Artificial Intelligence

A Course About Foundations

Reinforcement Learning

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Multi-Armed Bandit

Multi-Armed Bandit


[image from wikipedia]

A row of N old-style slot machines

- Basic definitions

  N arms or *bandits*

  Each arm a yields a random reward r with probability distribution $P(r \mid a)$

  For simplicity, only Bernoullian rewards (i.e. either 0 or 1) will be considered here

  Each time t in a sequence, the player (i.e. the *agent*) selects the arm $\pi(t)$

  In other words, π is the *policy* adopted by the agent

- Problem

  Find a policy π that maximizes the *total reward* over time

  The policy will include random choices i.e. it will be *stochastic*

Multi-Armed Bandit: strategies

- Informed (i.e. *optimal*) strategy

  At all times, select the bandit with higher probability of reward:

  $$\pi^*(t) = \mathrm{argmax}_a P(R = 1 \,|\, a)$$

  Clearly, this strategy is optimal but requires knowing all distributions $P(r \mid a)$

  With enough data (*e.g. from other players*), these distributions can be learnt

- Random strategy

  At all times, select a bandit a at random, with *uniform probability*

  How does the Random strategy compare with the optimal, informed strategy?

Multi-Armed Bandit: basic definitions

- *Actions, Rewards*

  $a \in \mathcal{A}$ *in the multi-armed bandit case* $a \in \{1, \ldots, N\}$

  $r \in \mathcal{R}$ *with Bernoulli (binary) reward* $r \in \{0, 1\}$

- *Probability distribution (unknown)*

  $P(R \,|\, A)$ *probability of reward* R *for action* A *(i.e. two random variables)*

- *Policy*

  $\pi : \mathbb{N}^+ \to \mathcal{A}$ *at each time, defines which action will be taken, it may be* <u>*stochastic*</u>

- *Q-value*

  The <u>*expected*</u> *reward of action* a

  $$Q(a) := \mathbb{E}[R \,|\, A = a] = \sum_r r \, P(r \,|\, A = a)$$

- *Optimal Value*

  Maximum <u>*expected*</u> *reward*

  $$V^* := Q(a^*) = \max_{a \in \mathcal{A}} Q(a)$$

Multi-Armed Bandit: evaluating strategies

- *Total Expected Regret (TER)*

  How far from optimality a policy is, considering the total reward over T trials

  For just <u>one</u> sequence of T trials, the *Total Regret with <u>expected</u> rewards* is

  action taken at step t

$$L(T) := TV^* - \sum_{t=1}^{T} Q(\pi(t))$$

  In a more general definition, the *Total <u>Expected</u> Regret* is

$$\overline{L}(T) := TV^* - \sum_{a=1}^{N} \mathbb{E}[T_a(T)]Q(a) = \sum_{a=1}^{N} \mathbb{E}[T_a(T)]\Delta_a$$

  number of times action a is taken in T trials (i.e. *a random variable*)

  where:

$$\Delta_a := V^* - Q(a)$$

Multi-Armed Bandit: evaluating strategies

- *Total Expected Regret (TER)*

$$\overline{L}(T) := TV^* - \sum_{a=1}^{N} \mathbb{E}[T_a(T)]Q(a) = \sum_{a=1}^{N} \mathbb{E}[T_a(T)]\Delta_a$$

number of times action a is taken in T trials (i.e. *a random variable*)

where:

$$\Delta_a := V^* - Q(a)$$

With the optimal policy π^**, the total expected regret is* 0.

Whereas, with a <u>*random*</u> *policy* the total expected regret grows linearly over time:

$$\overline{L}(T) = \frac{T}{N} \sum_{a=1}^{N} \Delta_a$$

…since, with a random policy $\mathbb{E}[T_a(T)] = \dfrac{T}{N}$

Multi-Armed Bandit: *Online learning*

Adaptive policy: *exploration vs. exploitation*

exploration: make trials over the set of N arms to improve on estimates $\hat{Q}(a)$

exploitation: make use of the current best estimates $\hat{Q}(a)$

- Greedy policy

Initialize all the estimates $\hat{Q}(a)$ at random
Repeat:

1) select the bandit with the current best estimated reward $a = \mathrm{argmax}_a \hat{Q}(a)$
2) update the current estimate about a as

$$\hat{Q}(a) := \frac{\sum_{t=1}^{T_a} r_{a,t}}{T_a}$$

reward of arm ***a*** at trial *t*

Total number of times the arm ***a*** has been played

Multi-Armed Bandit: *Online learning*

Adaptive policy: *exploration vs. exploitation*

exploration: make trials over the set of N arms to improve on estimates $\hat{Q}(a)$

exploitation: make use of the current best estimates $\hat{Q}(a)$

- ε-greedy policy $(0 < \varepsilon < 1)$

  Initialize all the estimates $\hat{Q}(a)$ at random

  Repeat:

  1) with probability $(1 - \varepsilon)$ select the bandit $a = \mathrm{argmax}_a \hat{Q}(a)$
     else (*i.e. with probability* ε) select one bandit at random
  2) update the current estimate about a

$$\hat{Q}(a) := \frac{\sum_{t=1}^{T_a} r_{a,t}}{T_a}$$

reward of arm a at trial t

total number of times the arm a has been played

Multi-Armed Bandit: *Online learning*

Adaptive policy: *exploration vs. exploitation*

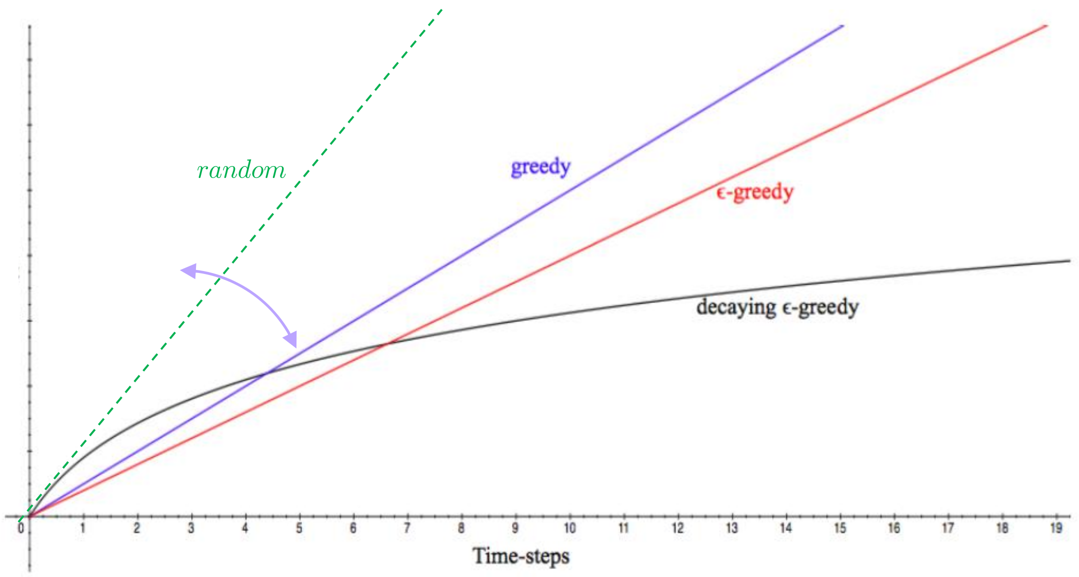
exploration: make trials over the set of N arms to improve on estimates $\hat{Q}(a)$

exploitation: make use of the current best estimates $\hat{Q}(a)$

- Theoretical comparison of different strategies (*Total Expected Regret*)

After a certain period of time, the *greedy* policy stops exploring and exploits its estimates

whereas the ε-greedy policy keeps exploring and improving



Decaying ε-greedy strategy: $\varepsilon = \dfrac{\varepsilon_{initial}}{t}$

Multi-Armed Bandit: evaluating strategies

- The two *greedy* strategies

  They are <u>*biased*</u>: they depend on the initial random estimates

  Optimistic variant: initially, set all $\hat{Q}(a) := 1$

  In this way, all action will be taken at least once

  The average total regret grows <u>linearly</u>, in the long run

  In fact:

  - on average, the *greedy* strategy will get stuck in a suboptimal choice
  - the ε-greedy strategy will continue to choose an arm at random (with probability ε)

  Can we do any better?

The decaying ε-greedy policy in fact does that…
Is there a minimum TER, that is, a lower bound?

Multi-Armed Bandit: Optimal *online learning*

- Lower bound theorem [Lai & Robbins 1985]

  Consider a generic, adaptive (i.e. learning) strategy for the multi-armed bandit problem with binary reward (i.e. $r \in \{0, 1\}$)

$$\lim_{T \to \infty} \overline{L}(T) \geq \ln T \sum_{a | \Delta_a > 0} \frac{\Delta_a}{\mathrm{kl}(Q(a), V^*)} \qquad \Delta_a := V^* - Q(a)$$

where:

$$\mathrm{kl}(Q(a), V^*) := Q(a) \ln \frac{Q(a)}{V^*} + (1 - Q(a)) \ln \frac{(1 - Q(a))}{(1 - V^*)}$$

Kullback-Leibler divergence (see Wikipedia)

In other words, we can achieve logarithmic growth for the total expected regret, but not better: on average, any adaptive strategy will choose suboptimal bandits a minimum number of times

$$\lim_{T \to \infty} \mathbb{E}[T_a(T)] \geq \frac{\ln T}{\mathrm{kl}(Q(a), V^*)}$$

Multi-Armed Bandit: UCB strategy

- Upper confidence bound (UCB) strategy [Auer, Cesa-Bianchi and Fisher 2002]

Initialize all the estimates of the expected reward $\hat{Q}(a) := 0$
Play each arm once (*to avoid zeroes in the formula below*)

total number of trials

number of times
the arm *a* has been played

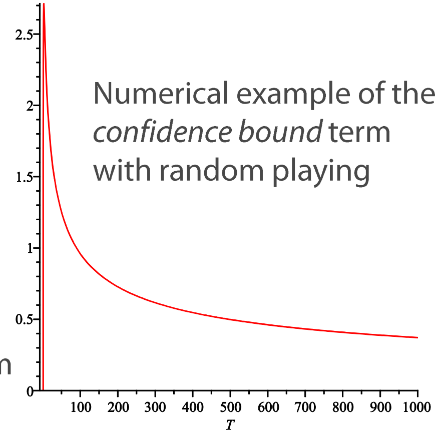
Repeat:

1) select the bandit $a = \text{argmax}_a \left(\hat{Q}(a) + \sqrt{\dfrac{2 \ln T}{T_a}} \right)$

2) update the current estimate $\hat{Q}(a)$
as the *average* reward

$$\sqrt{\frac{2 \ln T}{(T/N)}}, \quad N = 10$$

Arm chosen at random



Numerical example of the *confidence bound* term with random playing

Theorem

With the UCB strategy, $\lim_{T \to \infty} \mathbb{E}[T_a(T)] \leq \dfrac{8 \ln T}{\Delta_a^2} + c$

i.e. a (small) constant

where it can be shown that $\dfrac{8}{\Delta_a^2} \geq \dfrac{1}{\text{kl}(Q(a), V^*)}$

(*i.e. there is a reasonably small gap between the two bounds – near optimality*)

Multi-Armed Bandit: Thompson Sampling

- Thompson Sampling strategy (*also 'Bayesian Bandit'*) [Thompson, 1933]

  Initialize all the expected reward $\hat{Q}(a) :\sim \mathrm{Beta}(x; 1, 1)$

  i.e. assume this as a random variable with this distribution

  Repeat:

  1) <u>sample</u> each of the N distributions to obtain an estimate $\hat{Q}(a)$
  2) select the bandit $a = \mathrm{argmax}_a \hat{Q}(a)$
  3) update the *posterior* distribution

  $$\hat{Q}(a) :\sim \mathrm{Beta}(x; R_a + 1,\ T_a - R_a + 1)$$

  total number of times the arm has been played

  total (*Bernoulli*) reward from this arm (*i.e. number of wins*)

  Theorem [Kaufmann et al., 2012]

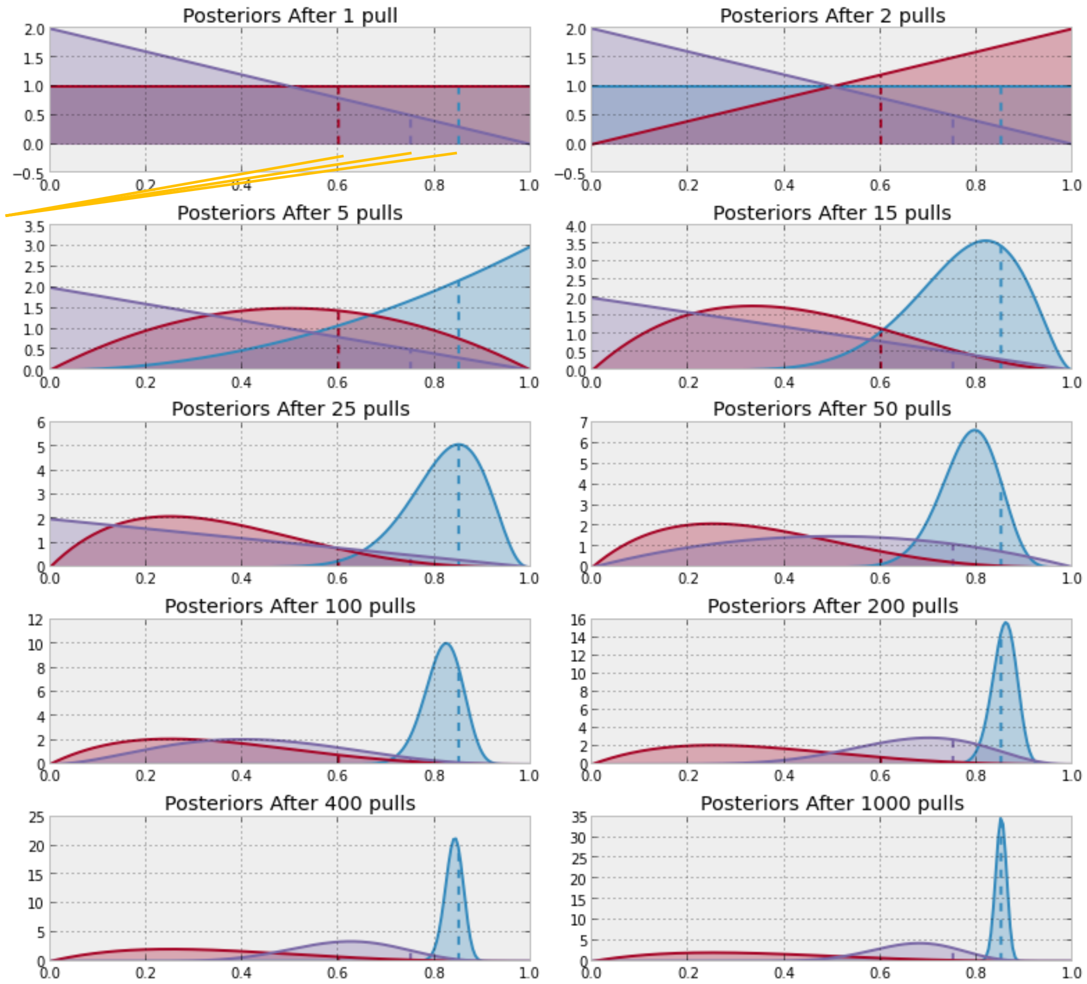
  The Thompson Sampling strategy has essentially the same theoretical bounds of the UCB strategy

Multi-Armed Bandit: Thompson Sampling

- Thompson Sampling strategy (*also 'Bayesian Bandit'*) [Thompson, 1933]

  Example run with 3 arms: trace of the posterior probabilities for each $\hat{Q}(a)$

ground truth: $Q(a)$

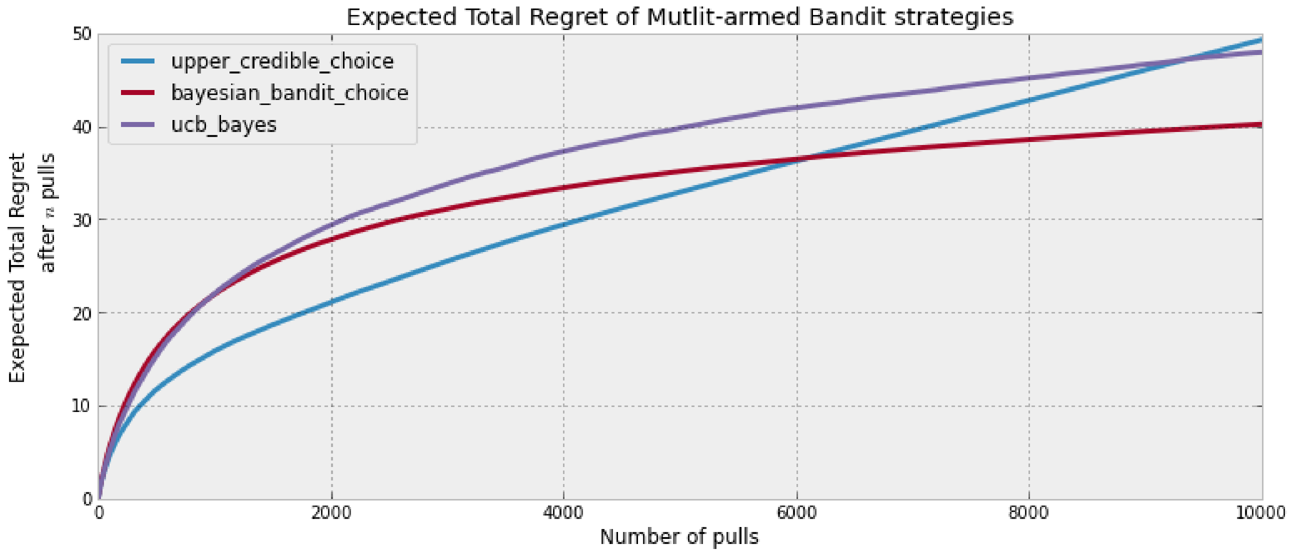


[image from: http://camdp.com/blogs/multi-armed-bandits]

Multi-Armed Bandit: Thompson Sampling

- Thompson Sampling strategy (*also 'Bayesian Bandit'*) [Thompson, 1933]

  In practical experiments, this strategy shows better performances in the long run
  [Chapelle & Li, 2011]



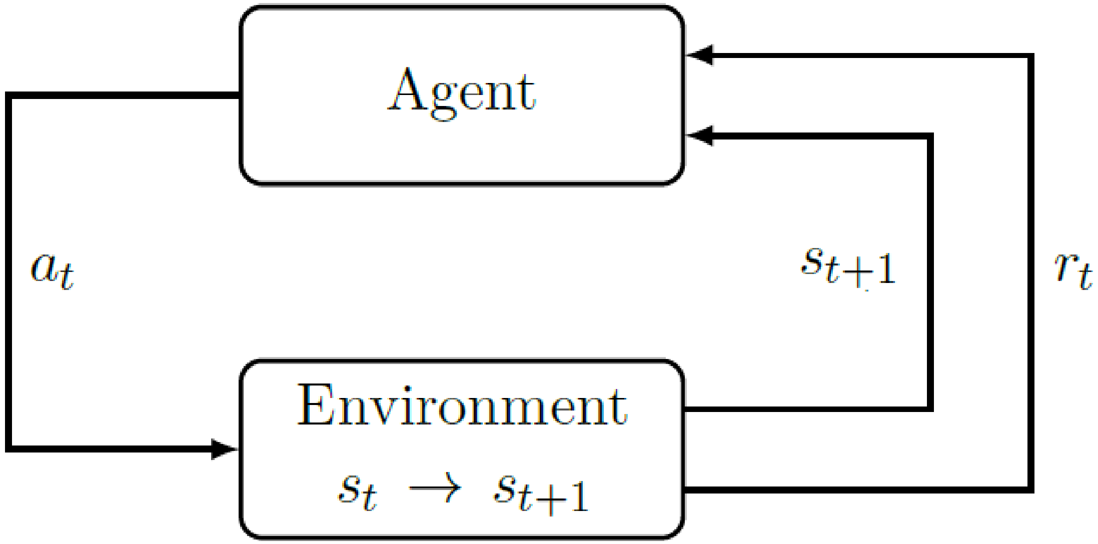
Actually, Thompson Sampling is a preferred strategy at Google Inc.
(see https://support.google.com/analytics/answer/2846882?hl=en)

[image from: http://camdp.com/blogs/multi-armed-bandits]

Markov Decision Process (MDP)

Basic assumptions

[image from: https://arxiv.org/pdf/1811.12560.pdf]



The **Environment**: is in *state* s_t time (discrete)

An **Agent** observes *state* s_t and performs *action* a_t

The **Environment** *state* transitions from $s_t \rightarrow s_{t+1}$

The **Agent** receives *reward* r_t

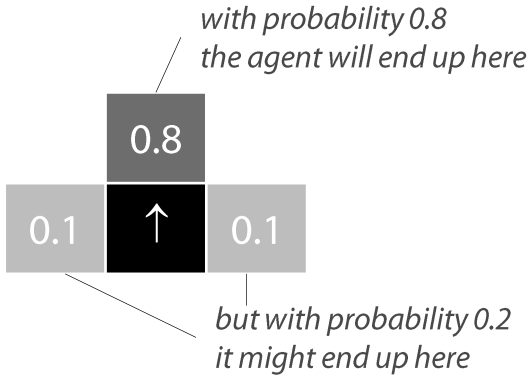
Cumulative reward: $R := \sum_{t=0}^{\infty} r_t$

An example: *gridworld*

	1	2	3	4
1	-0.02	-0.02	-0.02	1
2	-0.02		-0.02	-1
3	-0.02	-0.02	-0.02	-0.02

The *state* of the agent is the position on the grid: e.g. (1,1), (3,4), (2,3)

At each time step, the agent can *move* one box in the directions ←↑↓→



The effect of each move is somewhat stochastic, however: for example, a move ↑ has a slight probability of producing a different (*and perhaps unwanted*) effect

Entering each state yields the *reward* shown in each box above

There are two *absorbing states*: entering either the green or the red box means exiting the *gridworld* and completing the game

- What is the best (*i.e. maximally rewarding*) movement policy?

Markov Decision Process (MDP)

	1	2	3	4
1	-0.02	-0.02	-0.02	1
2	-0.02		-0.02	-1
3	-0.02	-0.02	-0.02	-0.02

Formalization and abstraction of the gridworld example

Markov Decision Process: $< \mathcal{S}, \mathcal{A}, r, P, \gamma >$

A set of *states* : $\mathcal{S} = \{s_1, s_2, \dots\}$

A set of *actions* : $\mathcal{A} = \{a_1, a_2, \dots\}$

A *reward function* : $r : \mathcal{S} \to \mathbb{R}$

A *transition probability distribution* : $P(S_{t+1} \mid S_t, A_t)$ (also called a *model*)

Markov property: the transition probability depends only on the previous state and action

$$P(S_{t+1} \mid S_t, A_t) = P(S_{t+1} \mid S_t, A_t, S_{t-1}, A_{t-1}, S_{t-2}, A_{t-2}, \dots)$$

A *discount factor* : $0 \leq \gamma < 1$

Markov Decision Process (MDP): policies and values

The agent is supposed to adopt a *deterministic* <u>*policy*</u> : $\pi : \mathcal{S} \rightarrow \mathcal{A}$

In other words, the agent always chooses its *action* depending on the *state* alone

Given a policy π , the ***state value function*** is defined, for each state s as:

$$V^\pi(s) := \mathbb{E}[r(S_t) + \gamma r(S_{t+1}) + \gamma^2 r(S_{t+2}) + \ldots \,|\, \pi, S_t = s]$$

Note the role of the *discount factor*: a value $\gamma < 1$ means that that future rewards could be weighted less (by the agent) than immediate ones

Note also that all states S_t must be described by *random variables* :
i.e. the policy is deterministic, yet the state transition is not

Note also that when the reward is *bounded*, i.e. $r(S) \leq r_{\max}$

$$\sum_{t=0}^{\infty} \gamma^t \, r(S_t) \;\leq\; r_{\max} \sum_{t=0}^{\infty} \gamma^t \;=\; r_{\max} \, \frac{1}{1-\gamma}$$

for $\gamma < 1$ this is the *geometric series*

Markov Decision Process (MDP): policies and values

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Note the role of the *discount factor*: a value $\gamma < 1$ means that that future rewards could be weighted less (by the agent) than immediate ones

Note also that all states S_t must be described by *random variables* :
i.e. the policy is deterministic, but the state transition is not

In the *gridworld* example:

- The set of states is finite
- The set of actions is finite
- For every policy, each entire story is finite

  Sooner or later the agent will fall into one of the absorbing states

Bellman equations

By working on the definition of value function:

$$
\begin{aligned}
V^\pi(s) &:= \mathbb{E}[r(S_t) + \gamma r(S_{t+1}) + \gamma^2 r(S_{t+2}) + \ldots \mid \pi, S_t = s] \\
&= \mathbb{E}[r(S_t) + \gamma(r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots) \mid \pi, S_t = s] \\
&= r(s) + \gamma \mathbb{E}[r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots \mid \pi, S_t = s] \\
&= r(s) + \gamma \sum_{s'} P(s' \mid s, \pi(s)) \cdot \mathbb{E}[r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots \mid \pi, S_{t+1} = s'] \\
&= r(s) + \gamma \sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^\pi(S_{t+1})
\end{aligned}
$$

This means that in a Markov Decision Process:

$$V^\pi(s) = r(s) + \gamma \sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^\pi(S_{t+1})$$

This is true for any *state*, so there is one such equation for each of those

If the set of states is <u>finite</u>, there are exactly $|S|$ (linear) Bellman equations for $|S|$ variables:
in general, for any <u>deterministic</u> policy, V^π <u>can</u> be computed analytically

Optimal policy – Optimal value function

- Basic definitions

$$V^*(s) := \max_\pi V^\pi(s),\ \forall s \in S$$
$$\pi^*(s) := \mathrm{argmax}_\pi V^\pi(s),\ \forall s \in S$$

Property: for every MDP, there exists such an optimal deterministic policy (*possibly non-unique*)

With Bellman Equations:

$$\begin{aligned}
\max_\pi V^\pi(s) &= r(s) + \gamma \max_\pi \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^\pi(S_{t+1})\right) \\
V^*(s) &= r(s) + \gamma \max_\pi \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^*(S_{t+1})\right) \\
&= r(s) + \gamma \max_a \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, a) \cdot V^*(S_{t+1})\right)
\end{aligned}$$

Therefore:

$$\pi^*(s) = \mathrm{argmax}_a \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, a) V^*(S_{t+1})\right)$$

once V^* *has been determined,*
π^* *can be determined as well*

Computing V^* *directly from these equations is unfeasible, however*

There are in fact $|\mathcal{A}|^{|\mathcal{S}|}$ *possible strategies ...*

Reinforcement Learning
(model-based)

Optimal value function: value iteration

- Value iteration algorithm

  Initialize: $V(s) := r(s), \ \forall s \in S$

  Note that there is no policy: all actions must be explored

  Repeat:

  1) For every state, update: $V(s) := r(s) + \gamma \max_a \sum_{s'} P(s' \mid s, a) V(s')$

  Theorem: for every fair way (*i.e. giving an equal chance*) of visiting the states in S, this algorithm converges to V^*

Value iteration and optimal policy

	1	2	3	4
1	-0.02	-0.02	-0.02	1
2	-0.02		-0.02	-1
3	-0.02	-0.02	-0.02	-0.02

Initialize states
(e.g. using rewards as initial values)

Iterate and compute

V^*

	1	2	3	4
1	0.86	0.90	0.93	1
2	0.82		0.69	-1
3	0.78	0.75	0.71	0.49

V^*

Define the optimal policy as:

$$\pi^*(s) := \text{argmax}_a \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, a) \cdot V^*(S_{t+1})\right)$$

	1	2	3	4
1	$\rightarrow$	$\rightarrow$	$\rightarrow$	1
2	$\uparrow$		$\uparrow$	-1
3	$\uparrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$

π^*

Optimal policy: policy iteration

- Policy iteration algorithm

Initialize $\pi(s), \forall s \in S$ at random
Repeat:

This step is computationally expensive: either solve the equations or use value iteration (with fixed policy π)

1) For each state, compute: $V(s) := V^{\pi}(s)$
2) For each state, define: $\pi(s) := \text{argmax}_a \sum_{s'} P(s' \mid s, a) V(s')$

Theorem: for every fair way (*i.e. giving an equal chance*) of visiting the states in S, this algorithm converges to π^*

As with the value iteration algorithm, this algorithm uses partial estimates to compute new estimates.
It is also <u>greedy</u>, in the sense that it exploits its current estimate $V^{\pi}(s)$

Policy iteration converges with very few number of iterations, but every iteration takes much longer time than that of *value iteration*

The tradeoff with value iteration is the <u>action space</u>:
when action space is large and state space is small, policy iteration could be better

Reinforcement Learning
(model-free)

Molde-based vs. model-free reinforcement learning

- *Value iteration* and *policy iteration* are offline algorithms

  The *model* , i.e. the Markov Decision Process is known

  What needs to be learnt is the optimal policy π^*

  In the algorithms, *visiting states* just means considering them:
  there needs not be an agent which *actually plays* the game

- Different conditions: *learning by doing …*

  Suppose the *model* (i.e. the MDP) is NOT known, or perhaps known only in part

  In particular, it might not be known the transition function $P(S_{t+1} \mid S_t, A_t)$

  Such scenario is also called 'model-free'

  The agent, then, must learn by doing… that is, actually playing the game

Action value function

An analogous of the value function V^π

Given a policy π , the ***action value function*** is defined, for each pair (s, a) as:

$$Q^\pi(s,a) := \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot V^\pi(S_{t+1})$$

$$= \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot \mathbb{E}[r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots \mid \pi, S_{t+1}]$$

$$= \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \mathbb{E}[\gamma r(S_{t+2}) + \ldots \mid \pi, S_{t+1}]]$$

$$= \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \gamma Q^\pi(S_{t+1}, \pi(S_{t+1}))]$$

In other words, $Q^\pi(s,a)$ is the expected value of the reward in S_{t+1}
by taking action a in state s and then following policy π from that point on

Following a similar line of reasoning, the ***optimal*** *action value function* is

$$Q^*(s,a) = \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \gamma \max_{a'} Q^*(S_{t+1}, a')]$$

Q-Learning

- Q-learning algorithm (ε*-greedy version*)

Initialize $\hat{Q}(s,a)$ at random, put the agent is in a random state s
Repeat:

1) Select the action $\mathrm{argmax}_a \hat{Q}(s,a)$ with probability $(1-\varepsilon)$
otherwise, select a at random
2) The agent is now in state s' and has received the reward r
3) Update $\hat{Q}(s,a)$ by

$$\Delta\hat{Q}(s,a) = \alpha[r + \gamma \max_{a'} \hat{Q}(s',a') - \hat{Q}(s,a)]$$

Exponential Moving Average
(see later …)

Note that step 1) is closely similar to a ***multi-armed bandit****:*
in each state, the agent has to choose one among all actions in $\mathcal{A}$
and this will produce a random reward…

Q-Learning

- Q-learning algorithm

  Theorem (Watkins, 1989): in the limit of that each action is played infinitely often and each state is visited infinitely often and $\alpha \to 0$ as experience progresses, then

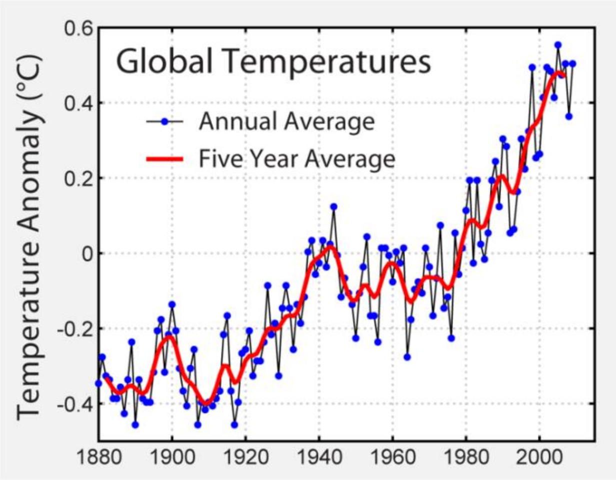
$$\hat{Q}(s, a) \to Q^*(s, a)$$

  with probability 1

The Q-learning algorithm bypasses the MDP entirely, in the sense that the optimal strategy is learnt without learning the model $P(S_{t+1} \mid S_t, A_t)$

An aside: *moving averages*

Following non-stationary phenomena



[image from wikipedia]

- Average

  Definition: $$\overline{v}_T := \frac{1}{T}\sum_{k=1}^{T} v_k$$

  Running implementation:

$$\overline{v}_T = \frac{1}{T}(v_T + \sum_{k=1}^{T-1} v_k) = \frac{1}{T}(v_T + (T-1)\overline{v}_{T-1})$$

$$= \overline{v}_{T-1} + \frac{1}{T}(v_T - \overline{v}_{T-1}) = \frac{1}{T}\, v_T + (1 - \frac{1}{T})\,\overline{v}_{T-1}$$

"the weight of newer observations diminishes with time"

- Simple Moving Average (SMA)

$$\overline{v}_{T,n} := \frac{1}{n}\sum_{k=T-n}^{T} v_k$$

- Exponential Moving Average (EMA)

$$\overline{v}_{T,\alpha} := \alpha\, v_T + (1-\alpha)\,\overline{v}_{T-1,\alpha},\ \ \alpha \in [0,1]$$

"the weight of newer observations remains constant"

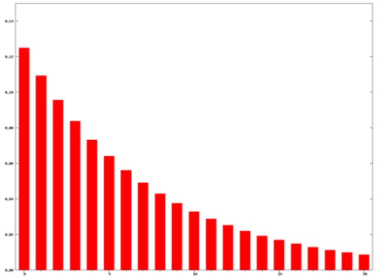
An aside: *moving averages*

- Exponential Moving Average (EMA)

$$\overline{v}_{T,\alpha} := \alpha\, v_T + (1-\alpha)\,\overline{v}_{T-1,\alpha},\;\; \alpha \in [0,1]$$

$(1-\alpha)^{\Delta_t}$

"the weight of older observations diminishes with time"



[image from wikipedia]

Expanding:

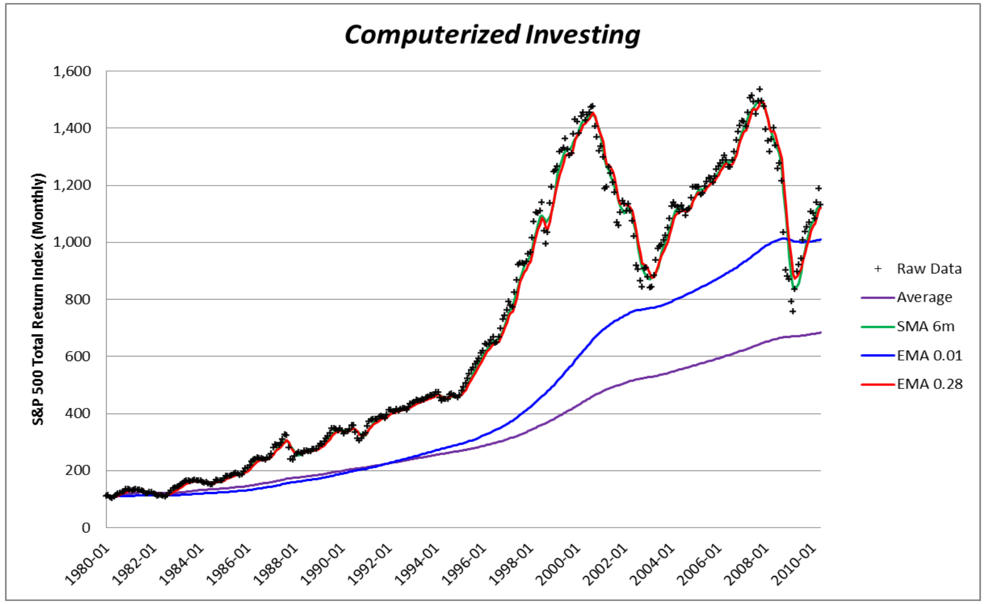
$$\begin{aligned}\overline{v}_{t,\alpha} &= \alpha\, v_t + (1-\alpha)\,\overline{v}_{t-1,\alpha}\\ &= \alpha\, v_t + (1-\alpha)(\alpha\, v_{t-1} + (1-\alpha)\overline{v}_{t-2,\alpha})\\ &= \alpha\, v_t + (1-\alpha)(\alpha\, v_{t-1} + (1-\alpha)(\alpha\, v_{t-2} + (1-\alpha)\overline{v}_{t-3,\alpha}))\\ &= \alpha\,(v_t + (1-\alpha)\, v_{t-1} + (1-\alpha)^2\, v_{t-2}) + (1-\alpha)^3\,\overline{v}_{t-3,\alpha}\end{aligned}$$

The weight of past contributions decays as

$$(1-\alpha)^{\Delta_t}$$

A SMA with n previous values is approximately equal to an EMA with

$$\alpha = \frac{2}{n+1}$$



Q-Learning revisited

- Q-learning algorithm (ε-*greedy version*)

Initialize $\hat{Q}(s,a)$ at random, put the agent is in a random state s
Repeat:

1) Select the action $a = \mathrm{argmax}_a \hat{Q}(s,a)$ with probability $(1-\varepsilon)$
   otherwise, select a at random
2) The agent is now in state s' and has received the reward r
3) Update $\hat{Q}(s,a)$ by

$$\Delta\hat{Q}(s,a) = \alpha[r + \gamma \max_{a'} \hat{Q}(s',a') - \hat{Q}(s,a)]$$

By rewriting step 3)

$$\begin{aligned}\hat{Q}(s,a) &= \hat{Q}(s,a) + \Delta\hat{Q}(s,a) = \hat{Q}(s,a) + \alpha[r + \gamma \max_{a'} \hat{Q}(s',a') - \hat{Q}(s,a)] \\ &= \underbrace{\alpha[r + \gamma \max_{a'} \hat{Q}(s',a')] + (1-\alpha)\hat{Q}(s,a)}_{\text{Exponential Moving Average}}\end{aligned}$$

compare with (see before):

$$Q^*(s,a) = \underbrace{\sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \gamma \max_{a'} Q^*(S_{t+1},a')]}_{\text{Expectation}}$$

SARSA

- SARSA algorithm (ε-*greedy version*)

Initialize $\hat{Q}(s,a)$ at random, put the agent is in a random state s
Repeat:

1) Select the action $a = \text{argmax}_a \hat{Q}(s,a)$ with probability $(1-\varepsilon)$
   otherwise, select a at random
2) The agent is now in state s' and has received the reward r
3) Select the action $a' = \text{argmax}_a \hat{Q}(s',a)$ with probability $(1-\varepsilon)$
   otherwise, select a' at random
4) Update $\hat{Q}(s,a)$ by

$$\Delta\hat{Q}(s,a) = \alpha[r + \gamma\hat{Q}(s',a') - \hat{Q}(s,a)]$$

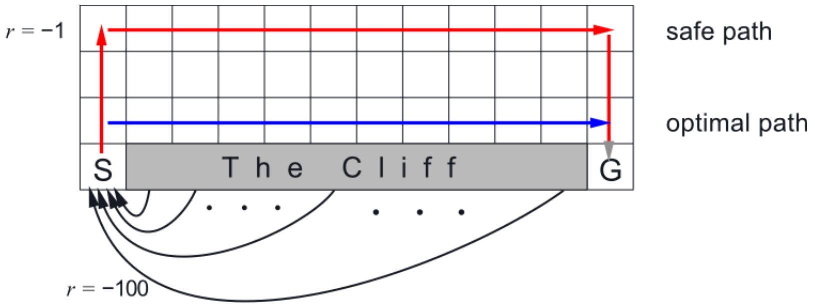
No more 'max' here

Q-learning is a an *off-policy* algorithm: each update involves $\max_{a'} \hat{Q}(s',a')$
(i.e. *exploration* is not taken into account)
SARSA is a an *on-policy* algorithm: each update involves $\hat{Q}(s',a')$
(which involves the next policy action, *exploration* included)

SARSA vs Q-Learning

- Cliff World

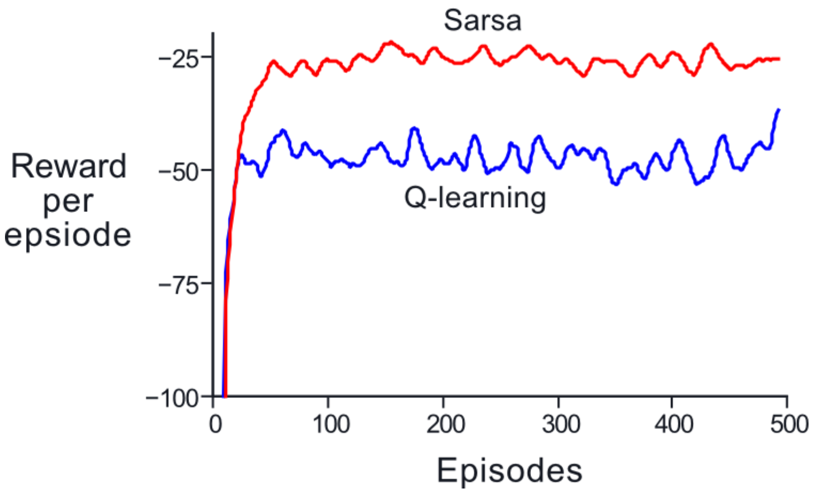
  'S' is the start
  'G' is the goal
  Each white box has $r = -1$
  'The Cliff' region has $r = -100$
  and entails going back to 'S'



- Experimental Results

  SARSA finds a sub-optimal but safer path
  since its learning takes into account
  the ε risk of going off the cliff

  Q-learning finds the optimal path
  but, occasionally, it falls off the cliff
  during learning due to the ε-greedy strategy



Reinforcement Learning Methods

[image from: https://arxiv.org/pdf/1811.12560.pdf]