Artificial Intelligence

A Course About Foundations

Entailment and Algorithms

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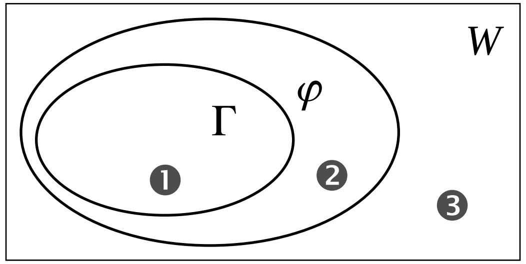
Entailment as *Satisfiability*

Transforming problems: entailment as satisfiability

- Step 1: the decision problem " $\Gamma \models \varphi$? " can be transformed into a *satisfiability* problem

  In fact, $\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is *not* satisfiable



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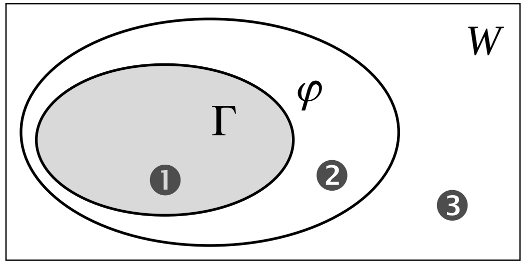

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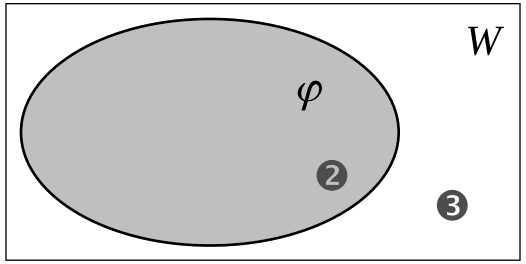

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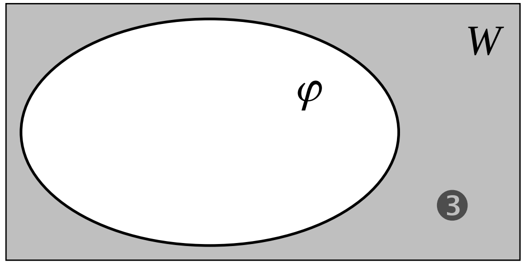

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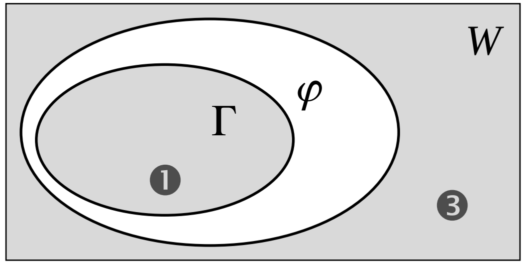

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$w(\{\varphi\})$ ❷

$w(\{\neg\varphi\})$ ❸

$w(\Gamma \cup \{\neg\varphi\}) = w(\Gamma) \cap w(\{\neg\varphi\})$
$w(\Gamma \cup \{\neg\varphi\}) = \varnothing$

❶ ∩ ❸ = ∅

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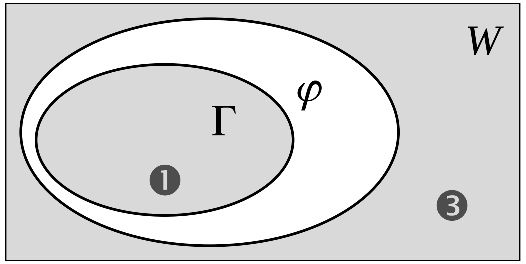


($w(\Gamma)$ is the set of possible worlds that satisfy Γ)

- Step 2: the decision problem " is $\Gamma \cup \{\neg\varphi\}$ satisfiable? "
  can be transformed into a wff *satisfiability* problem

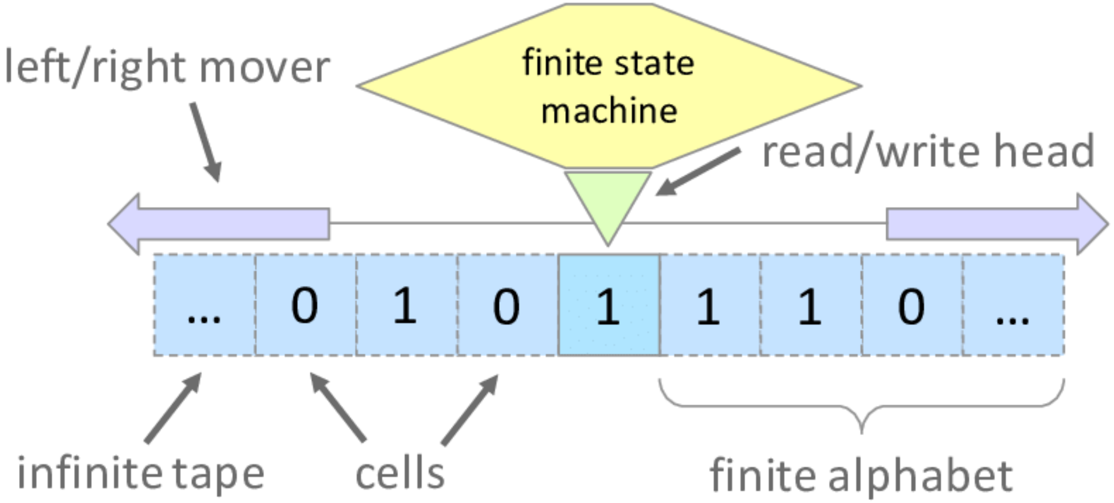
  Taking this one step further, we can transform $\Gamma \cup \{\neg\varphi\}$ into *just one formula*:

$$\bigwedge (\Gamma \cup \{\neg\varphi\})$$

This is the wff obtained by combing all the wffs in $\Gamma \cup \{\neg\varphi\}$ with $\wedge$,
it is called the *conjunctive closure* of the set $\Gamma \cup \{\neg\varphi\}$

"Algorithm"
(*Computational Complexity Theory in a Quick Ride*)

Turing Machine (A. Turing, 1937)



- A more precise definition

    A non-empty and finite set of *states* S

    At each instant the machine is in a state $s \in S$

    A non-empty and finite alphabet of *symbols* Q

    The alphabet Q includes a *blank*, default symbol b

    Each cell in the tape contains a symbol $q \in Q$

    A partial *transition* function

$$\tau : S \times Q \rightarrow S \times Q \times \{\text{Left}, \text{None}, \text{Right}\}$$

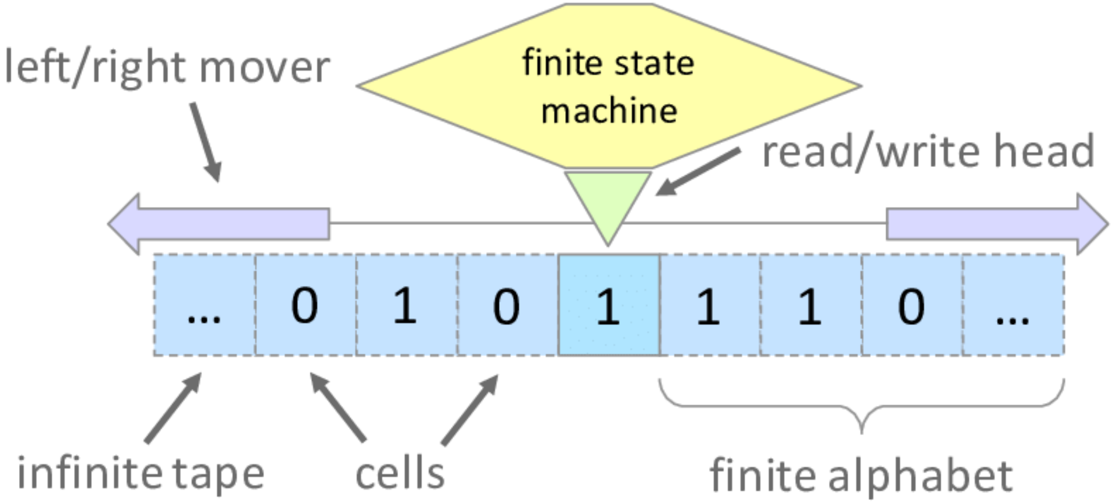
current state (S), *input symbol* (Q) → *next state* (S), *output symbol* (Q), *head move* ($\{\text{Left}, \text{None}, \text{Right}\}$)

It is partial in the sense it needs not be defined on any input tuple

    A subset of *terminal* states $T \subseteq S$

    An initial state $s_0 \in S$

Turing Machine (A. Turing, 1937)



- A ***busy beaver*** example (3 states)

$$S = \{A, B, C, \mathrm{HALT}\}$$
$$s_0 = A \qquad T = \{\mathrm{HALT}\}$$
$$Q = \{0, 1\} \qquad b = 0$$
$$\tau =$$
$$< A, 0 > \to < B, 1, \mathrm{Right} >$$
$$< A, 1 > \to < C, 1, \mathrm{Left} >$$
$$< B, 0 > \to < A, 1, \mathrm{Left} >$$
$$< B, 1 > \to < B, 1, \mathrm{Right} >$$
$$< C, 0 > \to < B, 1, \mathrm{Left} >$$
$$< C, 1 > \to < \mathrm{HALT}, 1, \mathrm{Right} >$$

Assume that the tape is infinite and plenty of blank symbols 0

What does this machine do?

Decisions and decidability (automation)

- What is a *problem*?

  A *problem* is a **relation** between *inputs* and *outputs* (= *solutions*)

  $K \subseteq I \times S$

- *Search problem*

  Typically, K may associate *one* input to *many* solutions

  Optimization problems

  A *search problem* plus an *objective* or *cost* function

  $c : S \to \mathbb{R}$ (from S to the set of real numbers)

  In general, the task in a search problem is finding the solution(s) having maximal or minimal cost

- ***Decision problem***

  The solution space S is *binary* $\{0, 1\}$
  and K associates each input to a *unique* solution: $K : I \to \{0, 1\}$

  Example of decision problem: $\Gamma \models \varphi$?

  The input space I contains all possible combinations of set Γ of wffs with individual wffs φ
  The solution is uniquely defined for any instance of such problems in I

Decisions and decidability (automation)

- ***Decidable problem***

  A decision problem K for which there exists an algorithm, i.e a *Turing machine*,

  (there are other ways of defining an algorithm or an *effective procedure*: they are all equivalent)

  that ***always terminates*** and produces the right answer in ***finite time***.

Example of an *undecidable* problem: The *Halting Problem*

Given the formal description of a particular Turing machine and a specific input,
is it possible to tell if whether it will either halt, eventually, or run forever?

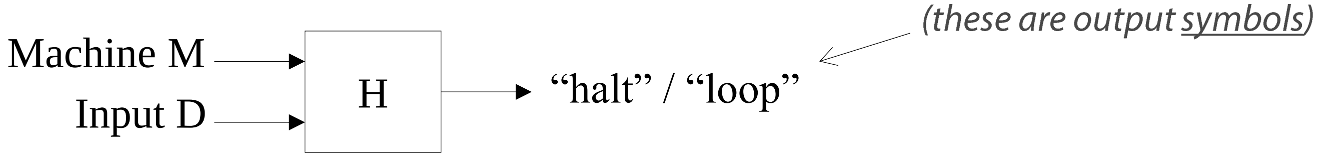
In other words, does it exist a Turing machine that, given in input the description of another Turing machine, will always produce the answer desired?

The answer is **no** (such a Turing machine *cannot* exist)

An aside: The *Halting Problem*

- Intuitive idea behind the proof (of the *undecidability* of this problem)

Let's assume there exists a Turing machine H that, given the description of any Turing machine M with input D always terminates producing an output "halt" or "loop" depending on whether M with input D will terminate or not



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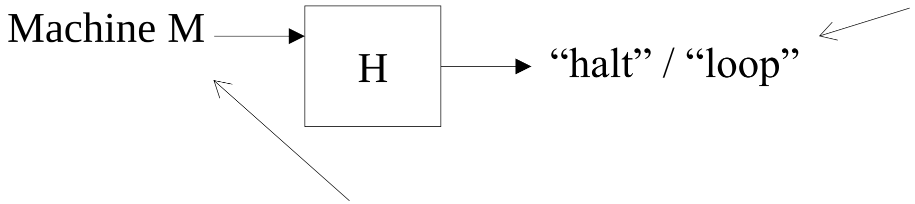
(these are output symbols)



Simplification: let's assume that H takes no input (i.e., the input is written inside the description of machine M)

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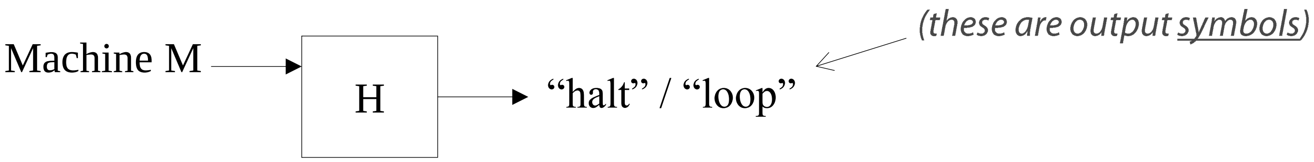
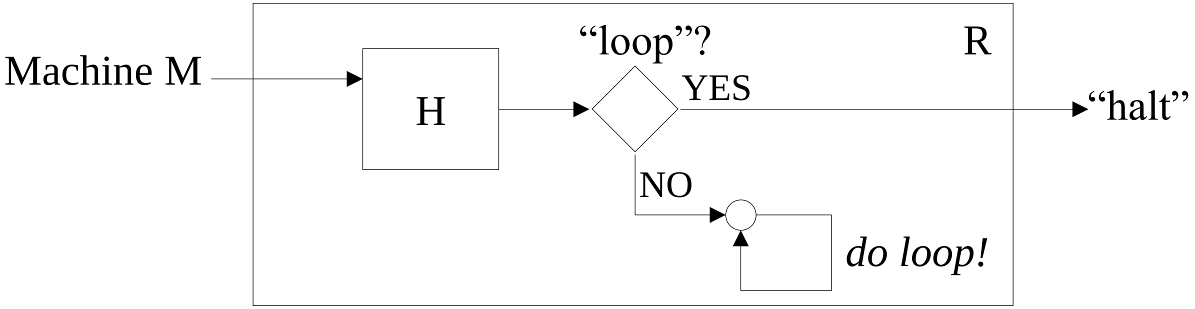


  Assume H existed
  We could build another Turing machine R that enters an infinite loop whenever the output of H is "halt" and that terminates, with output "halt", when H outputs "loop"



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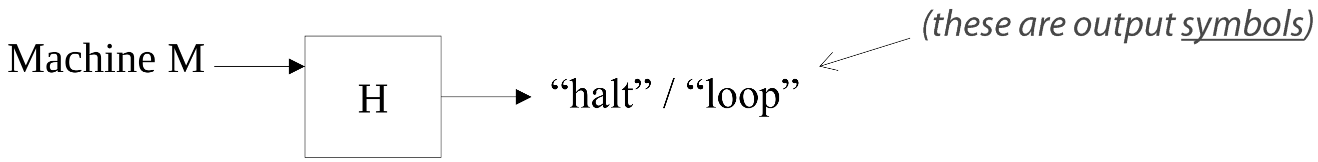

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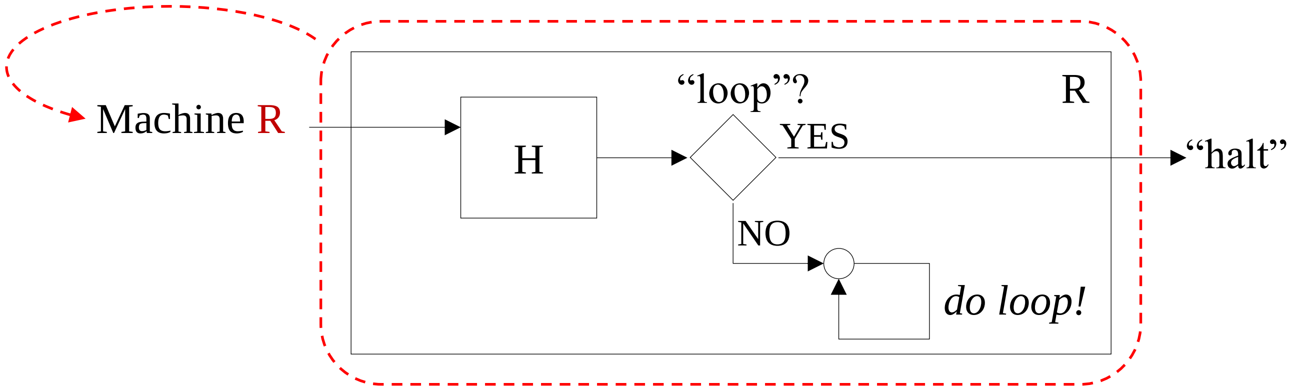


  What will be the output of R when given R *itself* as the input?
  R should *diverge* when R *terminates* and vice-versa: we have an absurdity

Computational complexity

CAUTION: *These notions apply to decidable problems only*

The benchmark is a (known) Turing machine that computes the correct answer in worst-case scenarios (= the least favorable inputs)

- *Time* complexity

  The number of *steps* that the Turing machine requires for computing the answer, as a function of some numerical dimension of the input (example: the number of atoms in a wff)

- *Memory* complexity

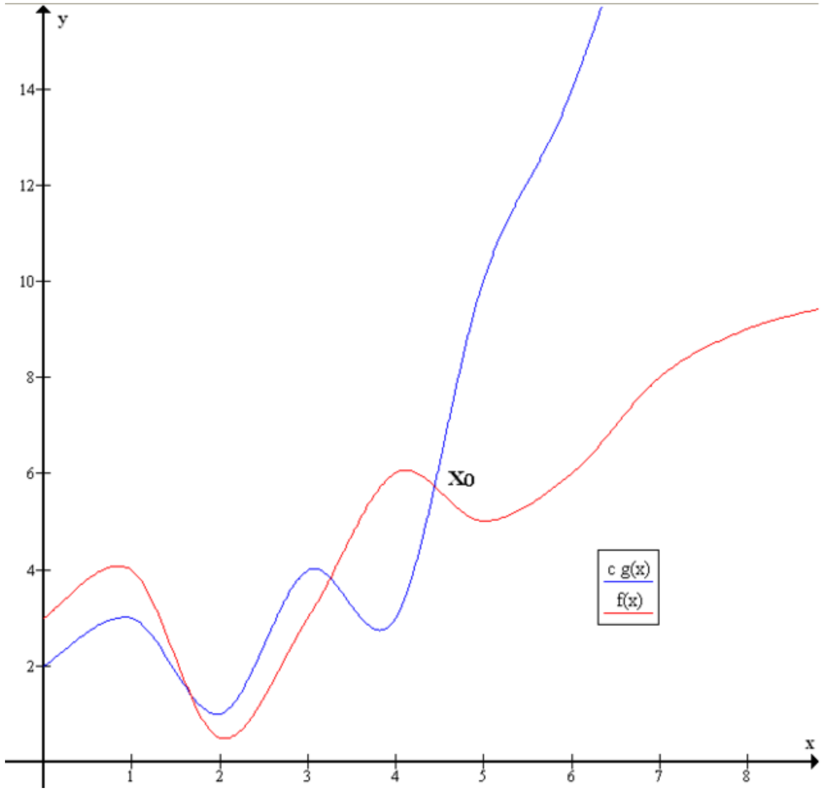
  The number of tape *cells* that the Turing machine requires for computing the answer, as a function of some numerical dimension of the input

- Big-O notation

$$f(x) = O(g(x))$$

  means that

$$\exists M > 0,\ \exists x_0 > 0 \quad \text{such that} \quad |f(x)| \leq M|g(x)|,\ \ \forall x > x_0$$



Classes P, NP and NP-complete – The SAT problem

- Class P

  The class of problems for which there is a Turing machine that requires $O(\mathrm{P}(n))$ time

  where P( ) is a polynomial of finite degree and n is the dimension of the (*worst-case*) input

- Class NP

  The class of all problems:

  a) A method for *enumerating* all possible answers (*recursive enumerability*)

  b) An algorithm in class P that *verifies* if a possible answer is also a *solution*

  It includes all problems in class P (that is, $\mathrm{P} \subseteq \mathrm{NP}$)

Classes P, NP and NP-complete - The SAT problem

- Class NP-complete

  It is a subclass of NP (NP-complete $\subseteq$ NP)

  A problem K is NP-complete if every problem in class NP is *reducible* to K

- Reducibility

  For class NP-complete

  Consider a problem K for which a decision algorithm $M(K)$ is known

  A problem J is *reducible* to K if there exist a decision algorithm $M(J)$ such that:

  a) algorithm $M(K)$ is called just once, as a "subroutine", at the end of $M(J)$

  b) apart from $M(K)$, $M(J)$ has polynomial complexity

- The problem SAT

  Is NP-complete (*historically, it is the first one to be known*)

  Moral: if we had a polynomial decision algorithm for SAT, we would also have that

  $\mathrm{P} = \mathrm{NP}$

  This is not known for certain: it is commonly believed that $\mathrm{P} \neq \mathrm{NP}$

  (*and quite a lot will change in the digital world, if this belief turns to be false*)

Exhaustive (Tree) Search

Satisfiability and decidability (in L_P)

- Is the decision problem " is the wff φ <u>satisfiable</u>? "?
  - It can be transformed into a *search* problem
    - that is, finding a possible world (in the set of all possible worlds) that satisfies φ
      - In the scientific literature, this problem is called "SAT"
    - *Intuition*: we can try every possible value assignment for the atoms in φ
    - *Hint:* the problem is NP-complete

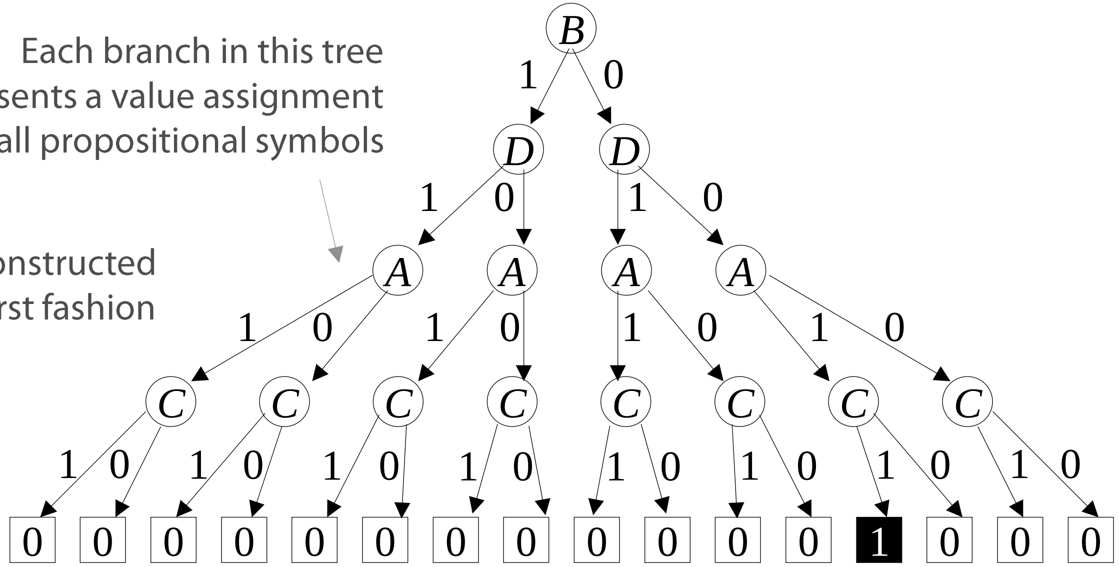
Satisfiability and decidability (in L_P)

Example: is this wff *satisfiable*?

$\varphi : \neg(B \vee D \vee \neg(A \wedge C))$

Each branch in this tree represents a value assignment to all propositional symbols

The tree can be constructed in a depth-first fashion



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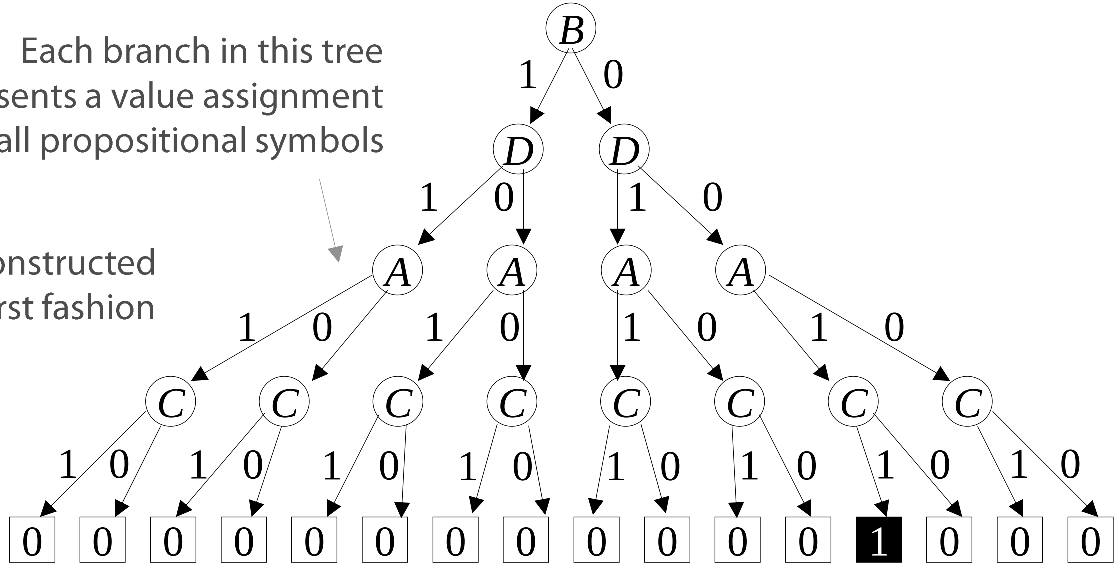

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In this case, a depth-first algorithm stops here

But the algorithm is forced to try all possible assignments when φ is *not* satisfiable, for example, with: $(\neg B \wedge \neg D \wedge \neg A \wedge \neg C)$

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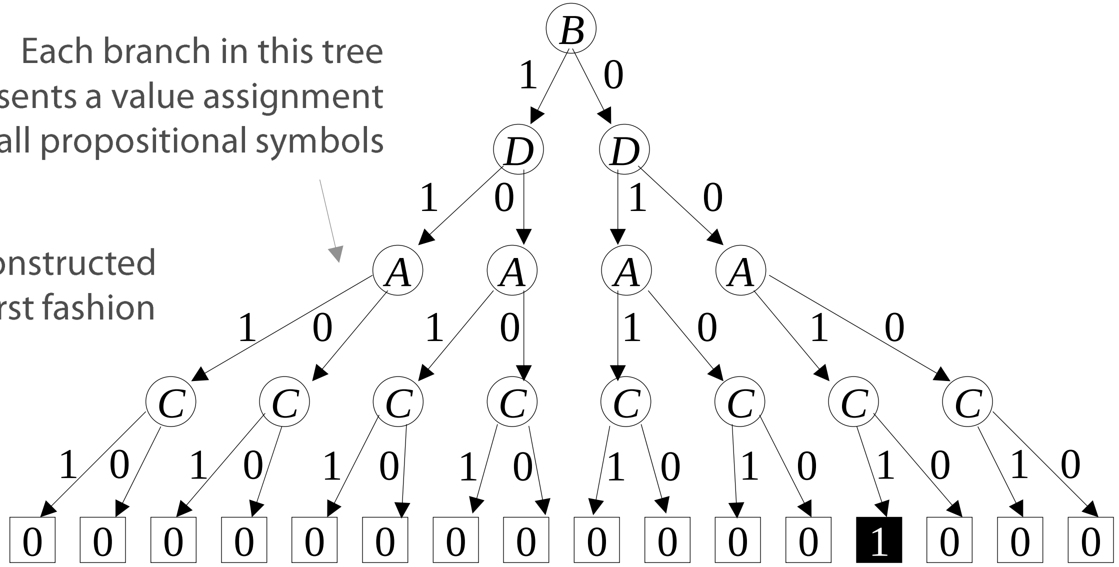

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This method has $O(2^n)$ time complexity, where n is the number of propositional symbols