

Artificial Intelligence

Probabilistic reasoning: *supervised learning*

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Machine Learning

Types of machine learning problems

Consider a number of observations (input data) made by an agent

$$\{D_1, D_2, \dots, D_N\}$$

■ **Supervised learning**

Learning from complete observations: each of the observations $\{D_1, D_2, \dots, D_N\}$ include values for all the random variables in the model

The objective is learning a *distribution* P

■ **Unsupervised learning**

Learning from incomplete observations: observations $\{D_1, D_2, \dots, D_N\}$ do not necessarily include values for all the random variables in the model

The objective is learning a *distribution* P

■ **Reinforcement learning**

The observations $\{D_1, D_2, \dots, D_n\}$ are *states* or *situations*,
at each state X_i the agent must perform an **action** a_i that produces a **result** r_i .

The objective is defining a *function* $a_i = \pi(D_i)$ that describes a strategy that the agent will follow

The strategy should be optimal, in the sense that it should maximize
the expected value of a *function* $v(< r_1, r_2, \dots, r_n >)$ of the sequence of *results*

Observations and Independence

About notation

Each observation could be the outcome of an experiment or a test

The outcome of a particular experiment can be represented by a set of *random variables*

For example, if the model makes use of the random variable $\{X, Y\}$, the N outcomes of the experiments are $D_1 = \{X^{(1)}, Y^{(1)}\}$, $D_2 = \{X^{(2)}, Y^{(2)}\}$, ..., $D_N = \{X^{(N)}, Y^{(N)}\}$

■ **Independent observations, same probability distribution**

Independent and Identically Distributed (IID) random variables

Definition

A sequence or a set of random variables $\{X_1, X_2, \dots, X_n\}$ is *Independent and Identically Distributed (IID)* iff:

- 1) $\langle X_i \perp X_j \rangle, \forall i \neq j$ (independence)
- 2) $P(X_i \leq x) = P(X_j \leq x), \forall i \neq j, \forall x$ (identical distribution)

CAUTION:

Being IID is not an obvious property of observations

Example: different measurements on different patients *may* be IID, but different measurements over time on the same patient are not IID

ML = Representation + Evaluation + Optimization

Assume that an I.I.D. dataset D is available

■ Representation

The objective is learning a specific distribution

$$P(\{X_i\}; \theta)$$

where $\{X_i\}$ are all the random variables of interest and θ is a set of parameters

Which kind of distribution (i.e. the *model* or also the *learner*) do we select?

Example: assume we select the anti-spam filter (i.e. Naïve Bayesian Classifier) as the model the parameters in such case are the numerical probabilities in the CPTs

■ Evaluation

Given a dataset D , how well does a specific set of parameter values $\hat{\theta}$ make the distribution P fit the dataset?

An estimator, i.e. a scoring function of some sort, must be selected

■ Optimization

How can we find the optimal set of parameter values θ^* with respect to the *estimator* of choice?

In general, this is an optimization problem

Maximum Likelihood Estimator (MLE)

Maximum Likelihood Estimation (MLE)

A probabilistic model $P(X)$, with parameters θ

θ is a set of values that characterizes $P(X)$ *completely*: once θ is defined, $P(X)$ is also defined.

A set of IID observations $D = \{D_1, D_2, \dots, D_N\}$

■ Likelihood function

A function, or a conditional probability, derived from the model $P(X)$

$$L(\theta | D) = P(D | \theta) = P(D_1, D_2, \dots, D_N | \theta)$$

where $P(D | \theta)$ is the conditional probability that the parameter θ , considered as a random variables, could generate the observations D

When the observations $\{D_1, D_2, \dots, D_N\}$ are IID:

$$P(D | \theta) = P(D_1 | \theta)P(D_2 | \theta) \dots P(D_N | \theta) = \prod_m P(D_m | \theta)$$

■ Maximum Likelihood Estimation

$$\theta_{ML}^* = \operatorname{argmax}_{\theta} L(\theta | D)$$

When the observations are IID, using *log-Likelihood* could ease computations:

$$\ell(\theta | D) = \log L(\theta | D) = \log \prod_m P(D_m | \theta) = \sum_m \log P(D_m | \theta)$$

$$\theta_{ML}^* = \operatorname{argmax}_{\theta} \ell(\theta | D) \quad \text{— true because log is monotonically increasing}$$

Example: coin tossing (*Bernoulli Trials*)

Experiment: tossing a coin X , not necessarily *fair* ($X = 1$ head, $X = 0$ tail)

Parameters: $\theta := \{ \pi \} \Leftrightarrow P(X = 1) = \pi, P(X = 0) = 1 - \pi$

Observations: a sequence of experimental outcomes

$$D = \{D_1 = \{X^{(1)} = x^{(1)}\}, D_2 = \{X^{(2)} = x^{(2)}\}, \dots, D_N = \{X^{(N)} = x^{(N)}\}\}$$

■ Binomial distribution

$$P(D | \theta) = \binom{N}{N_{X=1}} \prod_i P(X_i | \theta) = \binom{N}{N_{X=1}} P(X = 1 | \theta)^{N_{X=1}} P(X = 0 | \theta)^{N_{X=0}} = \binom{N}{N_{X=1}} \pi^{N_{X=1}} (1 - \pi)^{N_{X=0}}$$

$\binom{N}{k} := \frac{N!}{(N-k)!k!}$ *binomial coefficient*

$N_{X=1}$ is the number of
 $X=1$ (i.e. head) in a sequence of N trials

It is the probability of obtaining $N_{X=1}$ times 'head' in a sequence of N trials

In this case, it is assumed to be the likelihood of the sequence $\langle D_1, D_2, \dots, D_N \rangle$ given the parameters θ

Example: coin tossing (*Bernoulli Trials*)

■ (Log-)Likelihood Function

$$\ell(\theta | D) = \log P(D | \theta) = \log P(\{X_i\} | \theta) = \log \binom{N}{N_{X=1}} \prod_i P(X_i | \theta) = \log \binom{N}{N_{X=1}} + \sum_i \log P(X_i | \theta)$$

Rewrite $P(X | \theta)$ as:

$$P(X | \theta) = \pi^{[X=1]} (1 - \pi)^{[X=0]} \quad \text{where: } [X_i = v] := \begin{cases} 1 & \text{if } X_i = v \\ 0 & \text{if } X_i \neq v \end{cases} \quad \begin{array}{l} \text{Also called} \\ \text{indicator function} \end{array}$$

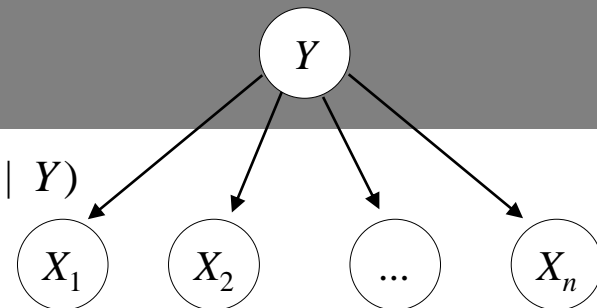
$$\begin{aligned} \ell(\theta | D) &= \log \binom{N}{N_{X=1}} + \sum_i \log \left(\pi^{[X_i=1]} (1 - \pi)^{[X_i=0]} \right) = \\ &= \log \binom{N}{N_{X=1}} + \log \pi \sum_i [X_i = 1] + \log (1 - \pi) \sum_i [X_i = 0] = \log \binom{N}{N_{X=1}} + N_{X=1} \log \pi + N_{X=0} \log (1 - \pi) \end{aligned}$$

■ Maximum Likelihood Estimation

$$\frac{\partial \ell}{\partial \theta} = \frac{\partial \ell}{\partial \pi} = \frac{N_{X=1}}{\pi} - \frac{N_{X=0}}{(1 - \pi)} \frac{\partial \ell}{\partial \theta} = 0 \quad \Rightarrow \quad \theta_{ML}^* = \frac{N_{X=1}}{N_{X=1} + N_{X=0}} = \frac{N_{X=1}}{N}$$

Anti-spam filter

$$P(Y, \{X_i\}) = P(Y) \prod_{i=1}^n P(X_i | Y)$$



Parameters: the *conditional probability tables* in the graphical model

$$\theta := \{ \pi_k, \pi_{ijk} \} \Leftrightarrow P(Y = k) = \pi_k, \quad P(X_i = j | Y = k) = \pi_{ijk}$$

Observations: a set of messages *with classification*

$$D = \{ D_1 = \{ Y^{(1)} = 1, X_1^{(1)} = 1, X_2^{(1)} = 1, \dots, X_n^{(1)} = 0 \}, \dots, \\ D_N = \{ Y_2^{(N)} = y^{(N)}, X_1^{(N)} = x_1^{(N)}, X_2^{(N)} = x_2^{(N)}, \dots, X_n^{(N)} = x_n^{(N)} \} \}$$

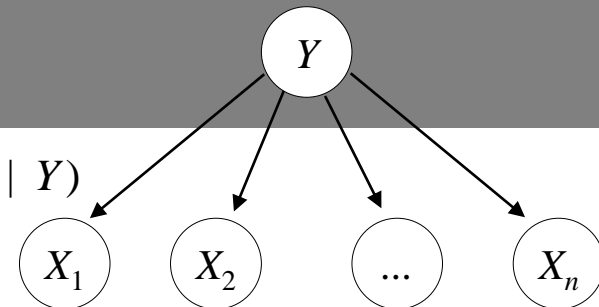
■ Likelihood Function

Sequence of messages

$$\begin{aligned} L(\theta | D) &= P(D | \theta) = P(\{D_m\} | \{\pi_k, \pi_{ijk}\}) = \prod_m P(D_m | \{\pi_k, \pi_{ijk}\}) && \text{(messages are IID)} \\ &= \prod_m P(\{Y^{(m)} = y^{(m)}, X_i^{(m)} = x_i^{(m)}\} | \{\pi_k, \pi_{ijk}\}) \\ &= \prod_m P(Y^{(m)} = y^{(m)} | \{\pi_k, \pi_{ijk}\}) P(\{X_i^{(m)} = x_i^{(m)}\} | Y^{(m)} = y^{(m)}, \{\pi_k, \pi_{ijk}\}) && \text{(factorization)} \\ &= \prod_m P(Y^{(m)} = y^{(m)} | \{\pi_k\}) P(\{X_i^{(m)} = x_i^{(m)}\} | Y^{(m)} = y^{(m)}, \{\pi_{ijk}\}) && \text{(cond. independence)} \\ &= \prod_m P(Y^{(m)} = y^{(m)} | \{\pi_k\}) \prod_i P(X_i^{(m)} = x_i^{(m)} | Y^{(m)} = y^{(m)}, \{\pi_{ijk}\}) && (<X_i \perp X_j | Y>) \end{aligned}$$

Anti-spam filter

$$P(Y, \{X_i\}) = P(Y) \prod_{i=1}^n P(X_i | Y)$$



■ Log-Likelihood Function

$$\ell(\{\pi_k, \pi_{ijk}\} | D) = \sum_m \log P(Y^{(m)} = y^{(m)} | \{\pi_k\}) + \sum_m \sum_i \log P(X_i^{(m)} = x_i^{(m)} | Y^{(m)} = y^{(m)}, \{\pi_{ijk}\})$$

Alternative form for P : (i.e. rewritten using indicator functions)

$$P(Y = k | \{\pi_k\}) = \pi_k = \prod_k \pi_k^{[Y=k]}$$

$$P(X_i = j | Y = k, \{\pi_{ijk}\}) = \pi_{ijk} = \prod_j \prod_k \pi_{ijk}^{[X_i=j][Y=k]}$$

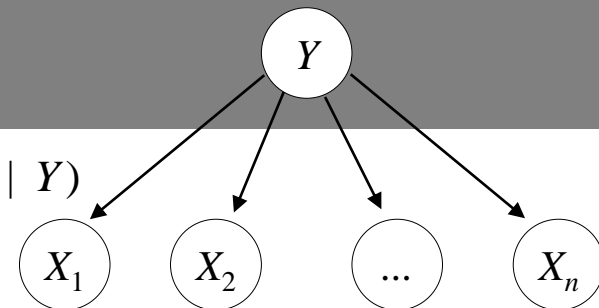
$$\ell(\{\pi_k, \pi_{ijk}\} | D) = \sum_m \sum_k [Y^{(m)} = k] \log \pi_k + \sum_m \sum_i \sum_j \sum_k [X_i^{(m)} = j][Y^{(m)} = k] \log \pi_{ijk}$$

■ Maximum Likelihood Estimation

*Being both positive and depending on different variables,
the two terms above can be optimized separately*

Anti-spam filter

$$P(Y, \{X_i\}) = P(Y) \prod_{i=1}^n P(X_i | Y)$$



■ Maximum Likelihood Estimation

$$\ell(\{\pi_k, \pi_{ijk}\} | D) = \sum_m \sum_k [Y^{(m)} = k] \log \pi_k + \sum_m \sum_i \sum_j \sum_k [X_i^{(m)} = j][Y^{(m)} = k] \log \pi_{ijk}$$

First term:

$$\ell^*(\{\pi_k\} | D) = \sum_m \sum_k [Y^{(m)} = k] \log \pi_k + \lambda \left(1 - \sum_k \pi_k\right)$$

Lagrange multiplier

$$\frac{\partial \ell^*}{\partial \pi_k} = \frac{\sum_m [Y^{(m)} = k]}{\pi_k} - \lambda$$

number of messages in D classified as k

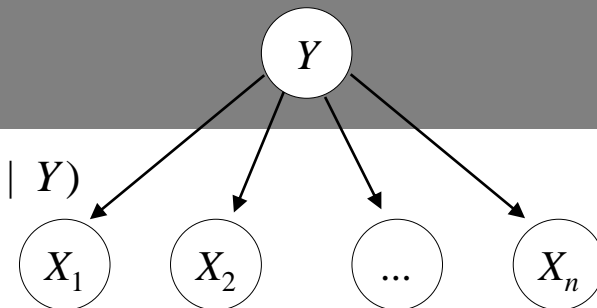
$$\frac{\partial \ell^*}{\partial \pi_k} = 0 \Rightarrow \pi_k = \frac{N_{Y=k}}{\lambda}$$

$$\sum_k \pi_k = 1 \Rightarrow \sum_k \frac{N_{Y=k}}{\lambda} = 1 \Rightarrow \lambda = \sum_k N_{Y=k} = N$$

$$\pi_k^* = \frac{N_{Y=k}}{N} \quad (\text{Maximum Likelihood Estimator of } \pi_k)$$

Anti-spam filter

$$P(Y, \{X_i\}) = P(Y) \prod_{i=1}^n P(X_i | Y)$$



■ Maximum Likelihood Estimation

$$\ell(\{\pi_k, \pi_{ijk}\} | D) = \sum_m \sum_k [Y^{(m)} = k] \log \pi_k + \sum_m \sum_i \sum_j \sum_k [X_i^{(m)} = j][Y^{(m)} = k] \log \pi_{ijk}$$

Second term:

$$\ell^*(\{\pi_{ijk}\} | D) = \sum_m \sum_i \sum_j \sum_k [X_i^{(m)} = j][Y^{(m)} = k] \log \pi_{ijk} + \sum_i \sum_k \lambda_{ik} (1 - \sum_j \pi_{ijk})$$

$$\frac{\partial \ell^*}{\partial \pi_{ijk}} = \frac{\sum_m [X_i^{(m)} = j][Y^{(m)} = k]}{\pi_{ijk}} - \lambda_{ik}$$

$$\frac{\partial \ell^*}{\partial \pi_{ijk}} = 0 \Rightarrow \pi_{ijk} = \frac{N_{X_i=j, Y=k}}{\lambda_{ik}}$$

$$\sum_j \pi_{ijk} = 1 \Rightarrow \sum_j \frac{N_{X_i=j, Y=k}}{\lambda_{ik}} = 1 \Rightarrow \lambda_{ik} = \sum_j N_{X_i=j, Y=k} = N_{Y=k}$$

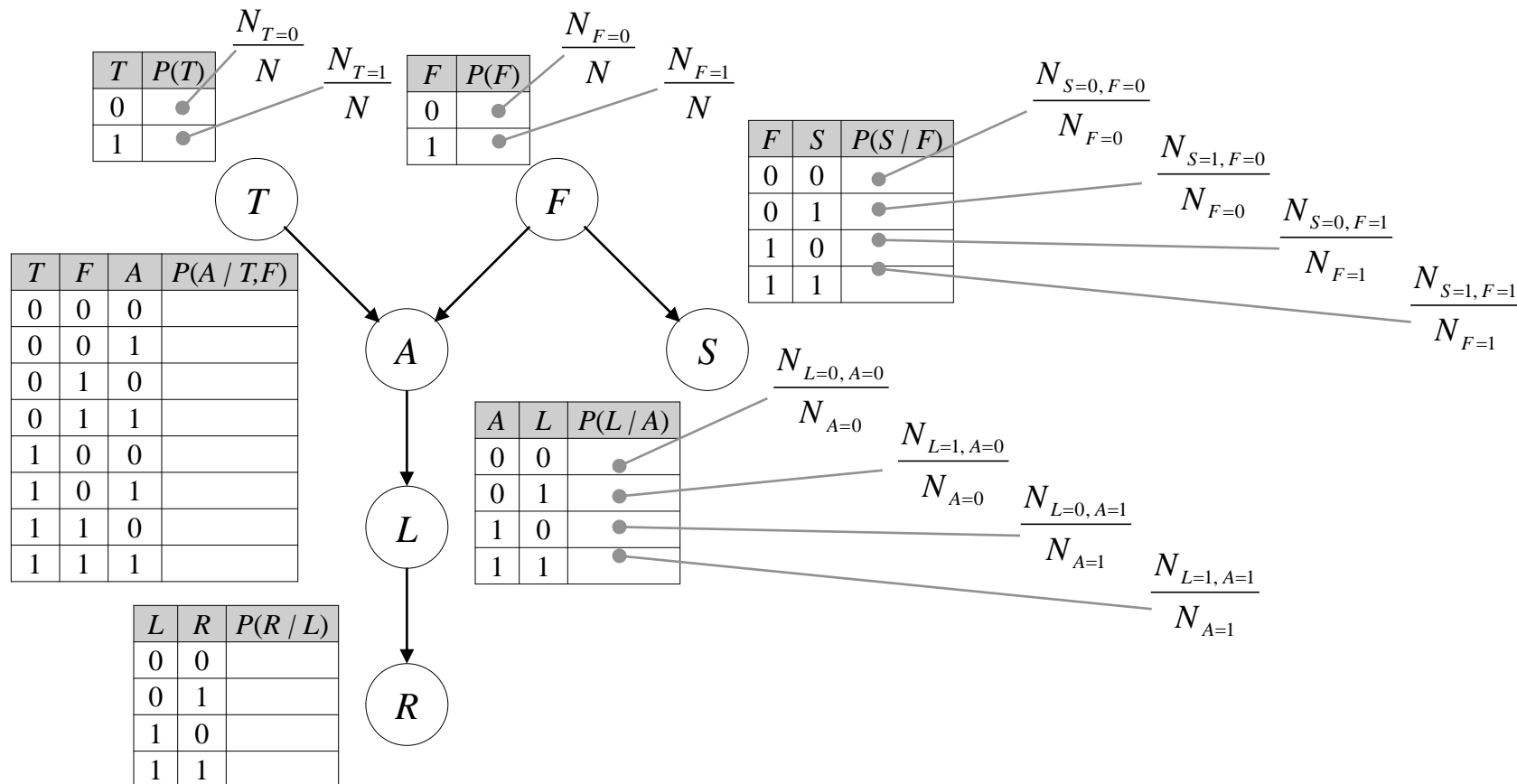
$$\pi_{ijk}^* = \frac{N_{X_i=j, Y=k}}{N_{Y=k}} \quad (\text{Maximum Likelihood Estimator of } \eta_{ijk})$$

Learning CPTs for a graphical model

As Maximum Likelihood Estimation

Parameters: the conditional probabilities (i.e. all CPTs)

Observations: sequence of sets of values, from completely observed situations

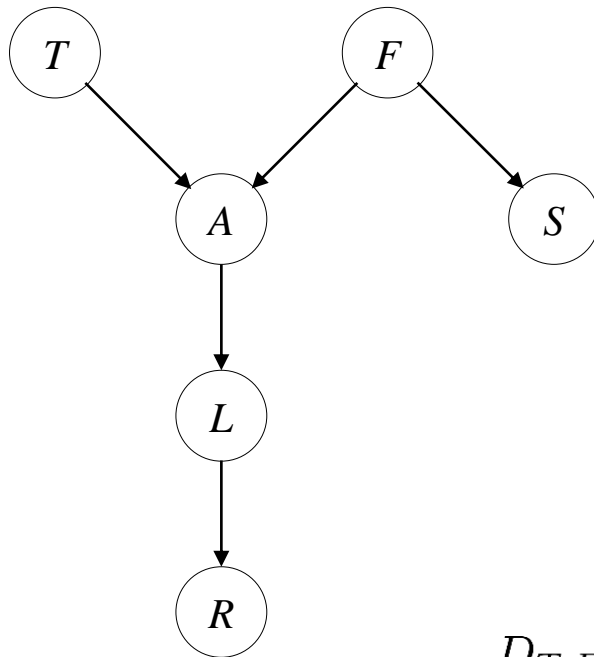


Learning CPTs for a graphical model

As Maximum Likelihood Estimation

More in general:

The MLE of a (directed) graphical model is the MLE of each node
(in each corresponding observation subset)



$$\theta_{ML}^* := \operatorname{argmax}_{\theta} P(D \mid \theta)$$

$$\theta = \{\pi_T, \pi_F, \pi_{S|F}, \pi_{A|S,F}, \pi_{L|A}, \pi_{R|L}\}$$

$$\pi_T^* := \operatorname{argmax}_{\pi_T} P(D \mid \pi_T)$$

$$\pi_F^* := \operatorname{argmax}_{\pi_F} P(D \mid \pi_F)$$

$$\pi_{S|F}^* := \operatorname{argmax}_{\pi_{S|F}} P(D_F \mid \pi_{S|F})$$

$$\pi_{A|T,F}^* := \operatorname{argmax}_{\pi_{A|T,F}} P(D_{T,F} \mid \pi_{A|T,F})$$

$$\pi_{L|A}^* := \operatorname{argmax}_{\pi_{L|A}} P(D_A \mid \pi_{L|A})$$

$$\pi_{R|L}^* := \operatorname{argmax}_{\pi_{R|L}} P(D_L \mid \pi_{R|L})$$

$D_{T,F}$ denotes the subset of complete observation in which
the random variables T, F have the corresponding values

Bayesian Learning:
Maximum a Posteriori (MAP)
estimator

Bayesian learning

■ *Maximum a Posteriori Estimation (MAP)*

Instead of a *likelihood function*, the a posteriori probability is maximized

$$P(\theta | D) = \frac{P(D | \theta) P(\theta)}{P(D)} = \frac{P(D | \theta) P(\theta)}{\sum_{\theta} P(D | \theta) P(\theta)}$$

Which is equivalent to optimize, w.r.t. θ :

$$P(D | \theta) P(\theta)$$

Advantages:

- Regularization: not all possible combinations of values might be present in D
- A formula for incremental learning:
a priori terms could represent what was known *before* observations D

Problem:

- Which *prior* distribution $P(\theta)$?

Beta distribution

(*) In a finitary setting

Gamma function (n integer > 0)

$$\Gamma(n) := (n-1)!$$

Beta function (α and β integers > 0)

$$B(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} = \frac{(\alpha-1)!(\beta-1)!}{(\alpha + \beta - 1)!}$$

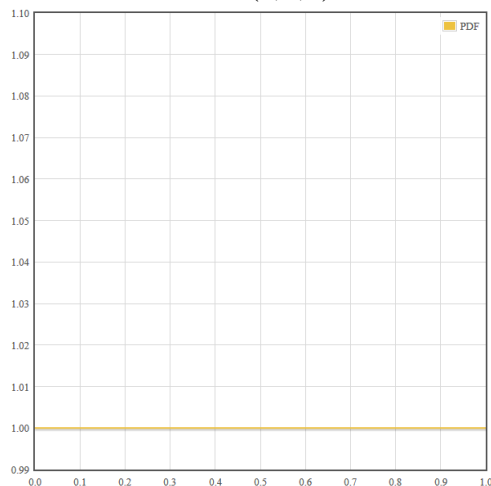
The definition is more complex when α and β are not integers (see Wikipedia)

■ Beta probability density function (pdf) (α and β integers > 0)

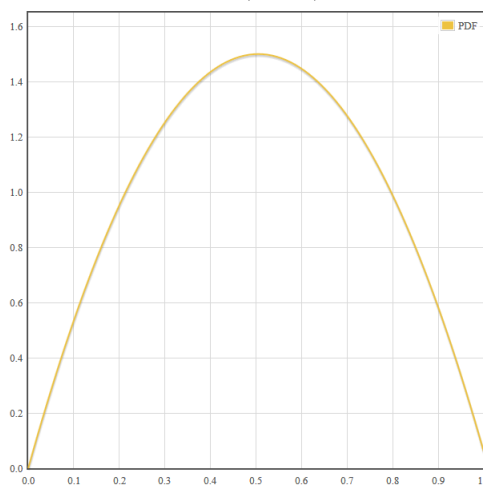
$$\text{Beta}(x; \alpha, \beta) := \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}$$

The maximum occurs at: $x = \frac{\alpha-1}{\alpha + \beta - 2}$

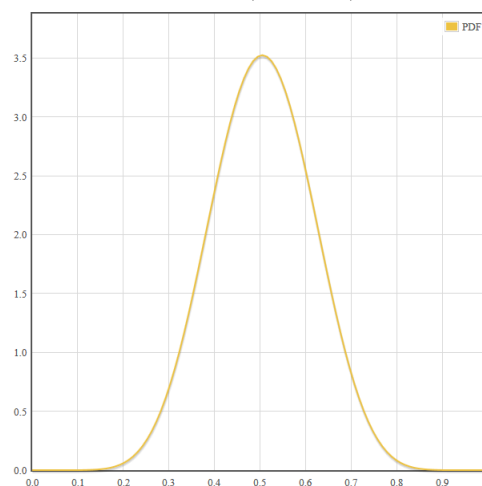
Beta($x;1,1$)



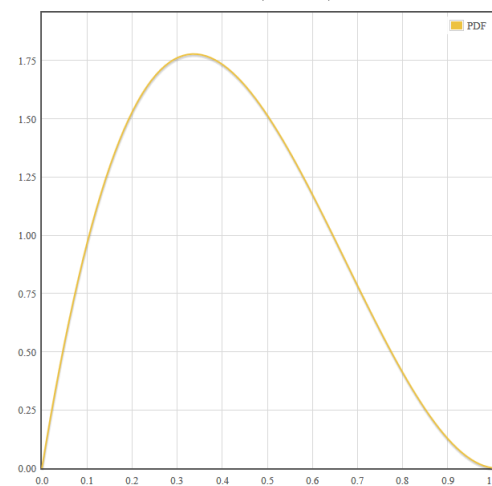
Beta($x;2,2$)



Beta($x;10,10$)



Beta($x;2,3$)



Conjugate prior distributions

Coin tossing (i.e. Binomial)

α_D and β_D are the result counts (i.e. heads and tails)

$$P(D | \theta) = \binom{\alpha_D + \beta_D}{\alpha_D} \prod_i P(X_i | \theta) = \binom{\alpha_D + \beta_D}{\alpha_D} \theta^{\alpha_D} (1 - \theta)^{\beta_D}$$

A posteriori probability with Beta prior

α_P and β_P are the **hyperparameters** of the prior

$$\begin{aligned} P(D | \theta) P(\theta) &= \binom{\alpha_D + \beta_D}{\alpha_D} \theta^{\alpha_D} (1 - \theta)^{\beta_D} \cdot \text{Beta}(\theta; \alpha_P, \beta_P) = \binom{\alpha_D + \beta_D}{\alpha_D} \theta^{\alpha_D} (1 - \theta)^{\beta_D} \cdot \frac{\theta^{\alpha_P - 1} (1 - \theta)^{\beta_P - 1}}{\text{B}(\alpha_P, \beta_P)} \\ &= \binom{\alpha_D + \beta_D}{\alpha_D} \frac{\theta^{\alpha_D + \alpha_P - 1} (1 - \theta)^{\beta_D + \beta_P - 1}}{\text{B}(\alpha_P, \beta_P)} = \binom{\alpha_D + \beta_D}{\alpha_D} \frac{\text{B}(\alpha_D + \alpha_P, \beta_D + \beta_P)}{\text{B}(\alpha_P, \beta_P)} \text{Beta}(\theta; \alpha_D + \alpha_P, \beta_D + \beta_P) \end{aligned}$$

this factor is a positive constant (for θ)

Conjugate prior distributions

Coin tossing (i.e. Binomial)

α_D and β_D are the result counts (i.e. heads and tails)

$$P(D | \theta) = \binom{\alpha_D + \beta_D}{\alpha_D} \prod_i P(X_i | \theta) = \binom{\alpha_D + \beta_D}{\alpha_D} \theta^{\alpha_D} (1 - \theta)^{\beta_D}$$

A posteriori probability with Beta prior

$$P(D | \theta) P(\theta) = \binom{\alpha_D + \beta_D}{\alpha_D} \frac{B(\alpha_D + \alpha_P, \beta_D + \beta_P)}{B(\alpha_P, \beta_P)} \text{Beta}(\theta; \alpha_D + \alpha_P, \beta_D + \beta_P)$$

“is proportional to”

$$P(D | \theta) P(\theta) \propto \text{Beta}(\theta; \alpha_D + \alpha_P, \beta_D + \beta_P)$$

Optimization:

$$\theta_{MAP}^* = \arg \max_{\theta} \text{Beta}(\theta; \alpha_D + \alpha_P, \beta_D + \beta_P) = \frac{\alpha_D + \alpha_P - 1}{\alpha_D + \alpha_P + \beta_D + \beta_P - 2}$$

which is the same as MLE but with the addition of $\alpha_P + \beta_P$ *pseudo-observations*

Being a **conjugate prior** $P(\theta)$ of a distribution $P(D | \theta)$ *in the above sense*
means that the posterior $P(D | \theta) P(\theta)$ is in *the same family* of $P(\theta)$

Conjugate prior distributions

Coin tossing (in a specific sequence)

$$P(D_i | \theta) = \theta^{[X_i=1]} (1 - \theta)^{[X_i=0]}$$

Likelihood

(a.k.a. the Bernoulli distribution)

α_D and β_D are the result counts (i.e. heads and tails)

$$P(D | \theta) = P(\{D_i\} | \theta) = \prod_i P(D_i | \theta) = \theta^{\alpha_D} (1 - \theta)^{\beta_D}$$

A posteriori probability with Beta prior

"is proportional to"

$$P(D | \theta) P(\theta) \propto \text{Beta}(\theta; \alpha_D + \alpha_P, \beta_D + \beta_P)$$

Therefore

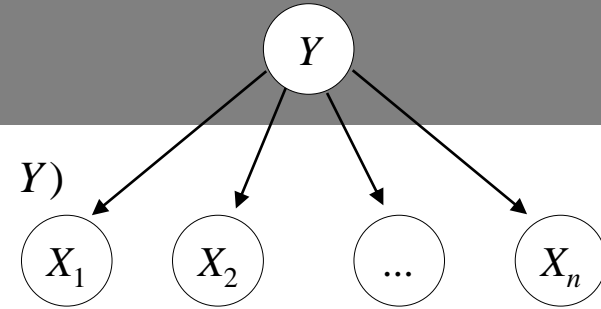
$$\theta_{MAP}^* = \arg \max_{\theta} \text{Beta}(\theta; \alpha_D + \alpha_P, \beta_D + \beta_P) = \frac{\alpha_D + \alpha_P - 1}{\alpha_D + \alpha_P + \beta_D + \beta_P - 2}$$

which is the same as MLE but with the addition of $\alpha_P + \beta_P$ *pseudo-observations*

Being a **conjugate prior** $P(\theta)$ of a distribution $P(D | \theta)$ *in the above sense* means that the posterior $P(D | \theta) P(\theta)$ is in *the same family* of $P(\theta)$

Anti-spam filter

$$P(Y, \{X_i\}) = P(Y) \prod_{i=1}^n P(X_i | Y)$$



■ Maximum a Posteriori (MAP) Estimation

The adapted computations for:

$$\theta_{MAP}^* = \arg \max_{\theta} P(D | \theta) P(\theta)$$

yield:

$$\pi_k^* = \frac{\alpha_k + N_{Y=k} - 1}{\alpha_k + \beta_k + N_D - 2} \quad (\text{MAP Estimator of } \pi_k)$$

$$\eta_{ijk}^* = \frac{\alpha_{ijk} + N_{X_i=j, Y=k} - 1}{\alpha_{ijk} + \beta_{ijk} + N_{Y=k} - 2} \quad (\text{MAP Estimator of } \eta_{ijk})$$

where the

$$\alpha_k, \beta_k, \alpha_{ijk}, \beta_{ijk}$$

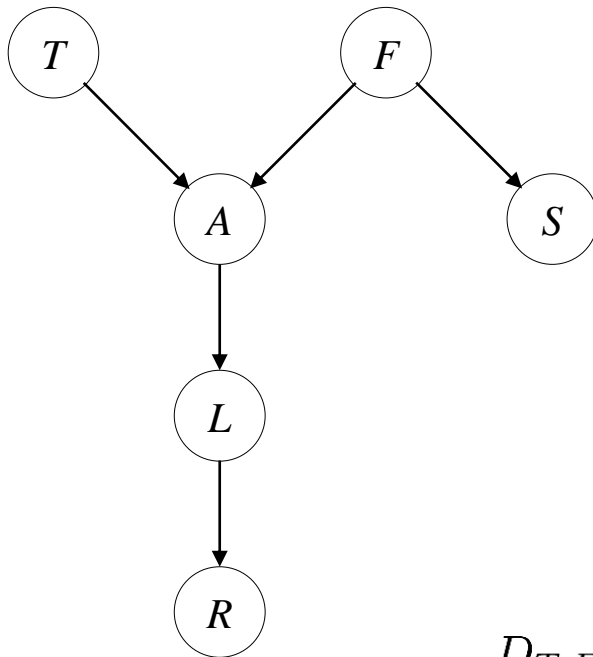
are the *hyperparameters* of the prior distribution
representing the *pseudo-observations*
made *before* the arrival of new, actual observations D

Learning CPTs for a graphical model

As Maximum a Posteriori Estimation

More in general:

The MAP of a (directed) graphical model is the MAP of each node
(in each corresponding observation subset)



$$\theta_{MAP}^* := \operatorname{argmax}_{\theta} P(D \mid \theta) P(\theta)$$

$$\theta = \{\pi_T, \pi_F, \pi_{S|F}, \pi_{A|S,F}, \pi_{L|A}, \pi_{R|L}\}$$

$$\pi_T^* := \operatorname{argmax}_{\pi_T} P(D \mid \pi_T) P(\pi_T)$$

$$\pi_F^* := \operatorname{argmax}_{\pi_F} P(D \mid \pi_F) P(\pi_F)$$

$$\pi_{S|F}^* := \operatorname{argmax}_{\pi_{S|F}} P(D_F \mid \pi_{S|F}) P(\pi_{S|F})$$

$$\pi_{A|T,F}^* := \operatorname{argmax}_{\pi_{A|T,F}} P(D_{T,F} \mid \pi_{A|T,F}) P(\pi_{A|T,F})$$

$$\pi_{L|A}^* := \operatorname{argmax}_{\pi_{L|A}} P(D_A \mid \pi_{L|A}) P(\pi_{L|A})$$

$$\pi_{R|L}^* := \operatorname{argmax}_{\pi_{R|L}} P(D_L \mid \pi_{R|L}) P(\pi_{R|L})$$

$D_{T,F}$ denotes the subset of complete observation in which
the random variables T, F have the corresponding value