First-Order Resolution

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Propositional Resolution

A decision method for $\Gamma \models \varphi$

a) Refutation $\Gamma \cup \{\neg\varphi\}$ and translation into *conjunctive normal form* (CNF)
   $\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_n$ where each β_i is a disjunction of literals (i.e. A or $\neg A$)

b) Translation of $\Gamma \cup \{\neg\varphi\}$ in *clausal form* (CF)
   $\{\beta_1, \beta_2, \ldots, \beta_n\}$ where each β_i is a *clause* (i.e. a set of literals, representing a disjunction)

c) Exhaustive application of the resolution rule
   1) Selection of two clauses $\{\beta_1, \beta_2, \ldots, \beta_n, \alpha\}, \{\neg\alpha, \gamma_1, \gamma_2, \ldots, \gamma_m\}$
   2) Generation of the *resolvent*
      $\{\beta_1, \beta_2, \ldots, \beta_n, \alpha\}, \{\neg\alpha, \gamma_1, \gamma_2, \ldots, \gamma_m\} \vdash \{\beta_1, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2, \ldots, \gamma_m\}$

Termination conditions:

1) The empty clause has been derived (*success*)
2) No further resolutions are possible – *fixed point* (*failure*)

Clausal Form in L_{FO}

1) Refutation: $\Gamma \cup \{\neg\varphi\}$

2) Translation into *prenex normal form* (PNF):

   All wff are now in the form:

   $Qx_1 Qx_2 \ldots Qx_n \psi$ (the *matrix* ψ does not contain quantifiers)

3) Removal of all existential quantifiers - *skolemization*:

   All wff are now in the form:

   $\forall x_1 \forall x_2 \ldots \forall x_m \chi$ (the *skolemized matrix* χ does not contain quantifiers)

   Given that all wffs are universal sentences, the universal quantifiers can just be omitted

Example:

1: $\forall x\, (P(x) \rightarrow (\exists y\, Q(x,y) \wedge R(y)))$	
2: $\forall x\, (\neg P(x) \vee (\exists y\, Q(x,y) \wedge R(y)))$	(removing $\rightarrow$)
2: $\forall x\, \exists y\, (\neg P(x) \vee (Q(x,y) \wedge R(y)))$	(PNF)
3: $\forall x\, (\neg P(x) \vee (Q(x, k(x)) \wedge R(k(x))))$	(Skolemization, with a new function $k/1$)
4: $\neg P(x) \vee (Q(x, k(x)) \wedge R(k(x)))$	(omitting universal quantifiers)

Just atoms, connectives and parentheses…

Clausal Form in L_{FO}

1) Refutation: $\Gamma \cup \{\neg\varphi\}$
2) Translation into PNF :

   All wff are now in the form:

   $Qx_1Qx_2 \ldots Qx_n\psi$ (the *matrix* ψ does not contain quantifiers)
3) Removal of all existential quantifiers - *skolemization*:

   All wff are now in the form:

   $\forall x_1 \forall x_2 \ldots \forall x_m \chi$ (the *skolemized matrix* χ does not contain quantifiers)

   Given that all wffs are universal sentences, the universal quantifiers can just be omitted
4) The *clausal form* can be obtained by just treating *atoms* as *propositions* and applying the rules of propositional logic

   First translate in *conjunctive normal form* (CNF) and then in *clausal form* (CF)

Example:

5: $\neg P(x) \vee (Q(x, k(x)) \wedge R(k(x)))$ (from before)

6: $(\neg P(x) \vee Q(x, k(x))) \wedge (\neg P(x) \vee R(k(x)))$ (CNF, by distributing $\vee$)

7: $\{\neg P(x), Q(x, k(x))\}, \{\neg P(x), R(k(x))\}$ (*Clausal Form*)

Unificare necesse est, for resolution

- Problem: $\Gamma \models \varphi$?

  $\Gamma \equiv \{\forall x\ (Greek(x) \to Human(x)), \forall x\ (Human(x) \to Mortal(x)), Greek(socrates)\}$

  $\varphi \equiv Mortal(socrates)$

  Refutation, translation, clausal form:

  1: $\{\forall x\ (Greek(x) \to Human(x)), \forall x\ (Human(x) \to Mortal(x)), Greek(socrates),$
  $\neg Mortal(socrates)\}$

  ($\Gamma \cup \{\neg\varphi\}$ is already in PNF, no skolemization is needed)

  2: $\{\{Human(x), \neg Greek(x)\}, \{Mortal(x), \neg Human(x)\}, \{Greek(socrates)\},$
  $\{\neg Mortal(socrates)\}\}$

  (*Clausal Form*)

  Resolution method (attempt):

  3: Try resolving: $\{\neg Mortal(socrates)\}, \{Mortal(x), \neg Human(x)\}$

  Technically, no resolution is applicable: no pairs of complementary literals

  Intuitively though,
  the two literals $\neg Mortal(socrates)$ *and* $Mortal(x)$ *<u>are</u> complementary, somehow…*

Unification

Replacing variables with terms to render two atoms identical

- **Unifier**

  A substitution of variables with terms $\sigma = [x_1 = t_1, \mathrm{x}_2 = t_2 \dots x_n = t_n]$ that makes two complementary literals α and $\neg\beta$ *resolvable*

  That is, it makes the two atoms *identical*: $\sigma(\alpha) = \sigma(\beta)$

  - Obviously, a unifier does not necessarily exist:
    for instance $P(g(x, f(a)), a)$ and $\neg P(g(b, f(w)), k(w))$ are not unifiable

- MGU - *most general unifier*

  It is the minimal *unifier* of α and $\neg\beta$

  $$\text{MGU } \mu \Leftrightarrow \forall\, \sigma\, \exists\, \sigma' : \sigma = \mu \cdot \sigma'$$

  Any other unifier can be obtained as a composition of μ

Constructing the MGU

- Martelli and Montanari's algorithm

  Input: $[s_1 = t_1, s_2 = t_2 \ldots s_n = t_n]$ (a system of *symbolic* equations)

  Procedure:

  Exhaustive application of the following rules to the system of symbolic equations (each rule *transforms* the original system)

(1)	$f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)$	*replace by the equations* $s_1 = t_1, \ldots, s_n = t_n$,
(2)	$f(s_1, \ldots, s_n) = g(t_1, \ldots, t_m)$ where $f \neq g$	*halt with failure*, ⟵ Applies even when either *m* or *n* are 0 (i.e. with *constants*)
(3)	$x = x$	*delete the equation*,
(4)	$t = x$ where t is not a variable	*replace by the equation* $x = t$,
(5)	$x = t$ where x does not occur in t and x occurs elsewhere	*apply the substitution* $\{x/t\}$ *to all other equations*
(6)	$x = t$ where x occurs in t and x differs from t	*halt with failure.*

  Unless an explicit failure occurs (i.e. by rules (2) or (6)),
  the procedure terminates with success when no further rule is applicable

Constructing the MGU: examples

Example: $[f(x, a) = f(g(z), y), h(u) = h(d)]$

$[x = g(z), y = a, h(u) = h(d)]$	Rule (1) on $f(x, a) = f(g(z), y)$
$[x = g(z), y = a, u = d]$	Rule (1) on $h(u) = h(d)$, MGU

Example: $[f(x, a) = f(g(z), y), h(x, z) = h(u, d)]$

$[x = g(z), y = a, h(x, z) = h(u, d)]$	Rule (1) on $f(x, a) = f(g(z), y)$
$[x = g(z), y = a, h(g(z), z) = h(u, d)]$	Rule (5) on $x = g(z)$
$[x = g(z), y = a, u = g(z), z = d]$	Rule (1) on $h(g(z), z) = h(u, d)$
$[x = g(d), y = a, u = g(d), z = d]$	Rule (5) on $z = d$, MGU

Example: $[f(x, a) = f(g(z), y), h(x, z) = h(d, u)]$

$[x = g(z), y = a, h(x, z) = h(d, u)]$	Rule (1) on $f(x, a) = f(g(z), y)$
$[x = g(z), y = a, h(g(z), z) = h(d, u)]$	Rule (5) on $x = g(z)$
$[x = g(z), y = a, g(z) = d, z = u]$	Rule (2) on $g(z) = d$ FAILURE

Standardization of variables is also necessary

- Example: $\Gamma \models \varphi$? (transitive property - in clausal form)

  $\Gamma \equiv \{\{\neg C(x,y), \neg C(y,z), C(x,z)\}, \{C(a,b)\}, \{C(b,c)\}, \{C(c,d)\}\}$

  $\varphi \equiv \{C(a,d)\}$

Refutation and resolution:

1: $\{\{\neg C(x,y), \neg C(y,z), C(x,z)\}, \{C(a,b)\}, \{C(b,c)\}, \{C(c,d)\}, \{\neg C(a,d)\}\}$

2: Unify and resolve $\{\neg C(x,y), \neg C(y,z), C(x,z)\}$ and $\{\neg C(a,d)\}$:
$[x=a, z=d]$ with resolvent $\{\neg C(a,y), \neg C(y,d)\}$

3: Unify and resolve $\{\neg C(x,y), \neg C(y,z), C(x,z)\}$ and $\{\neg C(a,y), \neg C(y,d)\}$:
$[x=a, z=y]$ with resolvent $\{\neg C(a,y), \neg C(y,y), \neg C(y,d)\}$

4: *This way seems to lead nowhere*: $\neg C(y,y)$ *will never be resolved in* $\Gamma \cup \{\neg\varphi\}$

Why is this??

Standardization of variables is also necessary

- Example: $\Gamma \models \varphi$? (transitive property - in clausal form)

  $\Gamma \equiv \{\{\neg C(x,y), \neg C(y,z), C(x,z)\}, \{C(a,b)\}, \{C(b,c)\}, \{C(c,d)\}\}$

  $\varphi \equiv \{C(a,d)\}$

  Refutation and resolution, standardize variables before each resolution
  (i.e. rename all variables with new, unique names)

  1: $\{\{\neg C(x,y), \neg C(y,z), C(x,z)\}, \{C(a,b)\}, \{C(b,c)\}, \{C(c,d)\}, \{\neg C(a,d)\}\}$

  2: Unify and resolve $\{\neg C(x_1,y_1), \neg C(y_1,z_1), C(x_1,z_1)\}$ and $\{\neg C(a,d)\}$:
  $[x_1=a, z_1=d]$ with resolvent $\{\neg C(a, y_1), \neg C(y_1,d)\}$

  3: Unify and resolve $\{\neg C(x_2,y_2), \neg C(y_2,z_2), C(x_2,z_2)\}$ and $\{\neg C(a,y_3), \neg C(y_3,d)\}$:
  $[x_2=a, z_2=y_3]$ with resolvent $\{\neg C(a, y_2), \neg C(y_2, y_3), \neg C(y_3,d)\}$

  4: Unify and resolve $\{\neg C(a, y_4), \neg C(y_4, y_5), \neg C(y_5,d)\}$ and $\{C(a,b)\}$:
  $[y_4=b]$ with resolvent $\{\neg C(b, y_5), \neg C(y_5,d)\}$

  5: Unify and resolve $\{\neg C(b, y_5), \neg C(y_5,d)\}$ and $\{C(b,c)\}$:
  $[y_4=c]$ with resolvent $\{\neg C(c,d)\}$

  5: Resolve $\{\neg C(c,d)\}$ and $\{C(c,d)\}$:
  resolvent $\{\}$ (*success*)

Resolution with unification for L_{FO}

A correct procedure for $\Gamma \vdash \varphi$ in L_{FO}

a) Refutation $\Gamma \cup \{\neg\varphi\}$,

b) Prenex normal form and skolemization $sko(\Gamma \cup \{\neg\varphi\})$

c) Translation of $sko(\Gamma \cup \{\neg\varphi\})$ into CNF hence into CF

d) Repeat application of the resolution method:

1) Selection of two clauses $\{\beta_1, \beta_2, \ldots, \beta_n, \alpha\}, \{\neg\alpha', \gamma_1, \gamma_2, \ldots, \gamma_m\}$
2) *Standardization* of variables
(i.e. create new copies of the two clauses having new and unique variables)
3) Construction of the MGU μ (if it exists) for the two literals α e α'
4) Application generation of the resolvent with the application of μ
$\{\beta_1, \beta_2, \ldots, \beta_n, \alpha\}[\mu], \{\neg\alpha', \gamma_1, \gamma_2, \ldots, \gamma_m\}[\mu] \vdash \{\beta_1, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2, \ldots, \gamma_m\}[\mu]$

e) Until

1) The empty clause has been derived (*success*)
2) No further resolutions are possible – *fixed point* (*failure*)

Note: the method is not guaranteed to terminate (i.e. it might *diverge*)

The method might diverge…

Problem: $\forall x\,(Q(f(x)) \rightarrow P(x)) \models \exists x\,(P(f(x)) \wedge \neg Q(f(x)))$?

(*The answer is <u>negative</u>: there is no entailment*)

Refutation:

$\{\ \forall x\,(Q(f(x)) \rightarrow P(x))\ \} \cup \{\neg\exists x\,(P(f(x)) \wedge \neg Q(f(x)))\ \}$

Prenex normal form:

$\{\ \forall x\,(Q(f(x)) \rightarrow P(x))\ \} \cup \{\ \forall x\ \neg(P(f(x)) \wedge \neg Q(f(x)))\ \}$

(no *skolemization* required)

Clausal form:

$\{\ Q(f(x)) \rightarrow P(x)\ \} \cup \{\neg(P(f(x)) \wedge \neg Q(f(x)))\ \}$

$\{\neg Q(f(x)) \vee P(x)\ \} \cup \{\ \neg P(f(x)) \vee Q(f(x))\ \}$

$\{\{\neg Q(f(x)) \vee P(x)\}, \{\neg P(f(x)) \vee Q(f(x))\}\}$

Resolution:

1: $\{\neg Q(f(x_1)), P(x_1)\}, \{\neg P(f(x_2)), Q(f(x_2))\}, [x_1/f(x_2)] \vdash \{\neg Q(f(f(x_2))), Q(f(x_2))\}$

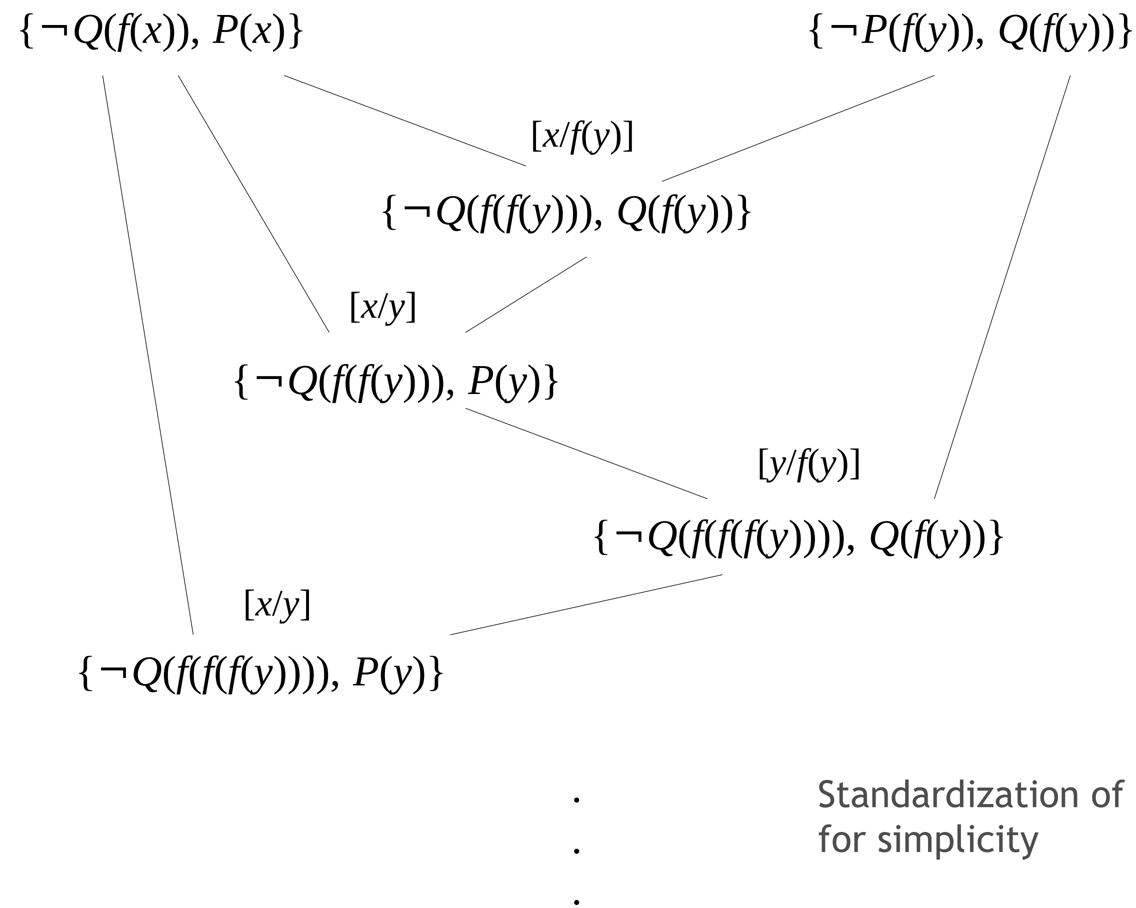
2: $\{\neg Q(f(x_3)), P(x_3)\}, \{\neg Q(f(f(x_4))), Q(f(x_4))\}, [x_3/x_4] \vdash \{\neg Q(f(f(x_4))), P(x_4)\}$

3: $\{\neg Q(f(f(x_5))), P(x_5)\}, \{\neg P(f(x_6)), Q(f(x_6))\}, [x_5/f(x_6)] \vdash \{\neg Q(f(f(f(x_6)))), Q(f(x_6))\}$

4: $\{\neg Q(f(x_7)), P(x_7)\}, , \{\neg Q(f(f(f(x_8)))), Q(f(x_8))\}, [x_7/x_8] \vdash \{\neg Q(f(f(f(x_8)))), P(x_8)\}$

…

The method might diverge…



Standardization of variables not applied here, for simplicity

Properties of resolution with unification

- The method is *correct* in L_{FO}

    If the method finds the empty clause for $sko(\Gamma \cup \{\neg\varphi\})$ then $\Gamma \models \varphi$

- Is the method *complete* in L_{FO}?

    Within the limits of semi-decidability, yes (Robinson, 1963)

    When $\Gamma \models \varphi$, the method will eventually find the empty clause for $sko(\Gamma \cup \{\neg\varphi\})$

    Very often (but not in the worst case) the method is more efficient than the one in the corollary of Herbrand's theorem

    The advantage is due to *lifting*
    (the method can resolve also non-ground clauses)

    When $\Gamma \not\models \varphi$, the method might diverge

    CAUTION: Unless the selection procedure is *fair* (more on this topic to follow) the method might diverge even when $\Gamma \models \varphi$

    Critical element:

    - Selecting the clauses and literals to be resolved