \text{ such that } \langle \mathbf{U}, \Sigma, v \rangle [s](x:\underline{d}) \models \varphi$$

Where $[s](x:\underline{d})$ is the *variant* of function s that assigns \underline{d} to x and remains unaltered for any other variables.

What is *true*?

- **A world of cats**

<i>Likes</i>	Tom	Spot	Kitty	Felix
Tom	x			
Spot	x		x	
Kitty		x	x	
Felix			x	

translates into $\langle \mathbf{U}, \Sigma, v \rangle$

- **Sentences**

$\langle \mathbf{U}, \Sigma, v \rangle [s] \models (\forall x (\exists y \text{Likes}(x, y)))$ because

FORALL cat1 $\in \mathbf{U}$, $\langle \mathbf{U}, \Sigma, v \rangle [s](x:\text{cat1}) \models (\exists y \text{Likes}(x, y))$ because

it EXISTS cat2 $\in \mathbf{U}$, $\langle \mathbf{U}, \Sigma, v \rangle ([s](x:\text{cat1}))(y:\text{cat2}) \models \text{Likes}(x, y)$

What is *true*?

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<i>Likes</i>	Tom	Spot	Kitty	Felix
Tom	x			
Spot	x		x	
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translates into $\langle \mathbf{U}, \Sigma, v \rangle$

- **Sentences**

$\langle \mathbf{U}, \Sigma, v \rangle [s] \not\models (\exists x (\forall y \text{ Likes}(x, y)))$ because

FORALL cat1 $\in \mathbf{U}$, $\langle \mathbf{U}, \Sigma, v \rangle [s](x:\text{cat1}) \not\models (\forall y \text{ Likes}(x, y))$ because

it EXISTS cat2 $\in \mathbf{U}$, $\langle \mathbf{U}, \Sigma, v \rangle ([s](x:\text{cat1}))(y:\text{cat2}) \not\models \text{ Likes}(x, y)$

Variables and quantifiers: further examples

- “Being brothers means being relatives”

$$\forall x \forall y (Brother(x, y) \rightarrow Relative(x, y))$$

- “Being relative is a symmetric relation”

$$\forall x \forall y (Relative(x, y) \leftrightarrow Relative(y, x))$$

- “By definition, being mother is being parent and female”

$$\forall x (Mother(x) \leftrightarrow (\exists y Parent(x, y) \wedge Female(x)))$$

- “A cousin is a son of either a brother or a sister of either parents”

$$\begin{aligned} \forall x \forall y (Cousin(x, y) \\ \leftrightarrow \exists z \exists w (Parent(z, x) \wedge Parent(w, y) \wedge (Brother(z, w) \vee Sister(z, w)))) \end{aligned}$$

- “Everyone has a mother”

$$\forall x \exists y Mother(y, x)$$

BE CAREFUL about the order of quantifiers, in fact:

$$\exists y \forall x Mother(y, x)$$

“There is one (common) mother to everyone”

Open formulae, sentences

- **Bound** and **free** variables

The occurrence of a *variable* in a wff is **bound** if it lies in the scope of a quantifier (for that *variable*)

The occurrence of a *variable* in a wff is **free** if it is not *bound*

Examples of bound variables:

$$\forall x P(x)$$

$$\exists x (P(x) \rightarrow (A(x) \wedge B(x)))$$

Examples of free variables:

$$P(x)$$

$$\exists y (P(y) \rightarrow (A(x,y) \wedge B(y)))$$

- **Open** and **closed** formulae: **sentences**

A wff is **open** if there is at least one free occurrence of a variable

Otherwise, the wff is **closed** (also called **sentence**)

Only *closed* wffs, i.e. *sentences*, have a truth value (see after)

Models

■ **Validity** in a possible world, **model**

A wff φ such that $\langle \mathbf{U}, \Sigma, \nu \rangle [s] \models \varphi$ for any *valuation* s is **valid** in $\langle \mathbf{U}, \Sigma, \nu \rangle$

$\langle \mathbf{U}, \Sigma, \nu \rangle$ is also a **model** of φ

and we write $\langle \mathbf{U}, \Sigma, \nu \rangle \models \varphi$ (i.e. the reference to s can be omitted)

A possible world $\langle \mathbf{U}, \Sigma, \nu \rangle$ is a **model** of a *set* of wff Γ iff it is a model for all the wffs in Γ

and we write $\langle \mathbf{U}, \Sigma, \nu \rangle \models \Gamma$

■ **Truth**

A **sentence** ψ such that $\langle \mathbf{U}, \Sigma, \nu \rangle [s] \models \psi$ for one valuation s is **valid** in $\langle \mathbf{U}, \Sigma, \nu \rangle$

If the sentence is true for one valuation s , then is true for all valuations

A **sentence** ψ is **true** in $\langle \mathbf{U}, \Sigma, \nu \rangle$ if it is **valid** in $\langle \mathbf{U}, \Sigma, \nu \rangle$

Validity in general

■ Validity and logical truth

A wff (either open or closed) is **valid** (also **logically valid**) if it is **valid** in any possible world $\langle \mathbf{U}, \Sigma, \nu \rangle$

Example:

$$(P(x) \vee \neg P(x))$$

A sentence ψ is a **logical truth** if it is **true** in any possible world $\langle \mathbf{U}, \Sigma, \nu \rangle$

we write then $\models \psi$ (i.e. no reference to $\langle \mathbf{U}, \Sigma, \nu \rangle$)

Examples:

$$\forall x (P(x) \vee \neg P(x))$$

$$\forall x \forall y (G(x,y) \rightarrow (H(x,y) \rightarrow G(x,y)))$$

■ Inconsistence

A wff (either open or closed) is **inconsistent** if its not *satisfiable*

Example:

$$\forall x (P(x) \wedge \neg P(x))$$

Entailment

- Definition

Given a set of wffs Γ and one wff φ , we have

$$\Gamma \models \varphi$$

iff all possible worlds $\langle \mathbf{U}, \Sigma, \nu \rangle [s]$ satisfying Γ also satisfy φ

This definition embraces all possible combinations $\langle \mathbf{U}, \Sigma, \nu \rangle [s]$

The only thing that does not vary is the language Σ

Is this problem decidable?

In general, in first-order logic, a direct calculus of entailment is impossible...