# Propositional Resolution

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# *Inference rule:* Resolution

$\varphi \vee \chi, \neg\chi \vee \psi \vdash \varphi \vee \psi$

$\varphi \vee \psi$ is also called the *resolvent* of $\varphi \vee \chi$ e $\neg\chi \vee \psi$

The resolution rule is *correct*

*In fact* $\varphi \vee \chi, \neg\chi \vee \psi \vdash \varphi \vee \psi \quad \Rightarrow \quad \varphi \vee \chi, \neg\chi \vee \psi \models \varphi \vee \psi$

| $\varphi$ | $\psi$ | $\chi$ | $\varphi \vee \chi$ | $\neg\chi \vee \psi$ | $\varphi \vee \psi$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

# Normal forms

= translation of each wff into an equivalent wff having a specific structure

- **Conjunctive Normal Form** (CNF)

  A wff with a structure

  $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n$

  where each $\alpha_i$ has a structure

  $(\beta_1 \vee \beta_2 \vee \ldots \vee \beta_n)$

  where each $\beta_j$ is a *literal* (i.e. an atomic symbol or the negation of an atomic symbol)

  Examples:

  $(B \vee D) \wedge (A \vee \neg C) \wedge C$

  $(B \vee \neg A \vee \neg C) \wedge (\neg D \vee \neg A \vee \neg C)$

- **Disjunctive Normal Form** (DNF)

  A wff with a structure

  $\beta_1 \vee \beta_2 \vee \ldots \vee \beta_n$

  where each $\beta_i$ has a structure

  $(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n)$

  where each $\alpha_j$ is a *literal*

# Conjunctive Normal Form

- Translation into CNF (it can be automated)

  Exhaustive application of the following rules:

  1) Rewrite $\rightarrow$ and $\leftrightarrow$ using $\wedge$, $\vee$, $\neg$

  2) Move $\neg$ inside composite formulae

     "De Morgan laws":
     $$\neg(\varphi \wedge \psi) \equiv (\neg\varphi \vee \neg\psi)$$
     $$\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$$

  3) Eliminate double negations: $\neg\neg$

  4) Distribute $\vee$
     $$((\varphi \wedge \psi) \vee \chi) \equiv ((\varphi \vee \chi) \wedge (\psi \vee \chi))$$

Examples:

$(\neg B \rightarrow D) \vee \neg(A \wedge C)$

| | |
|---|---|
| $B \vee D \vee \neg(A \wedge C)$ | (rewrite $\rightarrow$) |
| $B \vee D \vee \neg A \vee \neg C$ | (De Morgan) |

$\neg(B \rightarrow D) \vee \neg(A \wedge C)$

| | |
|---|---|
| $\neg(\neg B \vee D) \vee \neg(A \wedge C)$ | (rewrite $\rightarrow$) |
| $(B \wedge \neg D) \vee (\neg A \vee \neg C)$ | (De Morgan) |
| $(B \vee \neg A \vee \neg C) \wedge (\neg D \vee \neg A \vee \neg C)$ | (distribute $\vee$) |

# Clausal Forms

= each wff is translated into an equivalent <u>set</u> of wffs having a specific structure

- **Clausal Form** (CF)

  Starting from a wff in CNF

  $\alpha_1 \land \alpha_2 \land \ldots \land \alpha_n$

  the clausal form is simply the set of all *clauses*

  $\{\alpha_1, \alpha_2, \ldots , \alpha_n\}$

  Examples:

  $(B \lor D) \land (A \lor \neg C) \land C$

  $\{(B \lor D), (A \lor \neg C), C\}$

- **Special notation**

  Each clause is usually written as a *set*

  $\beta_1 \lor \beta_2 \lor \ldots \lor \beta_n$

  $\{\beta_1 , \beta_2 , \ldots , \beta_n\}$

  Example:

  $\{\{B, D\}, \{A, \neg C\}, \{C\}\}$

  **A set of *literals*:**
  ordering is irrelevant
  no multiple copies

# Resolution by refutation

- Algorithm

  Problem: "$\Gamma \vdash \varphi$" ?

  The problem is transformed into: is "$\Gamma \cup \{\neg\varphi\}$" *coherent*?

  If $\Gamma \vdash \varphi$ then $\Gamma \cup \{\neg\varphi\}$ is incoherent and therefore a contradiction can be derived

  $\Gamma \cup \{\neg\varphi\}$ is translated into CNF hence in CF

  The resolution algorithm is applied to the set of *clauses* $\Gamma \cup \{\neg\varphi\}$

  At each step:

  a) Select a pair of clauses $\{C_1,C_2\}$ containing a pair of *complementary literals* making sure that this combination has never been selected before

  b) Compute $C$ as the *resolvent* of $\{C_1,C_2\}$ according to the resolution rule.

  c) Add $C$ to the set of clauses

  Termination:

  When $C$ is the empty clause $\{ \}$

  or there are no more combinations to be selected in step a)

  Advantages:

  No axioms. Only one operation (i.e. the resolution rule).

# Resolution by refutation

- The same example as before

  $B \lor D \lor \neg A \lor \neg C,\ B \lor C,\ A \lor D,\ \neg B \vdash D$

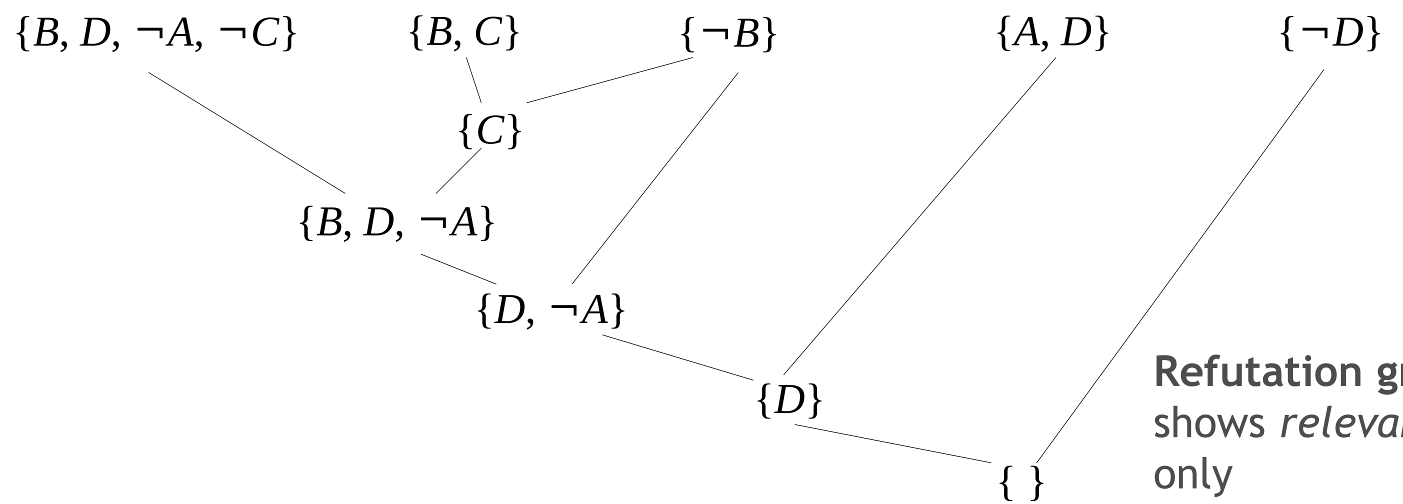
  Refutation + rewrite in CNF:

  $B \lor D \lor \neg A \lor \neg C,\ B \lor C,\ A \lor D,\ \neg B,\ \neg D$

  Rewrite in CF:

  $\{B, D, \neg A, \neg C\}, \{B, C\}, \{A, D\}, \{\neg B\}, \{\neg D\}$

  Applying the resolution rule:




**Refutation graph:** shows *relevant* resolutions only

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  Refutation + rewrite in CNF:

  $B \lor D \lor \neg A \lor \neg C, B \lor C, \ A \lor D, \ \neg B, \neg D$

  Rewrite in CF:

  $\{B, D, \neg A, \neg C\}, \{B, C\}, \{A, D\}, \{\neg B\}, \{\neg D\}$

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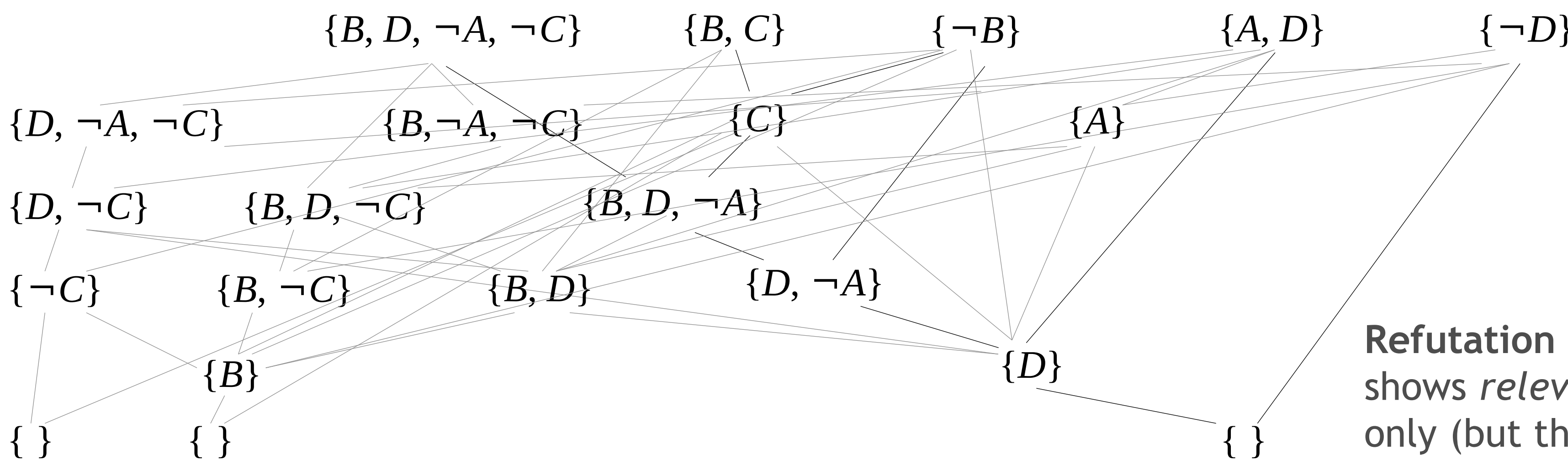


**Refutation graph:** shows *relevant* resolutions only (but there are more)

# Resolution by refutation

- Resolution by refutation for propositional logic

  Is correct: $\Gamma \vdash \varphi \;\Rightarrow\; \Gamma \models \varphi$

  Is complete: $\Gamma \models \varphi \;\Rightarrow\; \Gamma \vdash \varphi$

  In this sense: if $\Gamma \models \varphi$ then there exists a refutation graph

- Algorithm

  It is a decision procedure for the problem $\Gamma \models \varphi$

  It has time complexity $O(2^n)$

  where $n$ is the number of propositional symbols in $\Gamma \cup \{\neg\varphi\}$