Propositional Resolution

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Inference rule: Resolution

$\varphi \lor \chi, \neg\chi \lor \psi \vdash \varphi \lor \psi$

$\varphi \lor \psi$ is also called the *resolvent* of $\varphi \lor \chi$ e $\neg\chi \lor \psi$

The resolution rule is *correct*

φ	ψ	χ	$\varphi \lor \chi$	$\neg\chi \lor \psi$	$\varphi \lor \psi$
0	0	0	0	1	0
0	0	1	1	0	0
0	1	0	0	1	1
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	1

Normal forms

= translation of each wff into an equivalent wff having a specific structure

- **Conjunctive Normal Form** (CNF)

A wff with a structure

$\alpha_1 \land \alpha_2 \land \ldots \land \alpha_n$

where each α_i has a structure

$(\beta_1 \lor \beta_2 \lor \ldots \lor \beta_n)$

where each β_j is a *literal* (i.e. an atomic symbol or the negation of an atomic symbol)

Examples:

$(B \lor D) \land (A \lor \neg C) \land C$

$(B \lor \neg A \lor \neg C) \land (\neg D \lor \neg A \lor \neg C)$

- **Disjunctive Normal Form** (DNF)

A wff with a structure

$\beta_1 \lor \beta_2 \lor \ldots \lor \beta_n$

where each β_i has a structure

$(\alpha_1 \land \alpha_2 \land \ldots \land \alpha_n)$

where each α_j is a *literal*

Conjunctive Normal Form

- Translation into CNF (it can be automated)

  Exhaustive application of the following rules:

  1) Rewrite → and ↔ using $\land$, $\lor$, $\neg$

  2) Move $\neg$ inside composite formulae

  "De Morgan laws": $\neg(\varphi \land \psi) \equiv (\neg\varphi \lor \neg\psi)$

  $\neg(\varphi \lor \psi) \equiv (\neg\varphi \land \neg\psi)$

  3) Eliminate double negations: $\neg\neg$

  4) Distribute $\lor$

  $$((\varphi \land \psi) \lor \chi) \equiv ((\varphi \lor \chi) \land (\psi \lor \chi))$$

Examples:

$(\neg B \to D) \lor \neg(A \land C)$

$B \lor D \lor \neg(A \land C)$ (rewrite →)

$B \lor D \lor \neg A \lor \neg C$ (De Morgan)

$\neg(B \to D) \lor \neg(A \land C)$

$\neg(\neg B \lor D) \lor \neg(A \land C)$ (rewrite →)

$(B \land \neg D) \lor (\neg A \lor \neg C)$ (De Morgan)

$(B \lor \neg A \lor \neg C) \land (\neg D \lor \neg A \lor \neg C)$ (distribute $\lor$)

Clausal Forms

= each wff is translated into an equivalent <u>set</u> of wffs having a specific structure

- **Clausal Form** (CF)

  Starting from a wff in CNF

  $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n$

  the clausal form is simply the set of all *clauses*

  $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$

  Examples:

  $(B \vee D) \wedge (A \vee \neg C) \wedge C$

  $\{(B \vee D), (A \vee \neg C), C\}$

- **Special notation**

  Each clause is usually written as a *set*

  $\beta_1 \vee \beta_2 \vee \ldots \vee \beta_n$

  $\{\beta_1, \beta_2, \ldots, \beta_n\}$

  Example:

  $\{\{B, D\}, \{A, \neg C\}, \{C\}\}$

  A set of *literals*:
  ordering is irrelevant
  no multiple copies

Resolution by refutation

- Algorithm

  Problem: “$\Gamma \vdash \varphi$” ?

  The problem is transformed into: is “$\Gamma \cup \{\neg\varphi\}$” *coherent*?

  If $\Gamma \vdash \varphi$ then $\Gamma \cup \{\neg\varphi\}$ is incoherent and therefore a contradiction can be derived

  $\Gamma \cup \{\neg\varphi\}$ is translated into CNF hence in CF

The resolution algorithm is applied to the set of *clauses* $\Gamma \cup \{\neg\varphi\}$

At each step:

a) Select a pair of clauses $\{C_1,C_2\}$ containing a pair of *complementary literals* making sure that this combination has never been selected before

b) Compute C as the *resolvent* of $\{C_1,C_2\}$ according to the resolution rule.

c) Add C to the set of clauses

Termination:

When C is the empty clause $\{\ \}$

or there are no more combinations to be selected in step a)

Advantages:

No axioms. Only one operation (i.e. the resolution rule). It is a *native* algorithm

Resolution by refutation

- The same example as before

  $B \lor D \lor \neg A \lor \neg C, B \lor C,\ A \lor D,\ \neg B \vdash D$

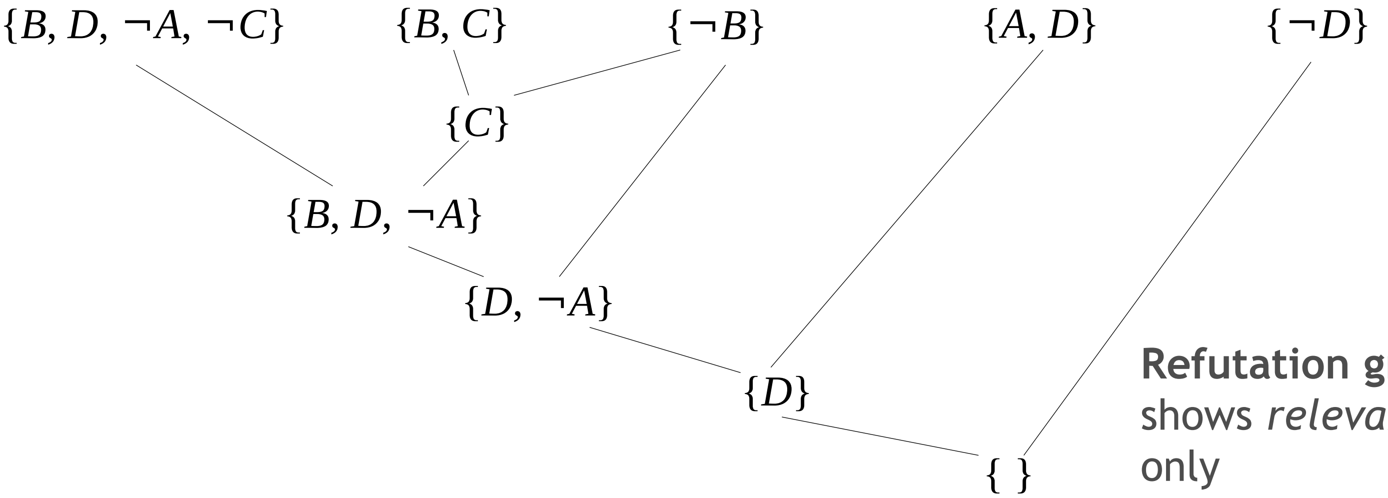
  Refutation + rewrite in CNF:

  $B \lor D \lor \neg A \lor \neg C, B \lor C,\ A \lor D,\ \neg B, \neg D$

  Rewrite in CF:

  $\{B, D, \neg A, \neg C\}, \{B, C\}, \{A, D\}, \{\neg B\}, \{\neg D\}$

  Applying the resolution rule:



$\{B, D, \neg A, \neg C\}$ $\{B, C\}$ $\{\neg B\}$ $\{A, D\}$ $\{\neg D\}$

$\{C\}$

$\{B, D, \neg A\}$

$\{D, \neg A\}$

$\{D\}$

$\{\ \}$

Refutation graph: shows *relevant* resolutions only

Resolution by refutation

- The same example as before

  $B \lor D \lor \neg A \lor \neg C, B \lor C,\ A \lor D,\ \neg B \vdash D$

  Refutation + rewrite in CNF:

  $B \lor D \lor \neg A \lor \neg C, B \lor C,\ A \lor D,\ \neg B, \neg D$

  Rewrite in CF:

  $\{B, D, \neg A, \neg C\}, \{B, C\}, \{A, D\}, \{\neg B\}, \{\neg D\}$

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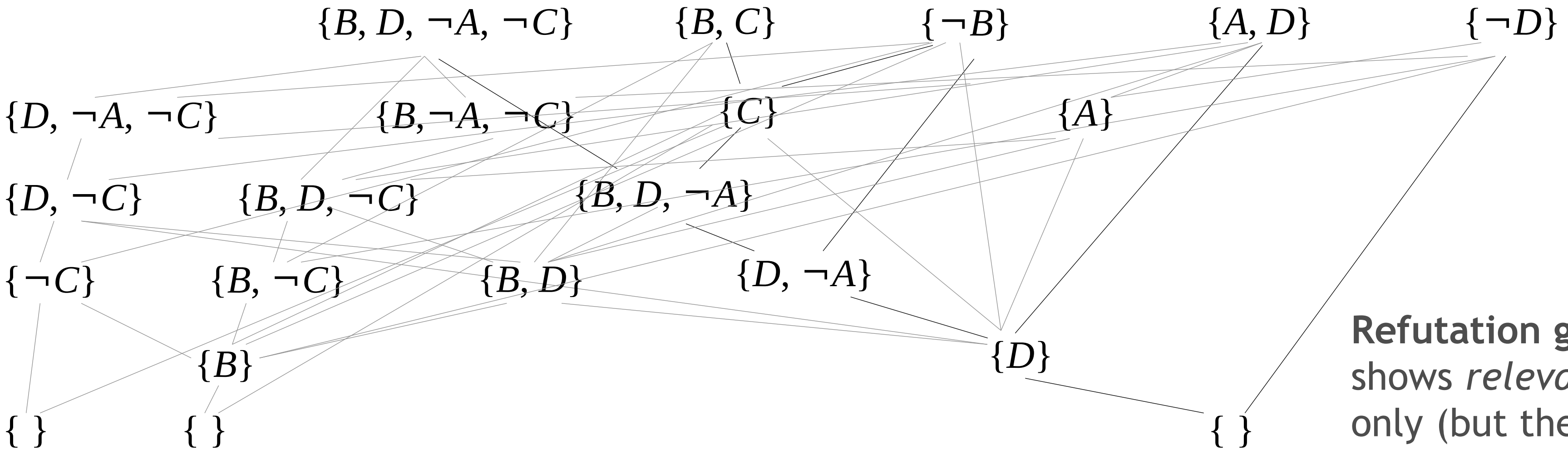

$\{B, D, \neg A, \neg C\}$ $\{B, C\}$ $\{\neg B\}$ $\{A, D\}$ $\{\neg D\}$

$\{D, \neg A, \neg C\}$ $\{B, \neg A, \neg C\}$ $\{C\}$ $\{A\}$

$\{D, \neg C\}$ $\{B, D, \neg C\}$ $\{B, D, \neg A\}$

$\{\neg C\}$ $\{B, \neg C\}$ $\{B, D\}$ $\{D, \neg A\}$

$\{B\}$ $\{D\}$

$\{\ \}$ $\{\ \}$ $\{\ \}$

Refutation graph:
shows *relevant* resolutions only (but there are more)

Resolution by refutation

- Resolution by refutation for propositional logic

  Is correct: $\Gamma \vdash \varphi \;\Rightarrow\; \Gamma \models \varphi$

  Is complete: $\Gamma \models \varphi \;\Rightarrow\; \Gamma \vdash \varphi$

    In this sense: if $\Gamma \models \varphi$ then there exists a refutation graph

- Algorithm

  It is a decision procedure for the problem $\Gamma \models \varphi$

  It has time complexity $O(2^n)$

    where n is the number of propositional symbols in $\Gamma \cup \{\neg\varphi\}$