

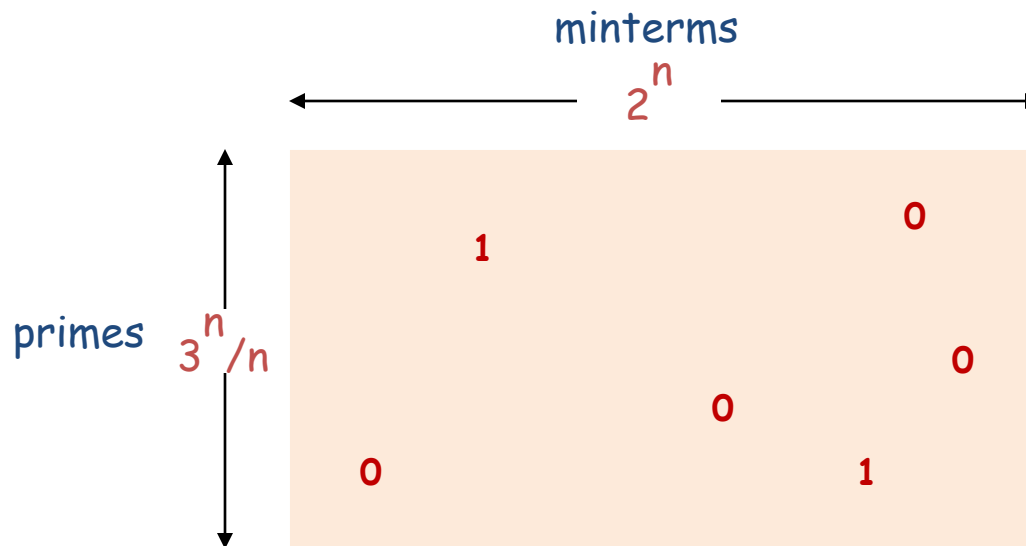
Quine-McCluskey (Tabular) Minimization

- ✓ Two step process utilizing tabular listings to:
 - Identify prime implicants (implicant tables)
 - Identify minimal PI set (cover tables)
- ✓ All work is done in tabular form
 - Number of variables is not a limitation
 - Basis for many computer implementations
 - Don't cares are easily handled
- ✓ Proper organization and term identification are **key factors for correct results**

Difficulty

Note:

- ✓ 2^n minterms
- ✓ $\sim 3^n/n$ primes

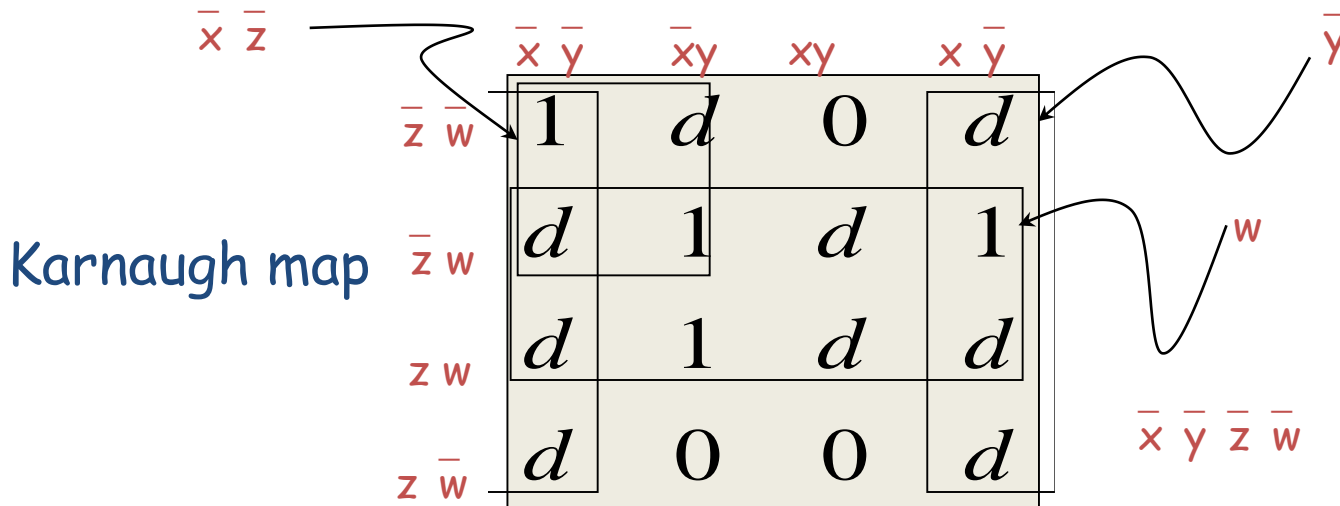


- ✓ Thus $O(2^n)$ rows and $O(3^n/n)$ columns and minimum covering problem is NP-complete.

Example

$$F = \bar{x}\bar{y}z\bar{w} + \bar{x}yz\bar{w} + x\bar{y}z\bar{w} + \bar{x}yzw$$

$$D = \bar{y}z + xyw + \bar{x}\bar{y}z\bar{w} + x\bar{y}w + \bar{x}yz\bar{w}$$



Primes: $\bar{y} + w + \bar{x}\bar{z}$

Covering Table

Solution: $\{1,2\} \Rightarrow \bar{y} + w$ is minimum prime cover.
(also $w + \bar{x}\bar{z}$)

	\bar{y}	w	$\bar{x}\bar{z}$
$\bar{x}\bar{y}\bar{z}\bar{w}$	1	0	1
$\bar{x}y\bar{z}w$	0	1	1
$x\bar{y}\bar{z}w$	1	1	0
$\bar{x}yzw$	0	1	0

Covering Table

		\bar{y}	w	$\bar{x} \bar{z}$	Primes of $f+d$
Minterms of f	$\bar{x} \bar{y} \bar{z} \bar{w}$	1	0	1	
	$\bar{x} y \bar{z} w$	0	1	1	
	$x \bar{y} \bar{z} w$	1	1	0	
	$\bar{x} y z w$	0	1	0	← Row singleton (essential minterm)

↑
Essential prime

- ✓ **Definition:** An essential prime is any prime that **uniquely** covers a minterm of f .

Quine-McCluskey Minimization (cont.)

- ✓ Terms are initially listed one per line in groups
 - Each group contains terms with the same number of **true and complemented variables**
 - Terms are listed in numerical order within group
- ✓ Terms and implicants are identified using one of three common notations
 - full variable form
 - cellular form
 - 1,0,- form

Example of Different Notations

$$F(A, B, C, D) = \sum_m (4,5,6,8,10,13)$$

	Full variable	Cellular	1,0,-
1	$\overline{A}\overline{B}\overline{C}\overline{D}$	4	0100
	$\overline{A}\overline{B}CD$	8	1000
2	$\overline{A}B\overline{C}\overline{D}$	5	0101
	$\overline{A}BC\overline{D}$	6	0110
	$\overline{A}BCD$	10	1010
3	$A\overline{B}\overline{C}\overline{D}$	13	1101

Notation Forms

- ✓ **Full variable form:** variables and complements in algebraic form
 - hard to identify when adjacency applies
 - very easy to make mistakes
- ✓ **Cellular form:** terms are identified by their decimal index value
 - easy to tell when adjacency applies; indexes must differ by power of two (one bit)
- ✓ **1,0,- form:** terms are identified by their binary index value
 - easier to translate to/from full variable form
 - easy to identify when adjacency applies, one bit is different
 - shows variable(s) dropped when adjacency is used
- ✓ Different forms may be mixed during the minimization

Implication Table (1,0,-)

✓ Quine-McCluskey Method

- Tabular method to systematically find all prime implicants

$$f(A,B,C,D) = \sum_m(1,2,5,6,7,9,10) + \sum_d(0,13,15)$$

- Part 1: Find all prime implicants
- Step 1: Fill Column 1 with **onset** and **DC-set** minterm indices.
Group by number of true variables (**# of 1's**).

NOTE THAT DCs ARE INCLUDED IN THIS STEP!

Implication Table		
Column I		
0000		
0001		
0010		
0101		
0110		
1001		
1010		
0111		
1101		
1111		

Minimization - First Pass (1,0,-)

✓ Quine-McCluskey Method

- Tabular method to systematically find all prime implicants
- $f(A,B,C,D) = \Sigma_m(1,2,5,6,7,9,10) + \Sigma_d(0,13,15)$
- Part 1: Find all prime implicants
- Step 2: Apply Adjacency - Compare elements of group with N 1's against those with N+1 1's. One bit difference implies adjacent. Eliminate variable and place in next column.

E.g., 0000 vs. 0100 yields 0-00

0000 vs. 1000 yields -000

When used in a combination, mark with a check. If cannot be combined, mark with a star. These are the prime implicants.

Repeat until nothing left.

Implication Table			
Column I		Column II	
0000	0	000-	0,1
		00-0	0,2
0001	1		
0010	2	0-01	1,5
		-001	1,9
0101	5	0-10	2,6
0110	6	-010	2,10
1001	9		
1010	10	01-1	5,7
		-101	5,13
0111	7	011-	6,7
1101	13	1-01	9,13
1111	15	-111	7,15
		11-1	13,15

Minimization - Second Pass (1,0,-)

✓ Quine-McCluskey Method

- Step 2 cont.: Apply Adjacency - Compare elements of group with N 1's against those with N+1 1's. One bit difference implies adjacent. Eliminate variable and place in next column.

E.g., 0000 vs. 0100 yields 0-00

00-0 vs. 10-0 yields -0-0

When used in a combination, mark with a check. If cannot be combined, mark with a star.

THESE ARE THE PRIME IMPLICANTS.

Repeat until nothing left.

- ✓ The set of * constitutes the Complete Sum Σ_c

Implication Table		
Column I	Column II	Column III
0000 ✓ 0	000- 0,1 00-0 0,2	--01 1,5,9,13 - 1-1 5,7,13,15
0001 ✓ 1		
0010 ✓ 2	0-01 1,5 -001 1,9	
0101 ✓ 5	0-10 2,6	
0110 ✓ 6	-010 2,10	
1001 ✓ 9		
1010 ✓ 10	01-1 5,7 -101 5,13	
0111 ✓ 7	011- 6,7	
1101 ✓ 13	1-01 9,13	
1111 ✓ 15	-111 7,15 11-1 13,15	

Prime Implicants

$$f(A,B,C,D) = \Sigma_m(1,2,5,6,7,9,10) + \Sigma_d(0,13,15)$$

A \ B		C D			
		00	01	11	10
A	00	X	1	0	1
	01	0	1	1	1
	11	0	X	X	0
	10	0	1	0	1

Diagram illustrating the Karnaugh map for the function $f(A,B,C,D) = \Sigma_m(1,2,5,6,7,9,10) + \Sigma_d(0,13,15)$. The map is a 4x4 grid with rows labeled A (00, 01, 11, 10) and columns labeled B (00, 01, 11, 10). The cells contain values: 00 (X), 01 (1), 11 (0), 10 (1) for A=00; 01 (0), 01 (1), 11 (1), 10 (1) for A=01; 00 (0), 01 (X), 11 (X), 10 (0) for A=11; 00 (0), 01 (1), 11 (0), 10 (1) for A=10. Prime implicants are highlighted: a blue vertical rectangle covers cells (00,01), (01,01), (11,01), (10,01) and (00,01), (01,01), (11,01), (10,01); a blue horizontal rectangle covers cells (01,11), (01,10); a red horizontal rectangle covers cells (00,10), (10,10); and a red vertical rectangle covers cells (00,10), (10,10). Brackets labeled C, D, and B indicate groupings.

Stage 2: find smallest set of prime implicants that cover the active-set
 Note that essential prime implicants must be in the final expression

Coverage Table

rows = prime implicants

columns = ON-set elements (minterms)

place an "X" if ON-set element is covered by the prime implicant

NOTE: DON'T INCLUDE DCs IN COVERAGE TABLE; THEY DON'T HAVE TO BE MANDATORY COVERED

Coverage Chart

		1	2	5	6	7	9	10
0,1	000-	X						
0,2	00-0		X					
2,6	0-10		X		X			
2,10	-010		X					X
6,7	011-				X	X		
1,5,9,13	--01	X		X			X	
5,7,13,15	-1-1			X		X		

Row and Column Dominance

- ✓ **Definition:** Given two rows i_1 and i_2 , a row i_1 is said to **dominate** i_2 if it has checks in all columns in which i_2 has checks, i.e. it is a superset of i_2

Example:

i_1	x x	x	x x	x
i_2	x x		x x	

i_1 dominates i_2

- ✓ We can remove row i_2 , because we would never choose i_2 in a minimum cover since it can always be replaced by i_1 (i_2 is anymore a prime implicant).

DOMINATED ROWS CAN BE ELIMINATED

Row and Column Dominance

- ✓ **Definition:** Given two columns j_1 and j_2 , if the set of primes of column j_2 is contained in the set of primes of column j_1 , j_1 is said to be **dominated** by j_2 .

Example:

	j_1	j_2
	x	
	x	x
j_2 dominates j_1	x	
	x	x

- ✓ We can remove column j_1 since we have to choose a prime to cover j_2 , any such prime also covers j_1 , that would result covered as well.

DOMINATED COLUMNS CAN BE ELIMINATED

Pruning the Covering Table

1. Remove all rows covered by **essential primes** (columns in row **singletons**). Put these primes in the cover G .
 2. **Group identical rows together and remove dominated rows.**
 3. **Remove dominating columns.** For equal columns, keep just one to represent them.
 4. Newly formed row singletons define **n -ary essential primes.**
 5. Go to 1 if covering table decreased.
- ✓ The algorithm may terminate successfully with a set of primes and an empty table.
 - ✓ In case it terminate with a non empty table, the resulting reduced covering table is called the **cyclic core**. This has to be solved. A minimum solution for the cyclic core must be added to the resulting G .

Coverage Table (cont.)

Coverage Chart

		1	2	5	6	7	9	10
0,1	000-	X						
0,2	00-0		X					
2,6	0-10		X		X			
2,10	-010		X					X
6,7	011-				X	X		
1,5,9,13	--01	X		X			X	
5,7,13,15	-1-1			X		X		

		1	2	5	6	7	9	10
0,1	000-	X						
0,2	00-0		X					
2,6	0-10		X		X			
2,10	-010		X					X
6,7	011-				X	X		
1,5,9,13	--01	X		X			X	
5,7,13,15	-1-1			X		X		

If column has a single x, than the implicant associated with the row is **essential**.
It must appear in the minimum cover

Coverage Table (cont.)

		1	2	5	6	7	9	10
0,1	000-	X						
0,2	00-0		X					
2,6	0-10		X		X			
2,10	-010		X					X
6,7	011-				X	X		
1,5,9,13	--01	X		X			X	X
5,7,13,15	-1-1			X		X		

Eliminate all columns covered by essential primes

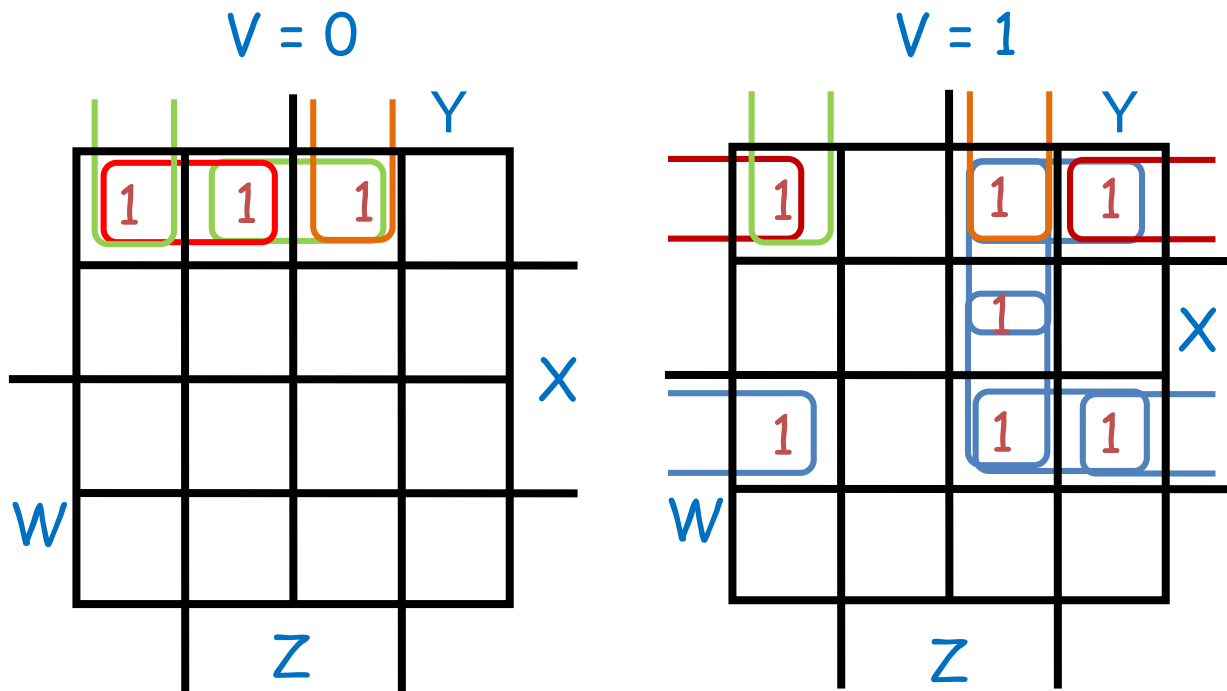
		1	2	5	6	7	9	10
0,1	000-	X						
0,2	00-0		X					
2,6	0-10		X		X			
2,10	-010		X					X
6,7	011-				X	X		
1,5,9,13	--01	X		X			X	X
5,7,13,15	-1-1			X		X		

Find minimum set of rows that cover the remaining columns

$$F = \bar{B}C\bar{D} + \bar{A}BC + \bar{C}D$$

Quine Mc Clunskey: *Cyclic Core example*

$$F = \sum_m (0, 1, 3, 16, 18, 19, 23, 28, 30, 31)$$



$$F = v'w'y'x' + v'w'x'z + w'x'y'z' + w'x'yz + vw'x'z' + vw'x'y + vwXZ' + vwxy + vw'yz + vxyz$$

A B C D E F G H I J

Implication Table (1,0,-)

✓ Quine-McCluskey Method

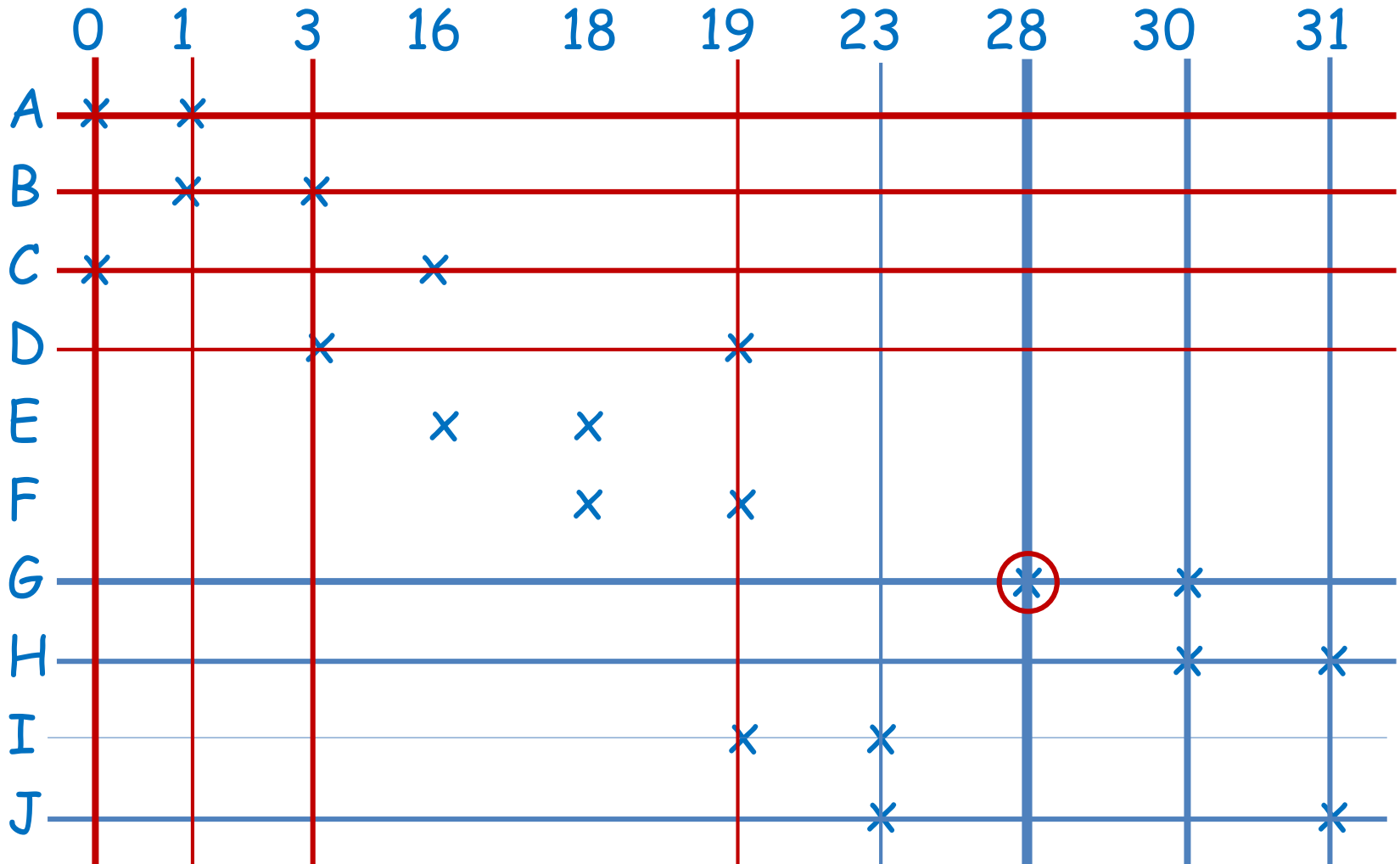
- Tabular method to systematically find all prime implicants
- $f(v,w,x,y,z) = \Sigma m(0,1,3,16,18,19,23,28,30,31)$
- Part 1: Find all prime implicants
- Step 1: Fill Column 1 with active-set and DC-set minterm indices. Group by number of true variables (# of 1's).

Implication Table			
	Column I	Column II	
0	00000	0000-	A: 0 1
1	00001	-0000	C: 0 16
16	1 0000	000-1	B: 1 3
3	00 01 1	100-0	E: 16 18
18	1001 0	-0011	D: 3 19
19	100 11	1001-	F: 18 19
28	1 1 100	10-11	I: 19 23
23	10 111	111-0	G: 28 30
30	111 10	1-111	J: 23 31
31	11111	1111-	H: 30 31

$$F = \underbrace{v'w'y'x'}_A + \underbrace{v'w'x'z}_B + \underbrace{w'x'y'z'}_C + \underbrace{w'x'yz}_D + \underbrace{vw'x'z'}_E + \underbrace{vw'x'y}_F + \underbrace{vwxz'}_G + \underbrace{vwxy}_H + \underbrace{vw'yz}_I + \underbrace{vxyz}_J$$

Quine Mc Clunskey

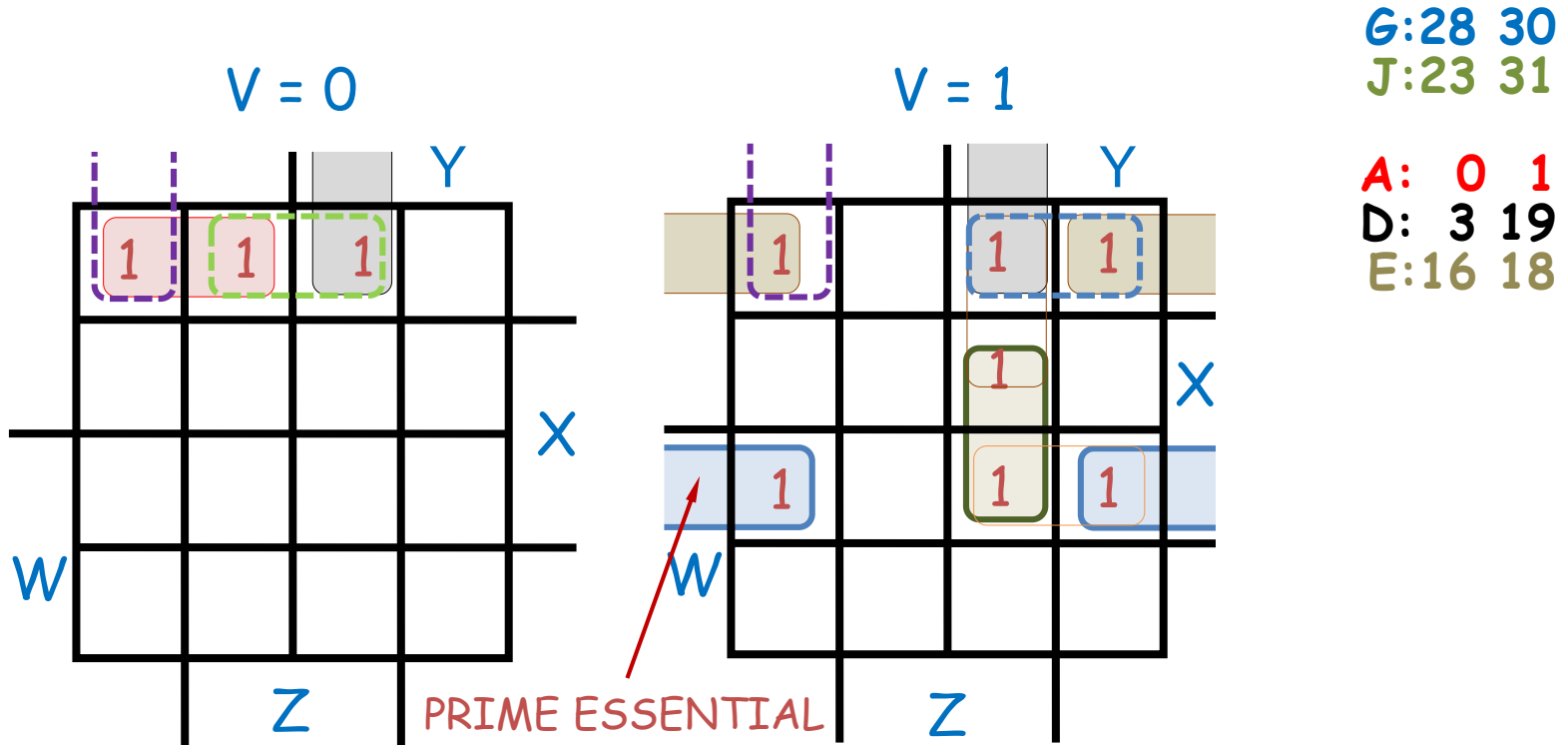
$$F = \sum_5(0,1,3,16,18,19,23,28,30,31)$$



G+J+A+D+E;

Quine Mc Clunskey: *Cyclic Core example*

$$F = \sum_m (0, 1, 3, 16, 18, 19, 23, 28, 30, 31)$$



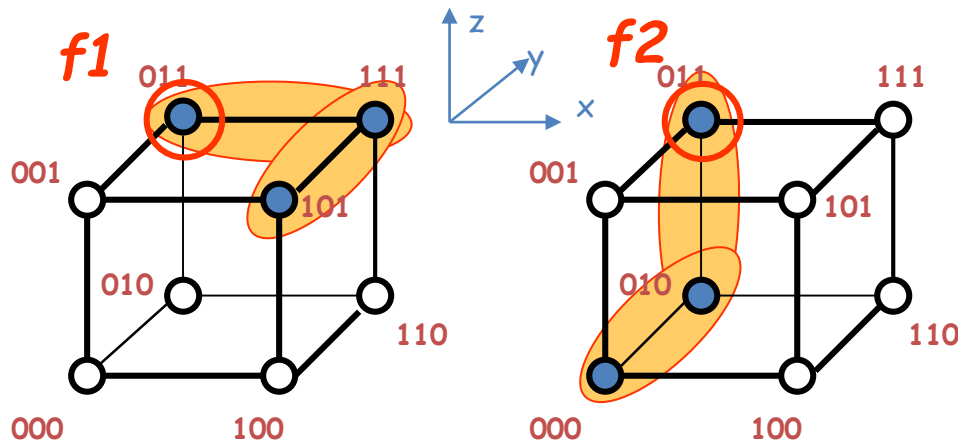
$$F = v'w'y'x' + \underline{v'w'x'z} + \underline{w'x'y'z'} + w'x'yz + \underline{vw'x'z'} + \underline{vw'x'y} + \underline{vwxz'} + vwxy + vw'yz + vxyz$$

A
B
C
D
E
F
G
H
I
J

$$G + J + A + D + E;$$

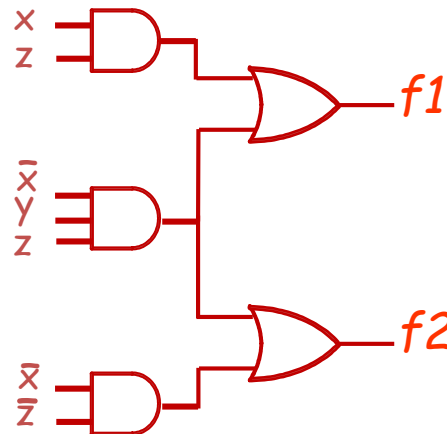
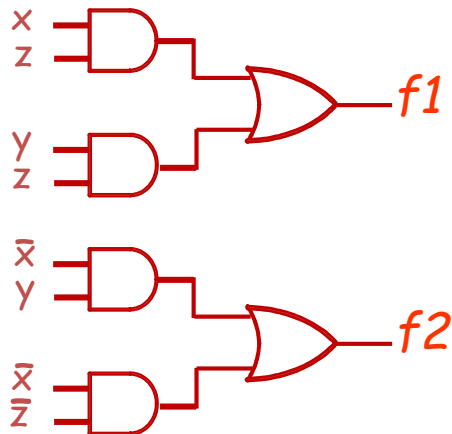
Generating Primes - multiple outputs

Example: $f_1(x, y, z) = \sum m(3,5,7)$, $f_2(x, y, z) = \sum m(0,2,3)$



$x y z$		$f_1 = yz + zx$
0 1 1	- 1 1	
1 0 1	1 - 1	
1 1 1		

$x y z$		$f_2 = \bar{x}\bar{z} + \bar{x}y$
0 0 0	0 - 0	
0 1 0	0 1 -	
0 1 1		



The idea is that we can share terms:

- using separate optimizations 6 gates and $G=12$
- sharing a term 5 gates and $G=11$.

$$f_1 f_2 = (yz + zx)(\bar{x}\bar{z} + \bar{x}y)$$

$$f_1 f_2 = \bar{x}yz$$

Generating Primes - multiple outputs

- ✓ Theorem: if p_1 is a prime implicant for f_1 , and p_2 is a prime implicant for f_2 , then if $p_1.p_2 \neq 0$, $p_1.p_2$ is a prime implicant of $f_1.f_2$
- ✓ Theorem: if p_3 is a prime implicant for $f_1.f_2$, then there exist p_1 for f_1 , and p_2 for f_2 , such that $p_3 = p_1.p_2$
- ✓ We can conclude that all prime implicants of $f_1.f_2$ are minimal sharable products for f_1 and f_2 : and that all prime implicants for $f_1.f_2$ are created by products of prime implicants for f_1 and f_2
- ✓ The way to use this is to make the prime implicants of $f_1.f_2$ available to the minimizations of f_1 and f_2 by extending the table concept

Generating Primes - multiple outputs

- ✓ Procedure similar to single-output function, except: include also the primes of the products of individual functions

	f_1 minterms	f_2 minterms
Rows for f_1 prime implicants: mark only f_1 columns		
Rows for f_2 prime implicants: mark only f_2 columns		
Rows for $f_1 f_2$ prime implicants: mark both f_1 and f_2 columns		

Minimize multiple-output cover

✓ Example, cont.

$$f_1 \quad \begin{array}{l} m_3 = 011 \quad p_1 = yz \\ m_5 = 101 \quad p_2 = xz \\ m_7 = 111 \end{array}$$

$$f_2 \quad \begin{array}{l} m_0 = 000 \quad p_3 = \bar{x}y \\ m_2 = 010 \quad p_4 = \bar{x}\bar{z} \\ m_3 = 011 \end{array}$$

$$f_1 f_2 \quad m_3 = 011 \quad p_5 = \bar{x}yz$$

Min cover has 3 primes:

$$F = \{ p_2, p_4, p_5 \}$$

	m_3	m_5	m_7	m_0	m_2	m_3
p_1	✓		✓			
p_2		✓	✓			
p_3					✓	✓
p_4				✓	✓	
p_5	✓					✓

	m_3	m_3
p_1	✓	
p_3		✓
p_5	✓	✓

Note that selecting p_5 and removing all columns the marks coverage is complete

