# Probabilistic reasoning: *representation & inference*

Marco Piastra

# Probability: events as *subsets of possible worlds*

- Boolean algebra

  A non-empty collection of subsets $\Sigma$ of a set $W$ such that:

  1) $A, B \in \Sigma \implies A \cup B \in \Sigma$

  2) $A \in \Sigma \implies A^c \in \Sigma$

  3) $\varnothing \in \Sigma$

  Corollary:
  The sets $\varnothing$ e $W$ belong to any Boolean algebra generated on $W$
  $\Sigma$ is also closed under <u>*binary*</u> *intersection*

- $\sigma$-algebra

  A non-empty collection of subsets $\Sigma$ of a set $W$ such that:

  1) $A_k \in \Sigma, \ \forall k \in \mathbb{N}^+ \implies (\bigcup_{k=1}^{\infty} A_k) \in \Sigma$

  2) $A \in \Sigma \implies A^c \in \Sigma$

  3) $\varnothing \in \Sigma$

  Corollary:
  The sets $\varnothing$ e $W$ belong to any $\sigma$- algebra generated on $W$
  $\Sigma$ is also closed under <u>*countable*</u> *intersection*

*This is a stronger requirement:*
*closeness under <u>countable</u> union*
*Hence a $\sigma$-algebra is a boolean algebra*
*but not vice-versa*

# Probability: events as *subsets of possible worlds*

- $\sigma$-algebra (*definition*)

  A non-empty collection of subsets $\Sigma$ of a set $W$ such that:

  1) $A_k \in \Sigma,\ \forall k \in \mathbb{N}^+ \implies (\bigcup_{k=1}^{\infty} A_k) \in \Sigma$
  2) $A \in \Sigma \implies A^c \in \Sigma$
  3) $\varnothing \in \Sigma$

- Probability *measure* over a $\sigma$-algebra

  A function $P : \Sigma \to [0, 1]$

  i.e. $P$ assigns a measure (i.e. a real number)
  to each elements of a *$\sigma$-algebra* $\Sigma$ of subsets of $W$

  1) $\forall A \in \Sigma, P(A) \geq 0$
  2) $A_k \in \Sigma,\ \forall k \in \mathbb{N}^+$ are *disjoint* $\implies P\,(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k)$
  3) $P(\varnothing) = 0$
  4) $P(A^c) = 1 - P(A)$ (which implies $P(W) = 1$)

- Probability *space*

  A triple $\langle W, \Sigma, P \rangle$

# Probability: events as *subsets of possible worlds*

- $\sigma$-algebra
- Probability *measure* over a $\sigma$-algebra
- Probability *space*

  A triple $\langle W, \Sigma, P \rangle$

*Why bothering so much with these (very) technical definitions?*

- Rationale *(just a few hints)*

  Closure w.r.t. *countable unions* of a $\sigma$-algebras
  (as well as *countable additivity* of $P$ )
  is required for dealing with *infinite sequences of events* and their properties

  However, assuming *countable* union and additivity is also a *restriction*,
  i.e. to ensure *measurability*
  *(see the so-called Banach-Tarski paradox for counterexamples)*

# Probability: events as *subsets of possible worlds*

- Probability *measure* over a $\sigma$-algebra
- *Disjoint* events

In general

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If $A \cap B = \varnothing$ then events $A$ and $B$ are ***disjoint***

$$P(A \cup B) = P(A) + P(B)$$

(*) Note that $A \cap B = \varnothing \implies P(A \cap B) = 0$
but not vice-versa: as an event can have zero probability without being empty

(**) Unlike in propositional logic, knowing $P(A)$ and $P(B)$ is not sufficient for determining $P(A \cup B)$
Namely, probability is not *truth-functional* …

# Studying basic properties: *a finitary setting*

*It can be useful to adopt, at least for a while, a simpler setting that allows a simpler definition of fundamental properties*

- Finite algebra of events

  $\Sigma$ is a finite collection of subsets

  *In this setting, boolean algebra = $\sigma$-algebra*

  *Events could also be defined via propositional logic*
  *(à la de Finetti, 1937)*

- Finitely additive probability measure

  *Just summations, no integrals*

  *Computability is always guaranteed*

# Partitions, random variables*

(*) In a finitary setting

- Partition

  A *finite* collection $A_i$ of *disjoint* subsets (i.e. events) such that

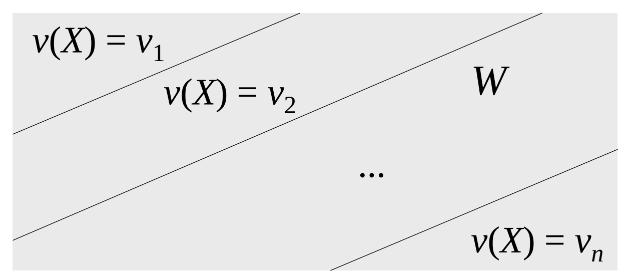
$$\bigcup_i P(A_i) = W$$

- Random Variable

  Let $X$ be a variable having a *finite* set of values $\{v_1, v_2, \ldots, v_n\}$

  - In each possible world, $X$ has a specific value $v_i$
  - The set of values $X = v_1, X = v_2, \ldots, X = v_n$ defines a *partition* of $W$
  - Each constraint $X = v_i$ defines an *event* (i.e. a subset of $W$)
  - *Given that* $X=v_i$ e $X=v_j$ are disjoint,
    $P(X=v_i \vee X=v_j) = P(X=v_i) + P(X=v_j)$ whenever $i \neq j$

Random variables having binary values are also said to be *binomial* (also *Bernoullian*)
Random variables with multiple values are also said to be *multinomial*



# Partitions, random variables*

(*) In a finitary setting

- Partition

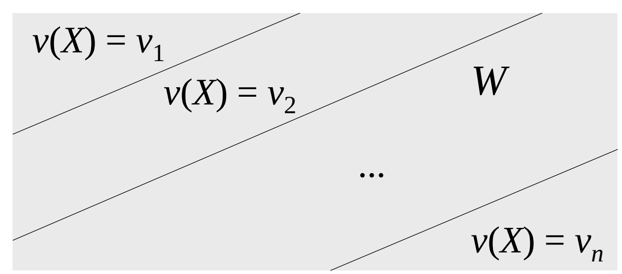
  A *finite* collection $A_i$ of *disjoint* subsets (i.e. events) such that

  $$\bigcup_i P(A_i) = W$$

  A $\sigma$-algebra can be generated from a partition
  by taking the closure of the partition under union and complement

- Random Variable

  Is a convenient way to define a $\sigma$-algebra over $W$



# Random variables, joint distribution*

(*) In a finitary setting

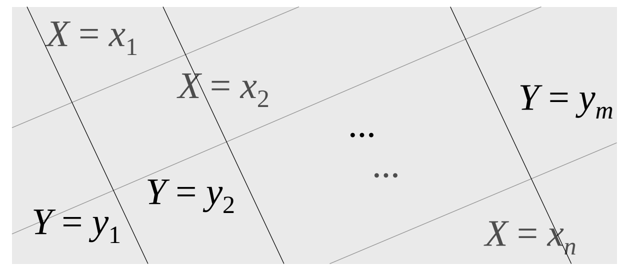
- Multiple random variables

  In practice, in a probabilistic representation, multiple random variables can coexist

  Example:
  $X_i$ occurrence of a word $i$ in the body of an email (binomial)
  $Y$ classification of that email as spam (binomial)

  Together, a collection of random variable defines
  a partition of $W$

  *The intersection of two or more* $\sigma$-algebras *is a* $\sigma$-algebra




- Joint probability distribution

  for a given set of random variables, e.g. $X, Y, Z$

  It is a <u>*function*</u> $P(X=x , Y=y , Z=z)$ that associates a value in $[0, 1]$
  to each individual combination of values $<x , y , z>$

  Given that $X$, $Y$ e $Z$ define a partition of $W$:

$$\sum_x \sum_y \sum_z P(X=x, Y=y, Z=z) = 1$$

# Random variables: notation

- Random variables, events and $\sigma$-algebras

(*sometimes the notation can be ambiguous*)

Examples:

$$P(X)$$

*This is the probability measure over the* $\sigma$-algebra *generated by random variable* $X$

$$P(X = x)$$

*This the probability (i.e. a value in* $[0,1]$ *) associated to the* event $X = x$

$$P(X, Y = y)$$

*This is the probability measure over the* $\sigma$-algebra *generated by random variable* $X$ *in the subspace of* $W$ *corresponding to the* event $Y = y$

# Marginalization

*Removing a random variable from a joint distribution*

Given a joint probability distribution

$$P(X = x, Y = y)$$

The *marginal probability* $P(X = x)$ is obtained via summation:

$$P(X = x) \; = \; \sum_y P(X = x, Y = y)$$

A marginal probability, in general, is still a joint probability

Most of times, the annotation is shortened as

$$P(X = x) \; = \; \sum_Y P(X = x, Y)$$

and, for the corresponding *measure*:

$$P(X) \; = \; \sum_Y P(X, Y)$$

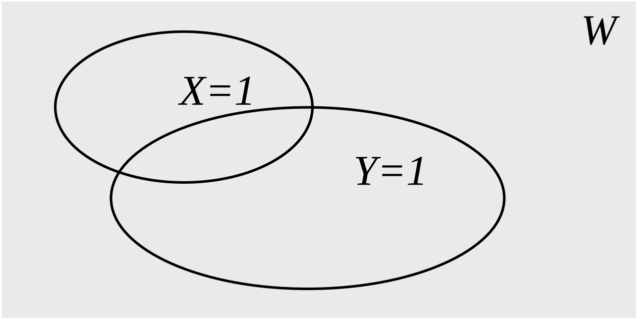
# Conditional probability

- Definition

$$P(X|Y = y) \;:=\; \frac{P(X, Y = y)}{P(Y = y)}$$

It is a form of *inference*: from a set $W$ to a set $W'$

*Therefore, from a probability space to another probability space*




Example: $W$ is the set of possible worlds, $X$, $Y$ are binary random variables and $P(X, Y)$ is the joint probability distribution

Suppose the agent learns that event $Y=1$ has occurred: the event $Y=0$ is now *impossible* (to him/her)

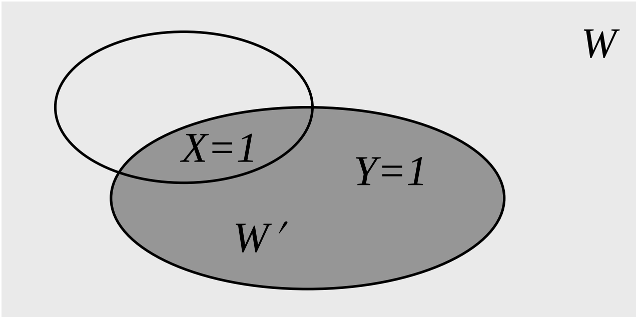
$W' \;:=\; \{w \in W | Y = 1\}$ is the new set of possible worlds

$P(X|Y = 1)$ is the new probability of $X$

More in general

$$P(X|Y) \;:=\; \frac{P(X, Y)}{P(Y)}$$

Denotes the conditional probabilities for the whole $\sigma$-algebra of events generated by $Y$



# Bayes' Theorem (T. Bayes, 1764)

- Definition

  A relation between conditional and marginal probabilities

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

$P(Y \mid X)$ is also called *likelihood* $L(X \mid Y)$

The theorem follows from the definition of conditional probability *(chain rule)*

$$P(X,Y) = P(X|Y)P(Y) = P(Y|X)P(X)$$

Furthermore, given the definition of marginalization:

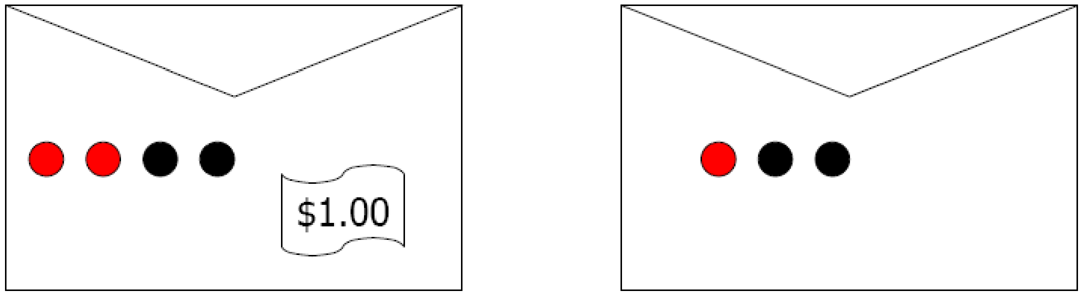
$$P(Y) = \sum_X P(X,Y) = \sum_X P(Y|X)P(X)$$

*Also called 'law of total probability'*

it follows an alternative formulation of the Bayes' theorem:

$$P(X|Y) = \frac{P(Y|X)P(X)}{\sum_X P(Y|X)P(X)}$$

# Example: information and bets




- Two envelopes, only one is extracted

  One envelope contains two red tokens and two black tokens, it is worth $1.00
  One envelope contains one red token and two black tokens, it is valueless

  The envelope has been extracted.

  Before posing you bet, you are allowed to extract on token from it

  a) The token is black. How much do you bet ?

  b) The token is red. How much do you bet ?

Purpose: showing that Bayes' Theorem makes the representation easier

# Independence, conditional independence

- **Independence** (also *marginal independence*)

  Two events are independent iff their joint probability is equal to the product of the marginals

$$< X \perp Y > \;\Rightarrow\; P(X,Y) = P(X)P(Y)$$

$$\Rightarrow\; P(X|Y) = \frac{P(X,Y)}{P(Y)} = \frac{P(X)P(Y)}{P(Y)} = P(X)$$

- **Conditional independence**

  Two events are conditional independent, given a third event, iff their joint conditional probability is equal to the product of the *conditional marginals*

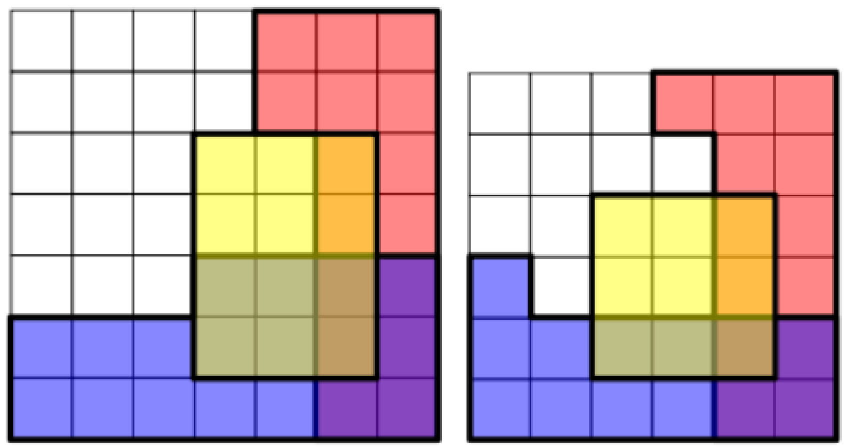
$$< X \perp Y \,|\, Z > \;\Rightarrow\; P(X,Y|Z) = P(X|Z)P(Y|Z)$$

$$\Rightarrow\; P(X|Y,Z) = \frac{P(X,Y|Z)}{P(Y|Z)} = \frac{P(X|Z)P(Y|Z)}{P(Y|Z)} = P(X|Z)$$

CAUTION: *the two forms of independence are distinct!*

$$< X \perp Y > \;\not\Rightarrow\; < X \perp Y \,|\, Z >, \quad < X \perp Y \,|\, Z > \;\not\Rightarrow\; < X \perp Y >$$

# Independence, conditional independence



These are two examples illustrating **conditional independence**. Each cell represents a possible outcome. The events $R$, $B$ and $Y$ are represented by the areas shaded red, blue and yellow respectively. And the probabilities of these events are shaded areas with respect to the total area. In both examples $R$ and $B$ are conditionally independent given $Y$ because:

$$\Pr(R \cap B \mid Y) = \Pr(R \mid Y)\Pr(B \mid Y)$$ [1]

but not conditionally independent given not $Y$ because:

$$\Pr(R \cap B \mid \text{not } Y) \neq \Pr(R \mid \text{not } Y)\Pr(B \mid \text{not } Y).$$

[from Wikipedia, "Conditional Independence"]

*$R$, $B$ and $Y$ here are subsets, i.e. <u>events</u>, not random variables*

*The example above shows that (conditional) independence of two specific <u>events</u> does NOT imply (conditional) independence of the whole $\sigma$-algebras*

# Probabilistic Inference (no *learning*)

- General setting

  The starting point is a fully-specified joint probability distribution

  $P(X_1, X_2, \ldots, X_n)$

  In an *inference* problem, the set of random variables $\{X_1, X_2, \ldots, X_n\}$ is divided into three categories:

  1) *Observed variables* $\{X_o\}$ , i.e. having a definite (and certain) value
  2) *Irrelevant variables* $\{X_i\}$ , i.e. which are not directly part of the answer
  3) *Relevant variables* $\{X_r\}$ , i.e. which are part of the answer we seek

  In general, the problem is finding:

$$P(\{X_r\}|\{X_o\}) = \sum_{\{X_i\}} P(\{X_r\}, \{X_i\}|\{X_o\})$$

  - "Decidability" (actually "computability") is not an issue (*in a finitary setting)

    Given that the joint probability distribution is completely specified
  - Computational efficiency can be a problem

    The number of value combinations grows exponentially with the number of random variables

# Continuous random variables (hint)

*Although conceptually the same, dealing with continuous random variables is technically difficult*

Consider a continuous random variable $X \in \mathcal{X}$ — *A continuous domain e.g. the real interval* $[0, 1]$

$X = x$ does not describe a proper *event*

Again for technical reasons (i.e. *measurability*) this must have probability zero

$X \leq a \quad X \leq b \qquad a < X \leq b$

(where $a < b$) these are *subsets* are proper *events* *(i.e. they may have non-zero probability)*

$$P(X \leq b) = P(X \leq a) + P(a < X \leq b)$$

*These two events are disjoint*

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a)$$

*Assume that the* derivative $p(X) := \dfrac{dP(X)}{dX}$ *exists*

*cumulative distribution function (cdf)*

$$P(a < X \leq b) = \int_a^b p(X)\, dX$$

*probability density function (pdf)*

# Expected value of a random variable

(also *expectation*)

## Basic definition

$$\mathbb{E}_X[X] := \sum_{x \in \mathcal{X}} x \; P(X = x)$$

*More concise notation*

$$\mathbb{E}[X] := \sum_{x \in \mathcal{X}} x \; P(x)$$

*Continuous case*

$$\mathbb{E}[X] := \int_{x \in \mathcal{X}} x \; p(x) dx$$

## A linear operator

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\mathbb{E}[cX] = c\mathbb{E}[X]$$

## Conditional expectation

$$\mathbb{E}_X[X|Y = y] = \mathbb{E}[X|Y = y] := \sum_{x \in \mathcal{X}} x \; P(X = x|Y = y)$$

## Iterated expectation (*see Wikipedia*)

$$\mathbb{E}_X[X] = \mathbb{E}_Y[\mathbb{E}_X[X|Y]]$$

# Variance of a random variable

Basic definition

$$\mathrm{Var}(X) := \mathbb{E}_X[(X - \mathbb{E}_X[X])^2] = \mathbb{E}_X[(X - \mu_X)^2]$$

where $\mu_X := \mathbb{E}_X[X]$

$$\mathrm{Var}(X) := \sum_{x \in \mathcal{X}} P(X = x)\ (x - \mu)^2$$

variance is *not* a linear operator

Conditional variance

$$\mathrm{Var}(X|Y = y) := \mathbb{E}_X[(X - \mathbb{E}_X[X|Y = y])^2\ |Y = y]$$

Variance lemma

$$\begin{aligned}\mathrm{Var}(X) = \mathbb{E}[(X - \mu_X)^2] &= \mathbb{E}[X^2] - 2\mu_X \mathbb{E}[X] + \mu_X^2 \\ &= \mathbb{E}[X^2] - 2\mu_X^2 + \mu_X^2 = \mathbb{E}[X^2] - \mu_X^2\end{aligned}$$

$$\mathbb{E}[X^2] = \mu_X^2 + \sigma_X^2$$

where $\sigma_X := \sqrt{\mathrm{Var}(X)}$ *standard deviation*