Logical Calculus and Algorithms

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Decisions and decidability (automation)

- What is a *problem*?

  A *problem* is an association, i.e. a **relation** between *inputs* and *solutions*

  $K : \mathrm{I} \rightarrow \mathrm{S}$ (K is the relation, I is the input space, S is the solution space)

- *Search* problem

  Relation K associates each input to many solutions (i.e. one-to-many)

  Optimization problems

  A search problem plus an *objective* or *cost* function

  $c : \mathrm{S} \rightarrow \boldsymbol{R}$ (from S to $\boldsymbol{R}$, the set of real number)

  In general, the task is finding the solution(s) having maximal or minimal cost

- ***Decision*** problem

  The solution space S coincides with $\{0, 1\}$
  and K associates each input to a unique solution

  Example: $\Gamma \models \varphi$?

  The input space I contains all possible combinations of set Γ of wffs with individual wffs φ

Decisions and decidability (automation)

- ***Decidable*** problem

  A decision problem K which there exists an algorithm, more precisely a *Turing machine*

  (there are other ways of defining an algorithm or an *effective procedure*: they are all equivalent)

  that ***always terminates*** and produces the right answer in ***finite time***.

  Example of an *undecidable* problem: The *Halting Problem*

  Given the formal description of a particular Turing machine with a specific input, is it possible to tell if whether it will eventually halt or run forever?

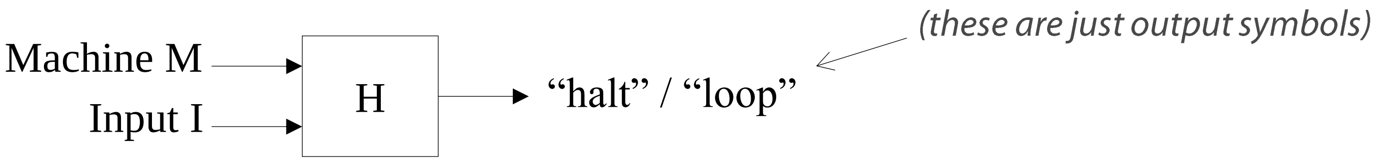
  In other words, does it exist a Turing machine that, given in input the description of *another* Turing machine, will always produce the answer desired?

  The answer is **no** (such a Turing machine *cannot* exist)

An aside: The *Halting Problem*

- Intuitive ideas behind the proof (i.e. of the *undecidability* of this problem)

  Let's assume there exists a Turing machine H that, given the description of a Turing machine M with input I always terminates producing an output "halt" or "loop" depending on whether M with input I will terminate or not



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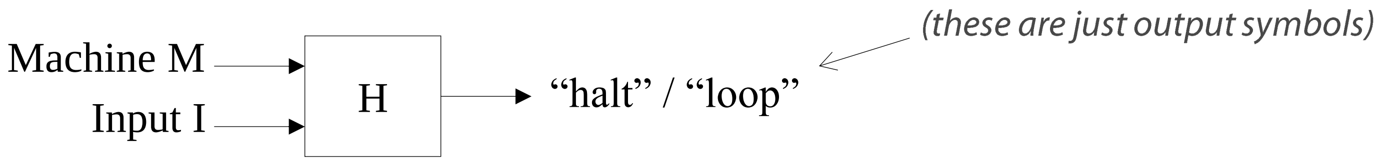
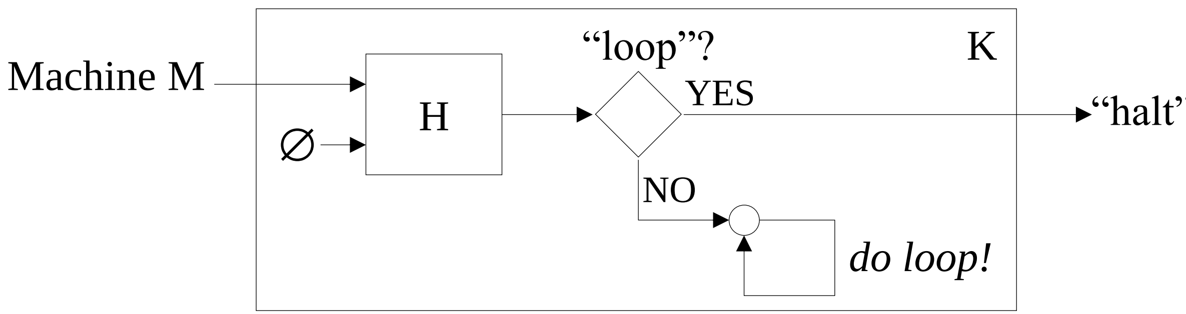


Assume H existed and consider the subclass of Turing machines having an empty input.
We can build another Turing machine K that enters an infinite loop whenever the output of H is "halt" and that terminates, with output "halt", when H outputs "loop"



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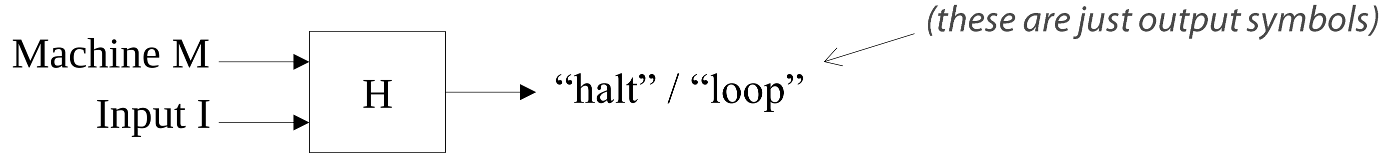

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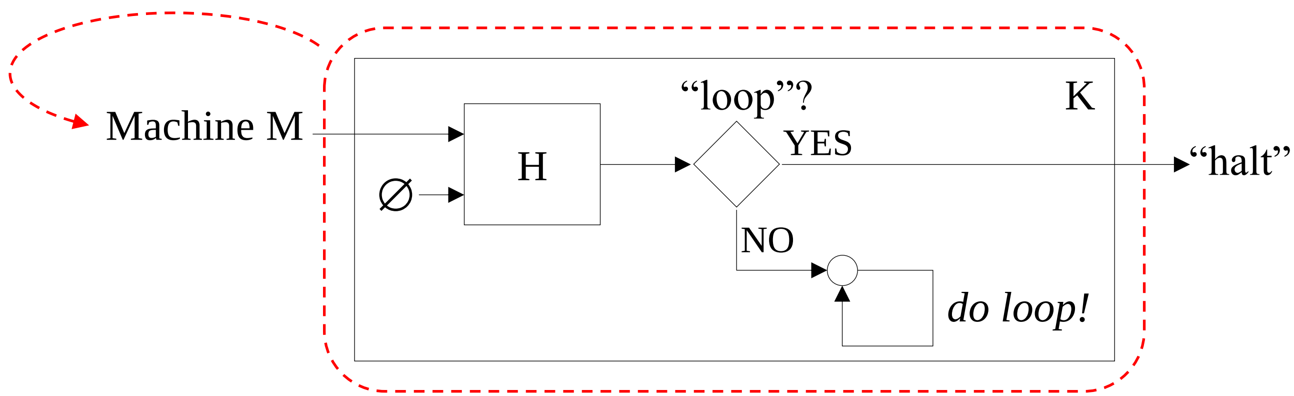
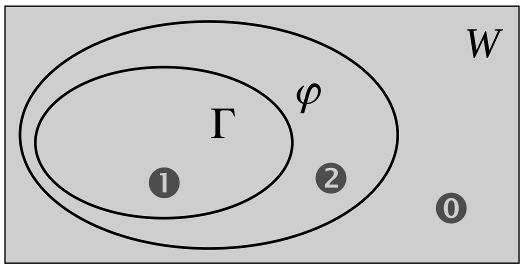


What will be the output of K when given K *itself* as the input?
K should *diverge* when K *terminates* and vice-versa: i.e. we have an absurdity

Transforming problems: entailment as satisfiability

- The decision problem " $\Gamma \models \varphi$? "
  can be transformed into a *satisfiability* problem

  In fact, $\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is *not* satisfiable



($w(\Gamma)$ is the set of possible worlds that satisfy Γ)

$\Gamma \models \varphi \;\Rightarrow\; w(\Gamma) \subseteq w(\{\varphi\})$ ❶ $\subseteq$ {❶, ❷}

$w(\{\neg\varphi\})$ = ⓿

$w(\Gamma \cup \{\neg\varphi\}) = w(\Gamma) \cap w(\{\neg\varphi\})$

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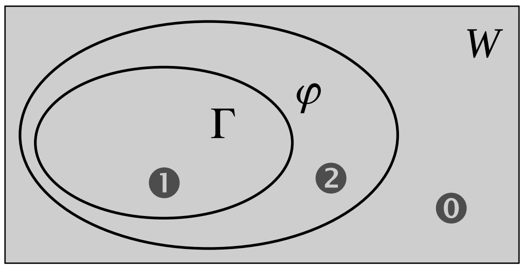

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- The decision problem "is $\Gamma \cup \{\neg\varphi\}$ satisfiable?"
  can be transformed into a wff *satisfiability* problem

  Taking this one step further, we can transform $\Gamma \cup \{\neg\varphi\}$ into *just one formula*:

$$\bigwedge (\Gamma \cup \{\neg\varphi\})$$

This is the wff obtained by combing all the wffs in $\Gamma \cup \{\neg\varphi\}$ with $\wedge$,
it is called the *conjunctive closure* of the set $\Gamma \cup \{\neg\varphi\}$

Satisfiability and decidability (in L_P)

- Is the decision problem "is the wff φ satisfiable?" *decidable*?

  It can be transformed into a *search* problem

  i.e. finding a possible world (in the set of all possible worlds) that satisfies φ

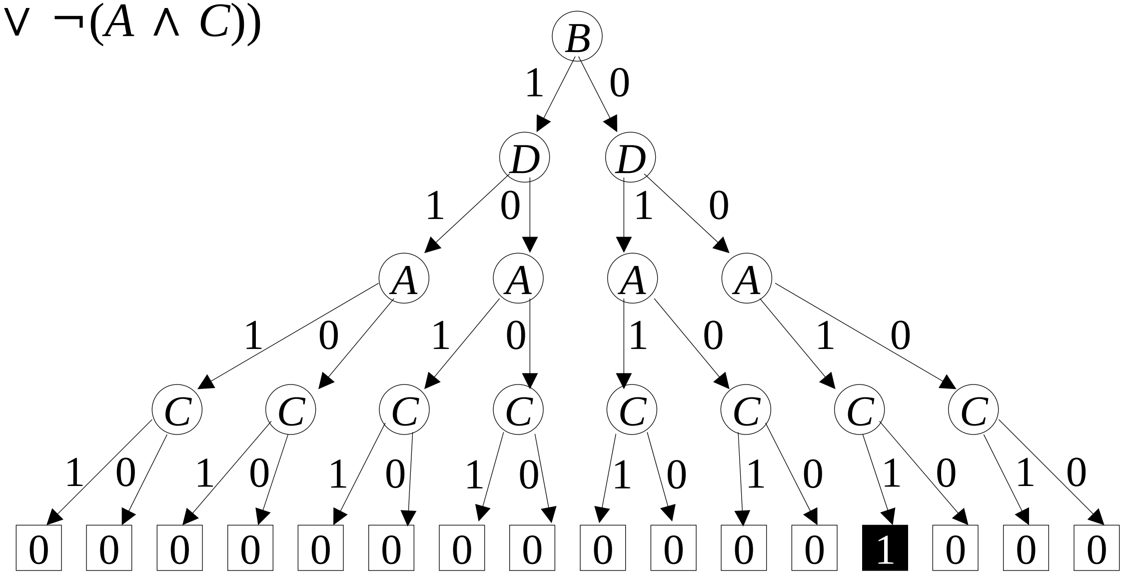
  In the scientific literature, this problem is called "SAT"

  Intuition: we can try every possible value assignment for the atoms in φ

Satisfiability and decidability (in L_P)

- Is the decision problem "is the wff φ satisfiable?" ***decidable***?
  - It can be transformed into a *search* problem
    - i.e. finding a possible world (in the set of all possible worlds) that satisfies φ
      - In the scientific literature, this problem is called "SAT"
    - *Intuition*: we can try every possible value assignment for the atoms in φ
  - Example:

$$\neg(B \vee D \vee \neg(A \wedge C))$$

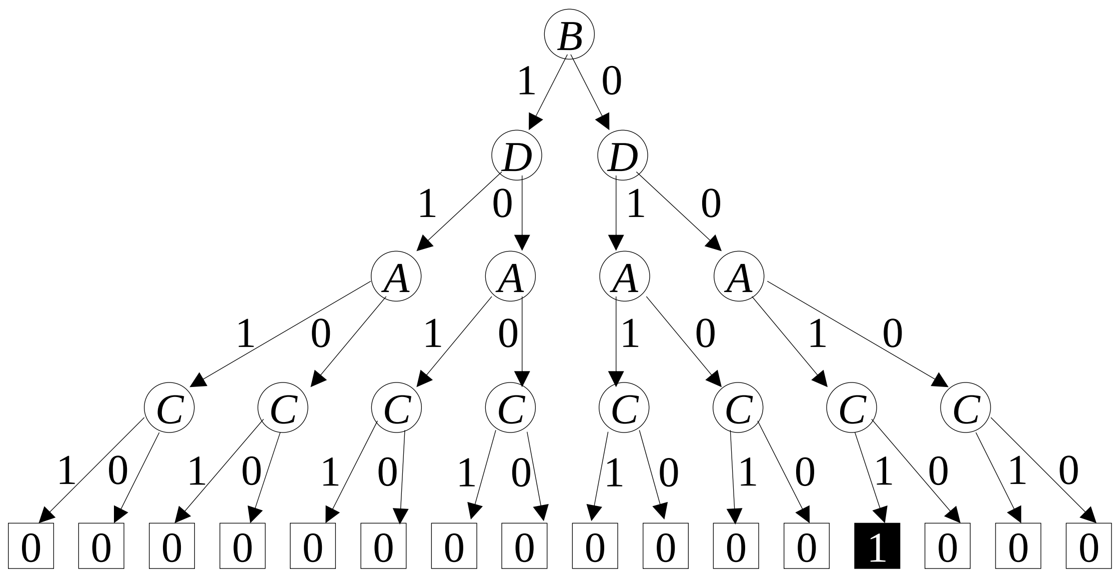


This method $O(2^n)$ time complexity, due to the number of value assignments

Example:

$\neg(B \wedge D \wedge \neg(A \wedge C))$ which is equivalent to
$(\neg B \vee \neg D \vee (A \wedge C))$

Each branch in the tree represents a possible assignment:



A real-world algorithm would stop here

The same algorithm is forced to try all possible assignments when ψ is *not* satisfiable.

For instance: $(\neg B \wedge \neg D \wedge \neg A \wedge \neg C)$

Computational complexity, classes P and NP

These notions apply to *decidable problems* only

It is based on the performances of a (known) Turing machine that gives the answer with respect to the *worst case* (i.e. the less favorable input for the specific problem)

- Time complexity

  The number of *steps* that the Turing machine requires for computing the answer, as a function of some numerical dimension of the input (e.g. the number of atoms in a wff)

- Memory complexity

  The number of *tape cells* that the Turing machine requires for computing the answer, as a function of some numerical dimension of the input

- Class P

  The class of problems for which there is a Turing machine that requires $O(\mathrm{P}(n))$ time

  where $\mathrm{P}(\ )$ is a polynomial of finite degree and n is the dimension of the (*worst-case*) input

- Class NP

  The class of all problems:

  a) A method for enumerating all possible answers (i.e. *recursive enumerability*)

  b) An algorithm in class P that *verifies* if a possible answer is also a *solution*

  It includes all problems in class P (that is, $\mathrm{P} \subseteq \mathrm{NP}$)

Class NP-complete and the SAT problem

- Class NP-complete

  It is a subclass of NP (NP-complete $\subseteq$ NP)

  A problem K is NP-complete if every problem in class NP is *reducible* to K

- Reducibility

  For class NP-complete

  Consider a problem K for which a decision algorithm $M(K)$ is known

  A problem J is *reducible* to K if there exist a decision algorithm $M(J)$ such that:

  a) algorithm $M(K)$ is called just once, as a “subroutine”, at the end of $M(J)$

  b) apart from $M(K)$, $M(J)$ has polynomial complexity

- The problem SAT

  Is NP-complete (*historically, it is the first one to be known*)

  Moral: if we had a polynomial decision algorithm for SAT, we would also have that

  $\mathrm{P} = \mathrm{NP}$

  This fact is not known, it is believed that: $\mathrm{P} \neq \mathrm{NP}$

  (and a lot will change in the digital world, if this proves to be false)

Semantic Tableau, alpha and beta rules

- *Semantic tableau* is a method

  which can be implemented as a Turing machine

- It is a decision algorithm for the problem "is Σ satisfiable?"

  where Σ is a set of wffs in $\boldsymbol{L_P}$

  In spite of its name, it is a *symbolic* method: it works on the structure of wffs only

    No explicit assignments of (semantic) values are involved

Semantic Tableau, alpha and beta rules

- A tableau is a set of wffs in $\boldsymbol{L_P}$

  The method starts from an *initial* tableau

  (i.e. the set Σ whose satisfiability is to be determined)

  It is based on rules that transform each one wff into two wffs

- Alpha rules (i.e. expansion)

(a1)	(a2)	(a3)	(a4)
$\neg(\neg\varphi)$	$\varphi \wedge \psi$	$\neg(\varphi \vee \psi)$	$\neg(\varphi \to \psi)$
φ	φ, ψ	$\neg\varphi, \neg\psi$	$\varphi, \neg\psi$

- Beta rules (i.e. bifurcation)

(b1)	(b2)	(b3)	(b4)	(b5)
$\varphi \vee \psi$	$\neg(\varphi \wedge \psi)$	$\varphi \to \psi$	$\varphi \leftrightarrow \psi$	$\neg(\varphi \leftrightarrow \psi)$
$\varphi \qquad \psi$	$\neg\varphi \qquad \neg\psi$	$\neg\varphi \qquad \psi$	$\neg\varphi, \neg\psi \qquad \varphi, \psi$	$\neg\varphi, \psi \qquad \varphi, \neg\psi$

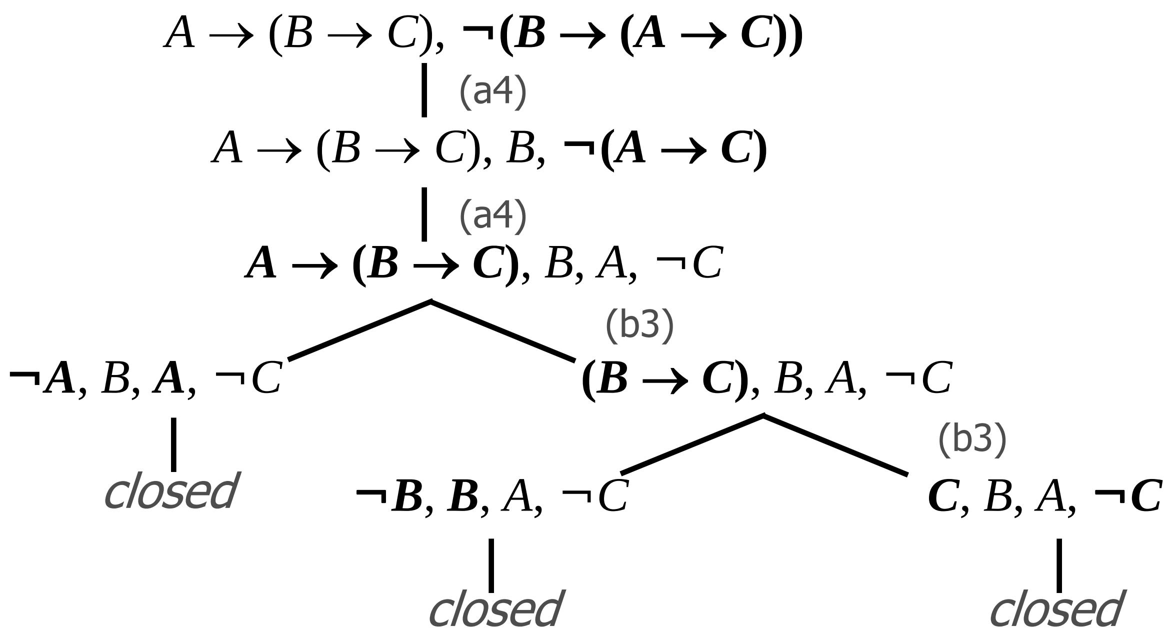
Semantic Tableau – a working example

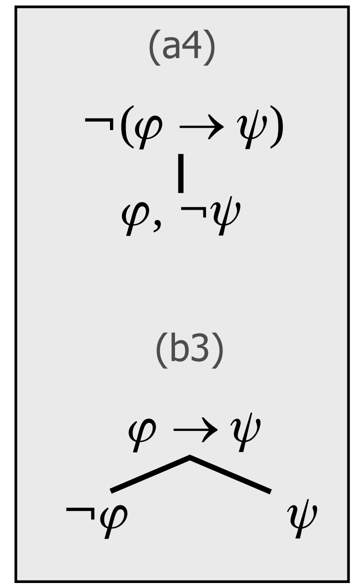
- Original problem: "$\Gamma \models \varphi$?"

  Example input: $A \to (B \to C) \models B \to (A \to C)$?

- Transformed problem: "is $\Gamma \cup \{\neg\varphi\}$ satisfiable?"

  Hence the initial tableau is $\Gamma \cup \{\neg\varphi\}$





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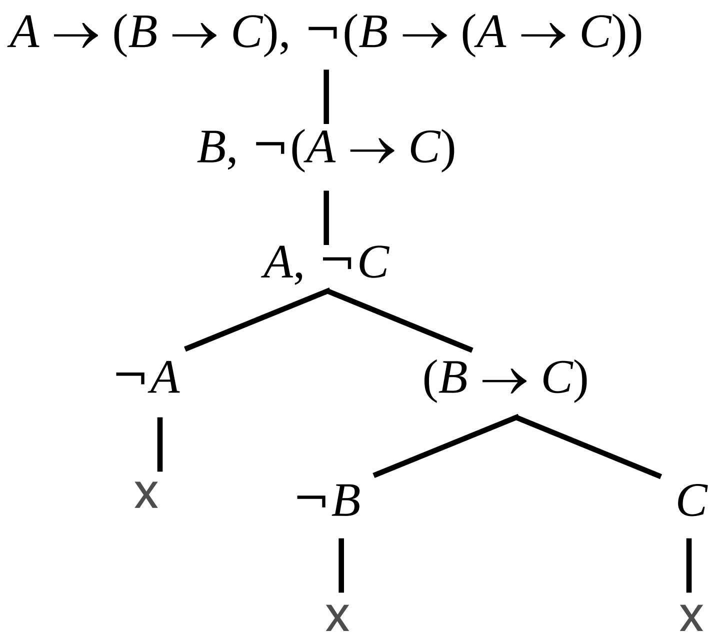

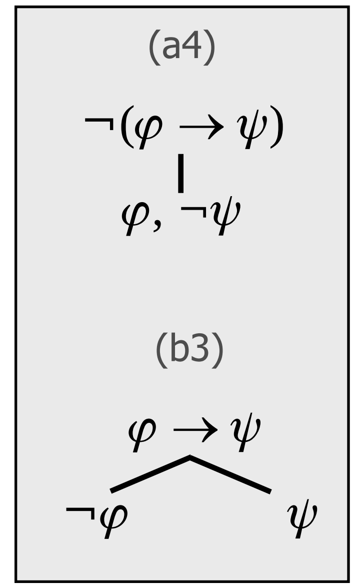

The usual notation in textbooks is even more concise:
only those wffs that are *added* to the initial tableau in each branch are shown in the tree

Semantic Tableau – algorithm recap

- Algorithm (informal description – see Lab for the implementation):

  Input problem: "$\Gamma \models \varphi$? "

  The input problem is transformed into "is $\Gamma \cup \{\neg\varphi\}$ satisfiable?"

  Methods of this type are also called '*by refutation*'

  For each active tableau (i.e. the *leaves* in the tree),

  There could be two cases:

  1) The tableau contains only *literals*
     If the tableau contains a *complementary pair of literals*
     then declare it *closed*
     else declare it *open* (i.e. failure)
  2) The tableau contains one or more *composite* wff
     First try to apply an *alpha* rule,
     otherwise, if this is not possible, try to apply a *beta* rule.
     In either case, two new tableau will be generated

  Output: the tree structure of tableau

Semantic Tableau – (required) algorithm properties

- **Termination**

  The algorithm never *diverges* (i.e. it never enters an infinite loop)

  Each application of either alpha or beta rule *simplifies* a wff (i.e. it makes it *less* composite): so the application of rules cannot continue forever

- ***Symbolic derivation***

  As already stated, in spite of its name, this is a *symbolic* method

  We write

  $$\Gamma \vdash_{ST} \varphi$$

  iff the *Semantic Tableau* method is successful (i.e. all leaves are *closed*) for $\Gamma \cup \{\neg\varphi\}$

  How do we know that $\Gamma \vdash_{ST} \varphi \;\Rightarrow\; \Gamma \models \varphi$?

  (*Soundness* - also *correctness* - of the method)

  Exercise: prove it
  (*hint*: consider the condition on $\Gamma \cup \{\neg\varphi\}$ and think about how it relates to each *rule*)

  How do we know that $\Gamma \models \varphi \;\Rightarrow\; \Gamma \vdash_{ST} \varphi$?

  (*Completeness* of the method)

  Proving it is definitely more difficult: see textbook (i.e. Ben-Ari)

Semantic Tableau – (required) algorithm properties

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- ***Soundness***

  $\Gamma \vdash_{ST} \varphi \;\Rightarrow\; \Gamma \models \varphi$

- **Completeness**

  $\Gamma \models \varphi \;\Rightarrow\; \Gamma \vdash_{ST} \varphi$

- **Termination + Soundness + Completeness = *Decision Algorithm***

  (for propositional logic)

Which method is faster?

- Time complexity (remember: consider the *worst case*)

  The `brute-force search' and *Semantic Tableau* have the same complexity : $O(2^n)$

- *How well do these method perform in practice?*

  It depends

Example 1 (try it)**:**

$$A \wedge B \wedge C \wedge \neg A$$

The `brute-force search' requires $2^3 = 8$ attempts

The Semantic Tableau method requires applying the same alpha rule 3 times

Example 2 (try it)**:**

$$(A \vee B) \wedge (A \vee \neg B) \wedge (\neg A \vee B) \wedge (\neg A \vee \neg B)$$

The `brute-force search' requires $2^2 = 4$ attempts

The Semantic Tableau method requires applying the same alpha rule 3 times; then the same beta rule is applied exhaustively producing a tree with 4 levels, with each node in a tree with a branching factor 2

At the end, the tree has $2^4 = 16$ leaves (all *closed* tableau)