

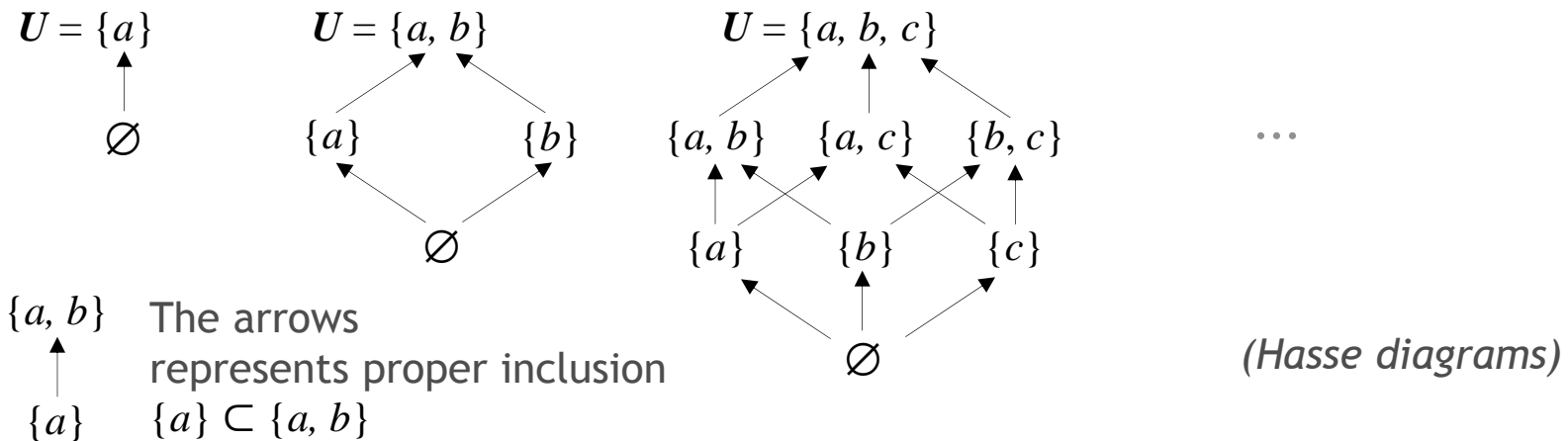
Propositional Logic

Marco Piastra

Boolean algebras by examples

Start from a *finite* set of objects U
and construct, in a *bottom-up fashion*, the collection X of all possible subsets of U

Examples:



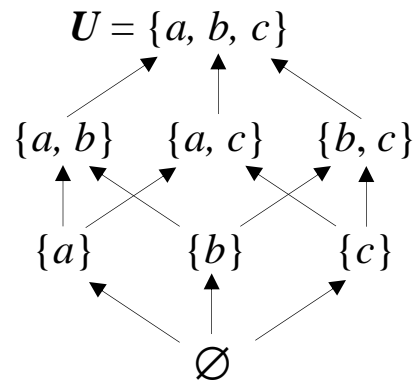
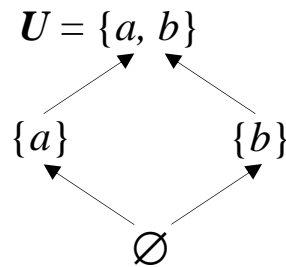
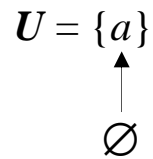
The collection X is also called the **power set** of U and is denoted as 2^U (i.e. $X = 2^U$)

Consider the operations $\cup, \cap, \setminus U$ (i.e. *union, intersection and absolute complement*):
the structure $\langle X, \cup, \cap, \setminus U, \emptyset, U \rangle$ is a **Boolean algebra**

Boolean algebras by examples

Start from a *finite* set of objects U
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Examples:



...

(Hasse diagrams)

Operations $\cup, \cap, \setminus U$ (union, intersection and absolute complement)

For these structures
properties
can be
checked
directly

$$A \cup A \setminus U = U$$

$$A = \{a\}$$

$$A \setminus U = \{b, c\}$$

$$A \cup A \setminus U = \{a, b, c\}$$

$$A \cap (A \cup B) = A$$

$$A = \{b\}$$

$$B = \{c\}$$

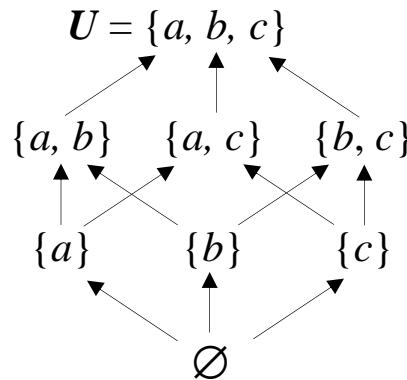
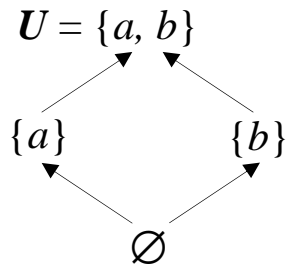
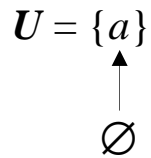
$$A \cup B = \{b, c\}$$

$$A \cap (A \cup B) = \{b\}$$

Boolean algebras by examples

Start from a *finite* set of objects U
and construct, in a *bottom-up fashion*, the collection X of all possible subsets of U

Examples:



...

(Hasse diagrams)

Operations $\cup, \cap, \setminus U$ (union, intersection and absolute complement)

For these structures
properties
can be
checked
directly

$$(A \cup B) \setminus U = A \setminus U \cap B \setminus U$$

$$A = \{b\}$$

$$A \setminus U = \{a, c\}$$

$$B = \{b, c\}$$

$$B \setminus U = \{a\}$$

$$A \cup B = \{b, c\}$$

$$(A \cup B) \setminus U = \{a\}$$

$$A \setminus U \cap B \setminus U = \{a\}$$

$$(A \cap B) \setminus U = A \setminus U \cap B \setminus U$$

$$A = \{b\}$$

$$A \setminus U = \{a, c\}$$

$$B = \{b, c\}$$

$$B \setminus U = \{a\}$$

$$A \cap B = \{b\}$$

$$(A \cap B) \setminus U = \{a, c\}$$

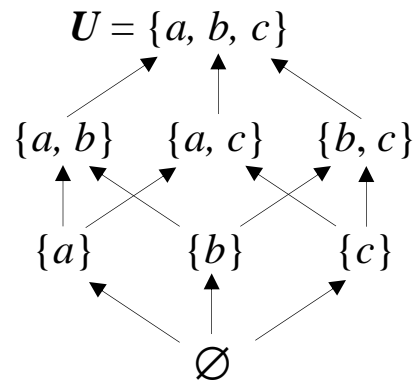
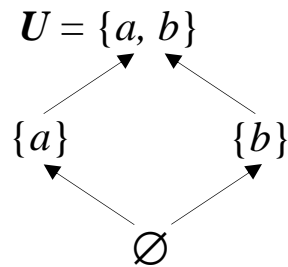
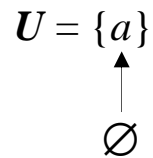
$$A \setminus U \cup B \setminus U = \{a, c\}$$

De Morgan's laws

Boolean algebras by examples

Start from a *finite* set of objects U
and construct, in a *bottom-up fashion*, the collection X of all possible subsets of U

Examples:



...

(Hasse diagrams)

Operations $\cup, \cap, \setminus U$ (union, intersection and absolute complement)

... but sometimes
we fail

$$A \setminus U \cup B = U$$

$$A = \{a\}$$

$$A \setminus U = \{b, c\}$$

$$B = \{b\}$$

$$A \setminus U \cup B = \{b, c\}$$

* Ouch!

This is NOT
true in general

It is only valid when
 $A \subseteq B$

Abstract Boolean Algebras

"This type of algebraic structure captures essential properties of both set operations and logic operations." [Wikipedia]

Boolean algebra

Any structure $\langle X, \cup, \cap, \setminus, \emptyset, U \rangle$ that has the following properties (for any $A, B, C \in X$):

$$A \cup A = A \cap A = A$$

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$\emptyset \cup A = A, \quad \emptyset \cap A = \emptyset, \quad U \cup A = U, \quad U \cap A = A$$

$$A \cup (A \setminus U) = U, \quad A \cap (A \setminus U) = \emptyset$$

idempotence

commutativity

associativity

absorption

distributivity

special elements

complement

Which Boolean algebra for logic?

* Given that all boolean algebras share the same properties (*see before*)
we can adopt the simplest one as reference, namely the one based on $\mathbf{X} = \{U, \emptyset\}$
i.e. a *two-valued* algebra: $\{\text{nothing}, \text{everything}\}$ or $\{\text{false}, \text{true}\}$ or $\{\perp, \top\}$ or $\{0, 1\}$

- Algebraic structure

$\langle \{0,1\}, OR, AND, NOT, 0, 1 \rangle$

- Boolean functions and *truth tables*

Boolean functions: $f: \{0, 1\}^n \rightarrow \{0, 1\}$

AND, OR and *NOT* are boolean functions, they are defined via *truth tables*

<i>A</i>	<i>B</i>	<i>OR</i>
0	0	0
0	1	1
1	0	1
1	1	1

<i>A</i>	<i>B</i>	<i>AND</i>
0	0	0
0	1	0
1	0	0
1	1	1

<i>A</i>	<i>NOT</i>
0	1
1	0

Composite functions

Truth tables can be defined also for composite functions

For example, to verify logical laws

De Morgan's laws

<i>A</i>	<i>B</i>	<i>NOT A</i>	<i>NOT B</i>	<i>A OR B</i>	<i>NOT(A OR B)</i>	<i>NOT A AND NOT B</i>
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

These columns are identical

Adequate basis

- How many *basic* boolean functions do we need to define *any* boolean function?

	A_1	A_2	...	A_n	$f(A_1, A_2, \dots, A_n)$
↑ 2^n rows ↓	0	0	...	0	f_1
	0	0	...	1	f_2

	1	1	...	1	f_{2^n}

Just *OR*, *AND* and *NOT*: any other function can be expressed as composite function

In the generic *truth table* above:

- For each row where $f = 1$, we compose by *AND* the n input variables taking either A_i when the i -th value is 1, or $\neg A_i$ when i -th value is 0
- We compose by *OR* all the composed expression obtained in the previous step

Other adequate basis

Also $\{OR, NOT\}$ o $\{AND, NOT\}$ sono basi adeguate

An adequate basis can be obtained by just one 'ad hoc' function: *NOR* or *NAND*

<i>A</i>	<i>B</i>	<i>A NOR B</i>
0	0	1
0	1	0
1	0	0
1	1	0

<i>A</i>	<i>B</i>	<i>A NAND B</i>
0	0	1
0	1	1
1	0	1
1	1	0

- Two remarkable functions: *implication* and *equivalence*

Logicians prefer the basis $\{IMP, NOT\}$

<i>A</i>	<i>B</i>	<i>A IMP B</i>
0	0	1
0	1	1
1	0	0
1	1	1

<i>A</i>	<i>B</i>	<i>A EQU B</i>
0	0	1
0	1	0
1	0	0
1	1	1

Identities: $A IMP B = NOT A OR B$

$$A EQU B = (A IMP B) AND (B IMP A)$$

Propositional logic

i.e. the simplest of 'classical' logics

■ Propositions

We consider all *possible worlds* that can be described via atomic **propositions**

"Today is Friday"

"Turkeys are birds with feathers"

"Man is a featherless biped"

■ Formal *language*

A precise and formal language in which *propositions* are the **atoms**
(i.e. no intention to represent the internal structure of *propositions*)

Atoms can be composed in complex formulae via **logical connectives**

■ Formal *semantics*

A class of formal structures, each representing a *possible world*

Fundamental: in each *possible world*, each formula of the language is either *true* or *false*

- *Atoms* are given a *truth value* (i.e. *false*, *true*)
- Logical connectives are associated to *boolean functions*: each *formula* corresponds to a functional composition in which *atoms* are the arguments (*truth-functionality*)

The class of propositional, semantic structures

They will define the meaning of the formal language (to be defined)

Each possible world is a structure $\langle \{0,1\}, \mathbf{P}, \nu \rangle$

$\{0,1\}$ are the *truth values*

\mathbf{P} is the **signature** of the formal language: a set of propositional symbols

ν is a *function*: $\mathbf{P} \rightarrow \{0,1\}$ assigning truth values to the symbols in \mathbf{P}

Propositional symbols (*signature*)

Each symbol in \mathbf{P} stands for an actual *proposition* (in natural language)

In the simple convention, we use the symbols A, B, C, D, \dots

Caution: \mathbf{P} is not necessarily *finite*

Possible worlds

The class of structures contains all possible worlds:

$\langle \{0,1\}, \mathbf{P}, \nu \rangle$

$\langle \{0,1\}, \mathbf{P}, \nu' \rangle$

$\langle \{0,1\}, \mathbf{P}, \nu'' \rangle$

...

Each class of structure shares \mathbf{P} and $\{0,1\}$

The functions ν are different: the assignment of truth values varies, depending on the possible world

If \mathbf{P} is finite, there are only *finitely* many distinct possible worlds (actually $2^{|\mathbf{P}|}$)

Propositional *language*

i.e. how we describe the world, by propositions

- In a propositional language L_P

A set P of propositional symbols: $P = \{A, B, C, \dots\}$

Two (primary) **logical connectives**: \neg, \rightarrow

Three (derived) **logical connectives**: $\wedge, \vee, \leftrightarrow$

Parenthesis: $(,)$ (there are no *precedence rules* in this language)

- Well-formed formulae (**wff**)

A set of syntactic rules

The set of all the **wff** of L_P is denoted as $\text{wff}(L_P)$

$A \in P \Rightarrow A \in \text{wff}(L_P)$

$\varphi \in \text{wff}(L_P) \Rightarrow (\neg\varphi) \in \text{wff}(L_P)$

$\varphi, \psi \in \text{wff}(L_P) \Rightarrow (\varphi \rightarrow \psi) \in \text{wff}(L_P)$

$\varphi, \psi \in \text{wff}(L_P) \Rightarrow (\varphi \vee \psi) \in \text{wff}(L_P), \quad (\varphi \vee \psi) \Leftrightarrow ((\neg\varphi) \rightarrow \psi)$

$\varphi, \psi \in \text{wff}(L_P) \Rightarrow (\varphi \wedge \psi) \in \text{wff}(L_P), \quad (\varphi \wedge \psi) \Leftrightarrow (\neg(\varphi \rightarrow (\neg\psi)))$

$\varphi, \psi \in \text{wff}(L_P) \Rightarrow (\varphi \leftrightarrow \psi) \in \text{wff}(L_P), \quad (\varphi \leftrightarrow \psi) \Leftrightarrow ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$

Semantics: interpretations

- Composite (i.e. *truth-functional*) semantics for wffs

Given a possible world $\langle \{0,1\}, \mathbf{P}, \nu \rangle$

the function $\nu : \mathbf{P} \rightarrow \{0,1\}$ can be extended to assign a value to every wff

Each logical connective is associated to a binary (i.e. *boolean*) function:

$$\nu(\neg\varphi) = NOT(\nu(\varphi))$$

$$\nu(\varphi \wedge \psi) = AND(\nu(\varphi), \nu(\psi))$$

$$\nu(\varphi \vee \psi) = OR(\nu(\varphi), \nu(\psi))$$

$$\nu(\varphi \rightarrow \psi) = OR(NOT(\nu(\varphi)), \nu(\psi)) \quad (\text{also } IMP(\nu(\varphi), \nu(\psi)) \text{)}$$

$$\nu(\varphi \leftrightarrow \psi) = AND(OR(NOT(\nu(\varphi)), \nu(\psi)), OR(NOT(\nu(\psi)), \nu(\varphi)))$$

- Interpretations

Function ν (extended as above) assigns a truth value to each $\varphi \in \text{wff}(L_P)$

$$\nu : \text{wff}(L_P) \rightarrow \{0,1\}$$

Then ν is said to be an *interpretation* of L_P

Note that the truth value of any wff φ is univocally determined by the values assigned to each symbol in the *signature* \mathbf{P}

Sometimes we will use just ν instead of $\langle \{0,1\}, \mathbf{P}, \nu \rangle$

Satisfaction, models

■ Possible worlds and *truth tables*

Examples: $\varphi = (A \vee B) \wedge C$

Different rows
different worlds

Caution: in each possible world
every $\varphi \in \text{wff}(L_p)$ has a truth value

A	B	C	$A \vee B$	$(A \vee B) \wedge C$
0	0	0	0	0
0	0	1	0	0
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

A possible world **satisfies** a wff φ iff $v(\varphi) = 1$

We also write $\langle \{0,1\}, \mathbf{P}, v \rangle \models \varphi$

In the truth table above, the rows that satisfy φ are in gray

Such possible world v is also said to be a **model** of φ

By extension, a possible world *satisfies* (i.e. is *model of*) a set of wff $\Gamma = \{\varphi_1, \varphi_2, \dots, \varphi_n\}$ iff v *satisfies* (i.e. is *model of*) each of its wff $\varphi_1, \varphi_2, \dots, \varphi_n$

Sometimes we will use $v \models \Gamma$ instead of $\langle \{0,1\}, \mathbf{P}, v \rangle \models \Gamma$

Tautologies, contradictions

■ A tautology

Is a (propositional) wff
that is always satisfied

It is also said to be **valid**

Any wff of the type $\varphi \vee \neg\varphi$
is a tautology

■ A contradiction

Is a (propositional) wff,
that cannot be satisfied

Any wff of the type $\varphi \wedge \neg\varphi$
is a contradiction

Note:

- Not all wffs are either tautologies or contradictions
- If φ is a tautology then $\neg\varphi$ is a contradiction and vice-versa

A	$A \wedge \neg A$	$A \vee \neg A$
0	0	1
1	0	1

A	B	$(\neg A \vee B) \vee (\neg B \vee A)$
0	0	1
0	1	1
1	0	1
1	1	1

A	B	$\neg((\neg A \vee B) \vee (\neg B \vee A))$
0	0	0
0	1	0
1	0	0
1	1	0

Formulae and subsets

- Consider the set W of all possible worlds

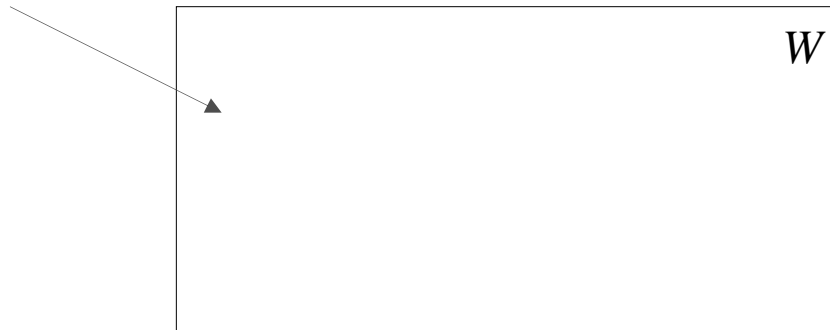
Each wff of L_P corresponds to a **subset** of W

i.e. the subset of possible worlds that *satisfy* it

For example, φ corresponds to $\{v : v(\varphi) = 1\}$ (it can be written also as $\{v : v \models \varphi\}$)

The corresponding subset may be empty (i.e. if φ is a contradiction)
or it may coincide with W (i.e. if φ is a tautology)

The set of all
possible worlds



Formulae and subsets

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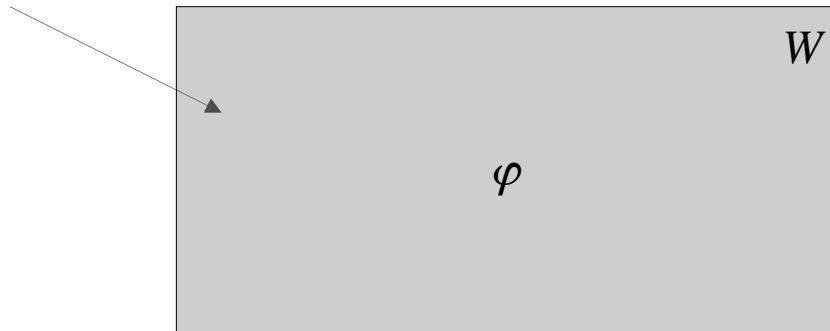
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The set of all possible worlds



“ φ is a tautology”

“any possible world in W is a *model* of φ ”

“ φ is (logically) *valid*”

Furthermore:

“ φ is *satisfiable*”

“ φ is not *falsifiable*”

Formulae and subsets

- Consider the set W of all possible worlds

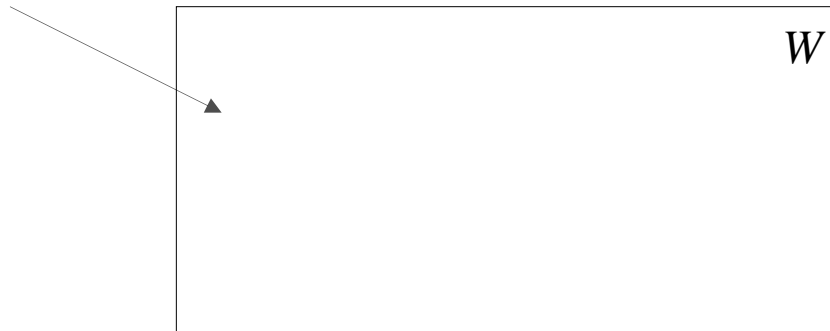
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The corresponding subset may be empty (i.e. if φ is a contradiction)
or it may coincide with W (i.e. if φ is a tautology)

The set of all possible worlds



“ φ is a contradiction”

“none of the possible worlds in W is a *model* of φ ”

“ φ is not (logically) *valid*”

Furthermore:

“ φ is not *satisfiable*”

“ φ is *falsifiable*”

Formulae and subsets

- Consider the set W of all possible worlds

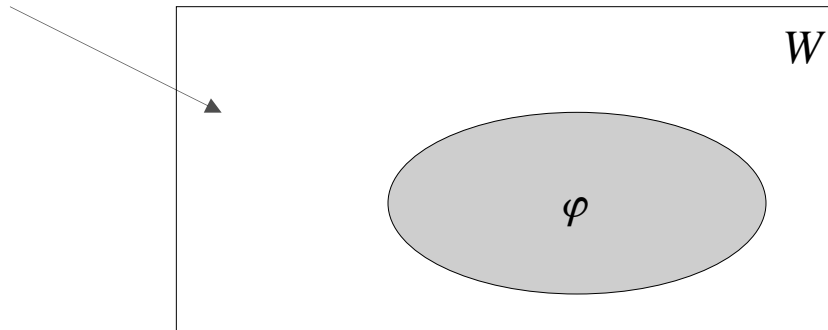
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The corresponding subset may be empty (i.e. if φ is a contradiction)
or it may coincide with W (i.e. if φ is a tautology)

The set of all possible worlds



“ φ is neither a contradiction nor a tautology”

“some possible worlds in W are *model* of φ , others are not”

“ φ is not (logically) **valid**”

Furthermore:

“ φ is **satisfiable**”

“ φ is **falsifiable**”

About formulae and their hidden relations

■ Hypothesis:

$$\varphi_1 = B \vee D \vee \neg(A \wedge C)$$

“Sally likes Harry” OR “Harry is happy”
OR NOT (“Harry is human” AND “Harry is a featherless biped”)

$$\varphi_2 = B \vee C$$

“Sally likes Harry” OR “Harry is a featherless biped”

$$\varphi_3 = A \vee D$$

“Harry is human” OR “Harry is happy”

$$\varphi_4 = \neg B$$

NOT “Sally likes Harry”

■ Thesis:

$$\psi = D$$

“Harry is happy”

Is there any **logical relation**
between hypothesis
and thesis?

And among the propositions
in the hypothesis?

Logical consequence

The overall truth table
for the wff in the example

$$\varphi_1 = B \vee D \vee \neg(A \wedge C)$$

$$\varphi_2 = B \vee C$$

$$\varphi_3 = A \vee D$$

$$\varphi_4 = \neg B$$

$$\psi = D$$

All the possible worlds that satisfy
 $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ satisfy ψ as well

A	B	C	D	φ_1	φ_2	φ_3	φ_4	ψ
0	0	0	0	1	0	0	1	0
0	0	0	1	1	0	1	1	1
0	0	1	0	1	1	0	1	0
0	0	1	1	1	1	1	1	1
0	1	0	0	1	1	0	0	0
0	1	0	1	1	1	1	0	1
0	1	1	0	1	1	0	0	0
0	1	1	1	1	1	1	0	1
1	0	0	0	1	0	1	1	0
1	0	0	1	1	0	1	1	1
1	0	1	0	0	1	1	1	0
1	0	1	1	1	1	1	1	1
1	1	0	0	1	1	1	0	0
1	1	0	1	1	1	1	0	1
1	1	1	0	1	1	1	0	0
1	1	1	1	1	1	1	0	1

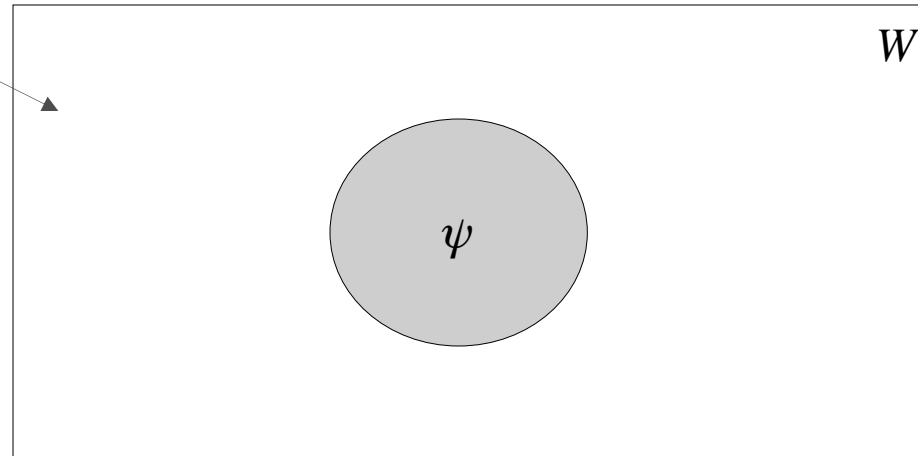
- This is the relation of **logical consequence**: $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \models \psi$
(also *logical entailment* or *entailment*)

(Pay attention to notation!)

Formulae, subsets and entailment

- Consider the set of all possible worlds W

All possible worlds

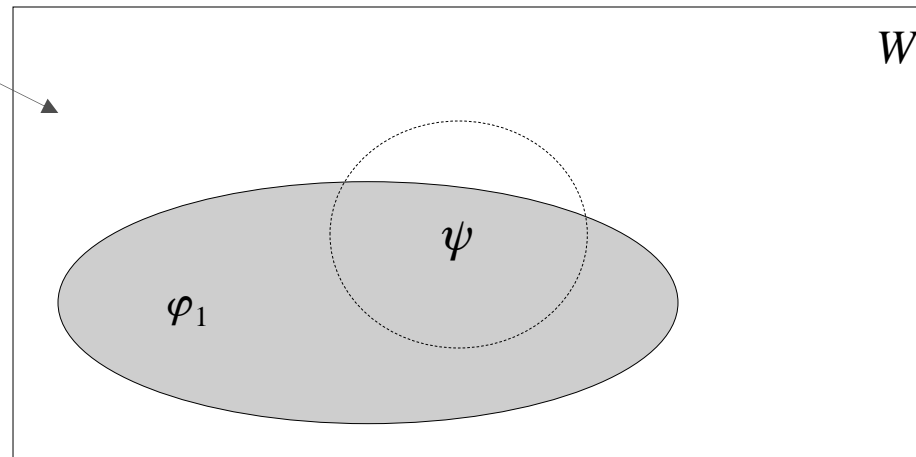


“All possible worlds that are *model* of ψ ”

Formulae, subsets and entailment

- Consider the set of all possible worlds W

All possible worlds



“All possible worlds that are model of φ_1 ”

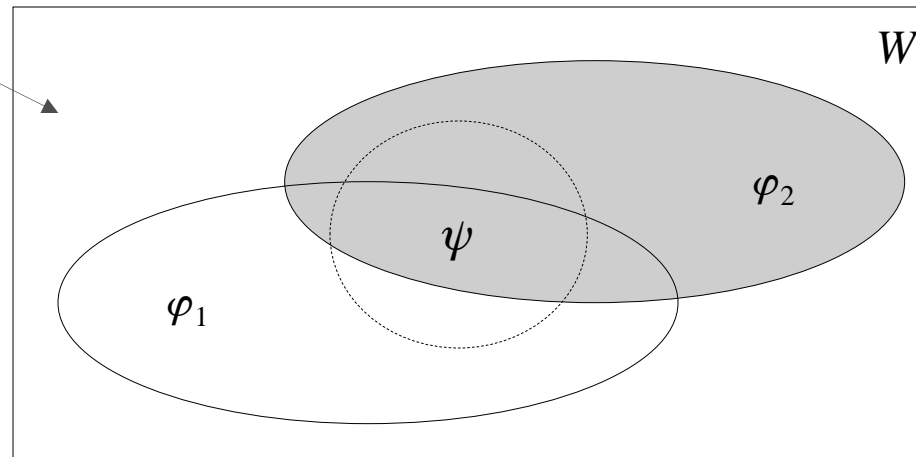
$\{\varphi_1\} \not\models \psi$

because the set of models of $\{\varphi_1\}$
is not contained in the set of models of ψ

Formulae, subsets and entailment

- Consider the set of all possible worlds W

All possible worlds



“All possible worlds that are models of φ_2 ”

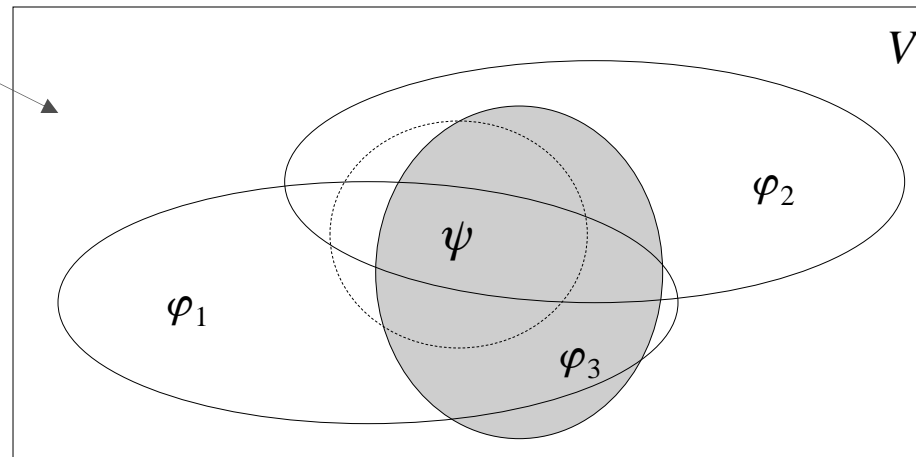
$\{\varphi_1, \varphi_2\} \not\models \psi$

because the set of models of $\{\varphi_1, \varphi_2\}$ (i.e. the *intersection* of the two subsets) is not contained in the set of models of ψ

Formulae, subsets and entailment

- Consider the set of all possible worlds W

All possible worlds



“All possible worlds that are models of φ_3 ”

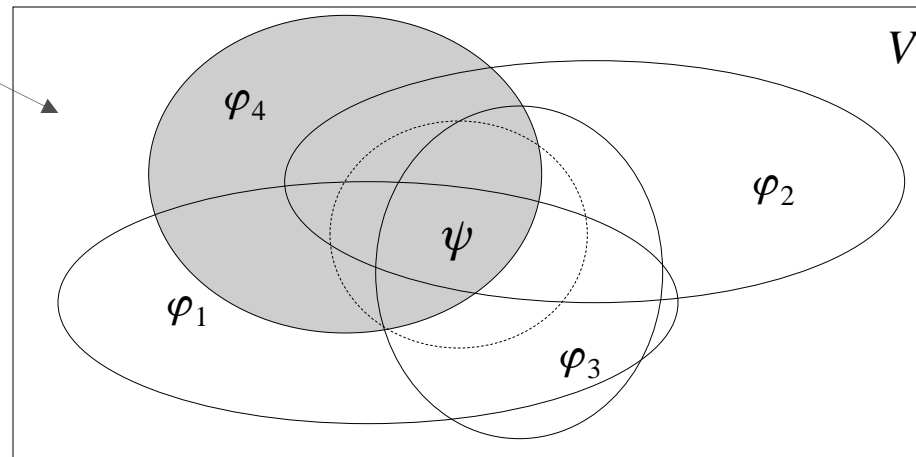
$\{\varphi_1, \varphi_2, \varphi_3\} \not\models \psi$

because the set of models of $\{\varphi_1, \varphi_2, \varphi_3\}$
is not contained in the set of models of ψ

Formulae, subsets and entailment

- Consider the set of all possible worlds W

All possible worlds



“All possible worlds that are models of φ_4 ”

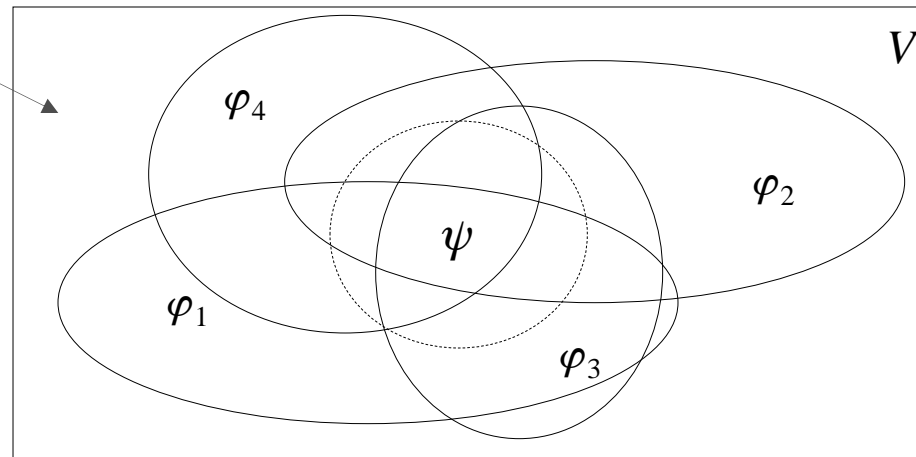
$$\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \models \psi$$

Because the set of models of $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$
is contained in the set of models of ψ

Formulae, subsets and entailment

- Consider the set of all possible worlds W

All possible worlds



“All possible worlds that are models of φ_4 ”

$$\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \models \psi$$

Because the set of models of $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ is contained in the set of models of ψ

In this case, all the wffs $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are needed for the relation of *entailment* to hold

Symmetric entailment = logical equivalence

■ Equivalence

Let φ and ψ be wffs such that:

$$\varphi \models \psi \text{ e } \psi \models \varphi$$

The two wffs are also said to be **logically equivalent**

In symbols: $\varphi \equiv \psi$

■ Substitutability

Two equivalent wffs have exactly the same **models**

In terms of entailment, equivalent wffs are substitutable

(even as sub-formulae)

In the example: $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \models \psi$

$$\varphi_1 = B \vee D \vee \neg(A \wedge C)$$

$$\varphi_2 = B \vee C$$

$$\varphi_3 = A \vee D$$

$$\varphi_4 = \neg B$$

$$\psi = D$$

$$\varphi_1 = B \vee D \vee (A \rightarrow \neg C)$$

$$\varphi_2 = B \vee C$$

$$\varphi_3 = \neg A \rightarrow D$$

$$\varphi_4 = \neg B$$

$$\psi = D$$

Implication

The wffs of the problem can be re-written using equivalent expressions:

(using the basis $\{\rightarrow, \neg\}$)

$$\varphi_1 = C \rightarrow (\neg B \rightarrow (A \rightarrow D))$$

$$\varphi_2 = \neg B \rightarrow C$$

$$\varphi_3 = \neg A \rightarrow D$$

$$\varphi_4 = \neg B$$

$$\psi = D$$

$$\varphi_1 = B \vee D \vee \neg(A \wedge C)$$

$$\varphi_2 = B \vee C$$

$$\varphi_3 = A \vee D$$

$$\varphi_4 = \neg B$$

$$\psi = D$$

- Some schemes are *valid* in terms of *entailment*:

$$\varphi \rightarrow \psi$$

$$\frac{\varphi}{\psi}$$

It can be verified that:

$$\varphi \rightarrow \psi, \varphi \models \psi$$

Analogously:

$$\varphi \rightarrow \psi, \neg\psi \models \neg\varphi$$

Modern formal logic: fundamentals

- **Formal language (*symbolic*)**

 - A set of symbols, not necessarily *finite*

 - Syntactic rules for composite formulae (wff)

- **Formal semantics**

 - For each formal language, a *class* of structures (i.e. a class of *possible worlds*)

 - In each possible world, every wff in the language is assigned a *value*

 - In classical propositional logic, the set of values is the simplest: $\{1, 0\}$

- **Satisfaction, entailment**

 - A wff is *satisfied* in a possible world if it is true in that possible world

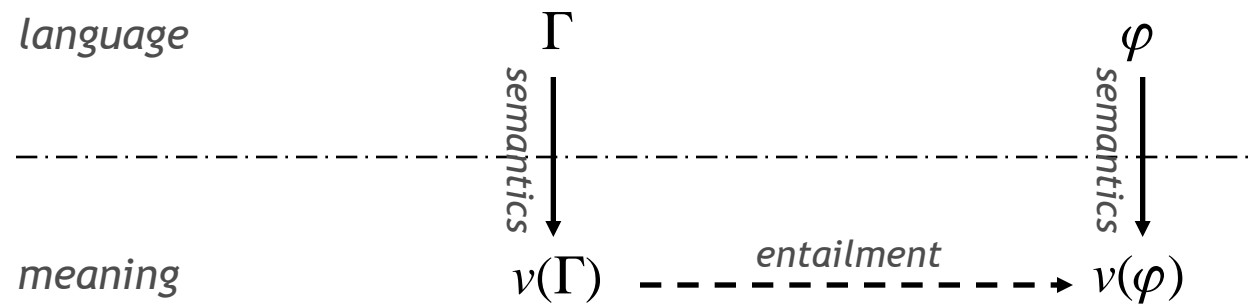
 - In classical propositional logic, iff the wff has value 1 in that world

 - (Caution: the definition of *satisfaction* will become definitely more complex with *first order logic*)

Entailment is a relation between a set of wffs and a wff

 - This relation holds when all possible worlds satisfying the set also satisfy the wff

What we have seen so far



Subtleties: object *language* and *metalanguage*

- The ***object language*** is L_P

It is the tool that we plan to use

It only contains the items just defined:

$P, \neg, \rightarrow, \wedge, \vee, \leftrightarrow, (,)$, plus *syntactic rules* (wff)

- ***Metalanguage***

Everything else we use to define the properties of the object language

Small greek letters ($\alpha, \beta, \chi, \varphi, \psi$) will be used to denote a generic formula (wff)

Capital greek letters (Γ, Δ, Σ) will be used to denote a set of formulae

Satisfaction, logical consequence (see after): \models

Derivability (see after): \vdash

Symbols for “iff” and “if and only if” (also “iff”): $\Rightarrow, \Leftrightarrow$