# Circuit Optimization

- ✓ Goal: To obtain the simplest implementation for a given function
- ✓ Optimization is a more formal approach to simplification that is performed using a specific procedure or algorithm
- ✓ Optimization requires a cost criterion to measure the simplicity of a circuit
- ✓ Distinct cost criteria we will use:
  - Literal cost (L)
  - Gate input cost (G)
  - Gate input cost with NOTs (GN)

# Literal Cost

- ✓ Literal – a variable or it complement
- ✓ Literal cost – the number of literal appearances in a Boolean expression corresponding to the logic circuit diagram
- ✓ Example, which solution is best?
  - $F = BD + A\overline{B}C + A\overline{C}\,\overline{D}$ L = 8
  - $F = BD + A\overline{B}C + A\overline{B}\,\overline{D} + AB\overline{C}$ L = 11
  - $F = (A + B)(A + D)(B + C + \overline{D})(\overline{B} + \overline{C} + D)$ L = 10

# Gate Input Cost

- ✓ Gate input costs - the number of inputs to the gates in the implementation corresponding exactly to the given equation or equations. (G - inverters not counted, GN - inverters counted)
- ✓ For SOP and POS equations, it can be found from the equation(s) by finding the sum of:
  - all literal appearances
  - the number of terms excluding single literal terms,(G) and
  - optionally, the number of distinct complemented single literals (GN).
- ✓ Example, which solution is best?
  - $F = BD + A\overline{B}C + A\overline{C}\,\overline{D}$ G =11, GN = 14
  - $F = BD + A\overline{B}C + A\overline{B}\,\overline{D} + AB\overline{C}$ G =15, GN = 18
  - $F = (A + \overline{B})(A + D)(B + C + \overline{D})(\overline{B} + \overline{C} + D)$ G =14, GN = 17

# Cost Criteria (continued)

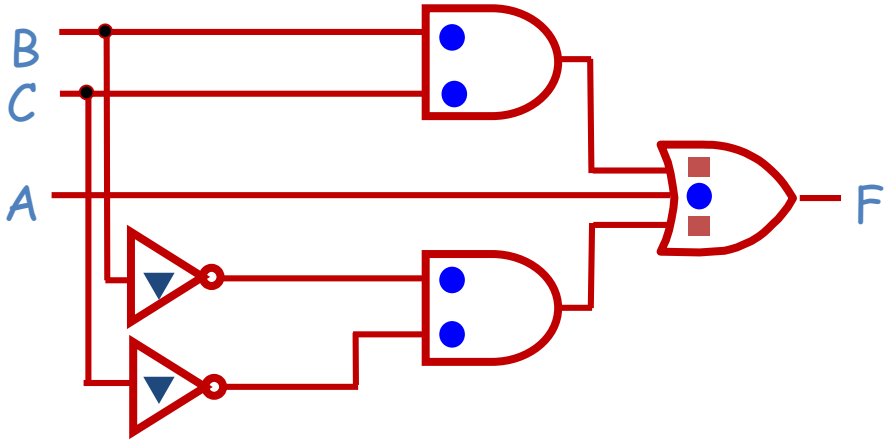
✓ Example 1:

✓ $F = A + B\,C + \overline{B}\,\overline{C}$

GN = G + 2 = 9

L = 5

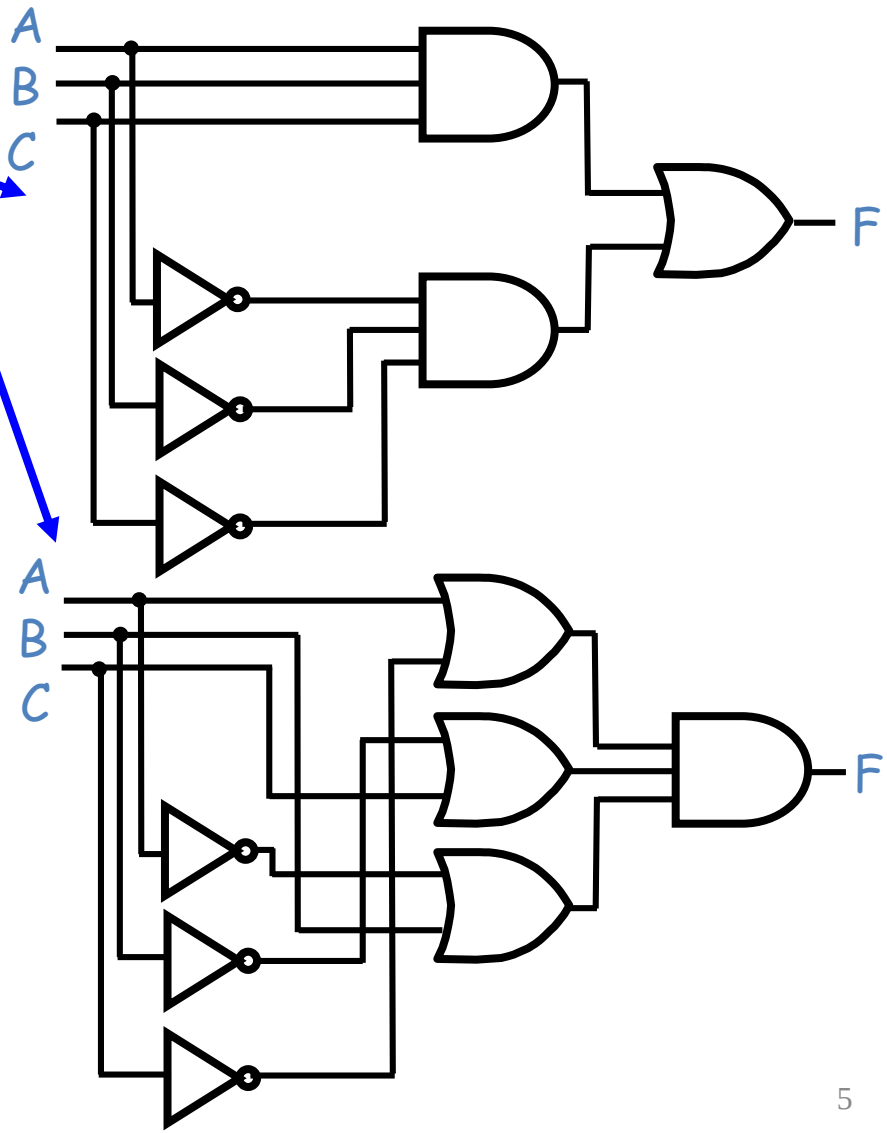
G = L + 2 = 7




- L (literal count) counts the AND inputs and the single literal OR input.
- G (gate input count) adds the remaining OR gate inputs
- GN(gate input count with NOTs) adds the inverter inputs

# Cost Criteria (continued)

- ✓ Example 2:
- ✓ $F = A\ B\ C + \overline{A}\ \overline{B}\ \overline{C}$
- ✓ L = 6 G = 8 GN = 11
- ✓ $F = (A + \overline{C})(\overline{B} + C)(\overline{A} + B)$
- ✓ L = 6 G = 9 GN = 12
- ✓ Same function and same literal cost
- ✓ But first circuit has better gate input count and better gate input count with NOTs



# Boolean Function Optimization

- ✓ Minimizing the gate input (or literal) cost of a (a set of) Boolean equation(s) reduces circuit cost.
- ✓ We choose gate input cost.
- ✓ Boolean Algebra and graphical techniques are tools to minimize cost criteria values.
- ✓ Some important questions:
  - When do we stop trying to reduce the cost?
  - Do we know when we have a minimum cost?
- ✓ Treat optimum or near-optimum cost functions for two-level (SOP and POS) circuits first.
- ✓ Introduce a graphical technique using Karnaugh maps (K-maps, for short)

# Karnaugh Maps (K-map)

- ✓ A K-map is a collection of squares
  - Each square represents a minterm
  - The collection of squares is a graphical representation of a Boolean function
  - Adjacent squares differ in the value of one variable
  - Alternative algebraic expressions for the same function are derived by recognizing patterns of squares (corresponding to cubes)
- ✓ The K-map can be viewed as
  - A reorganized version of the truth table or a particular cube representation

# Some Uses of K-Maps

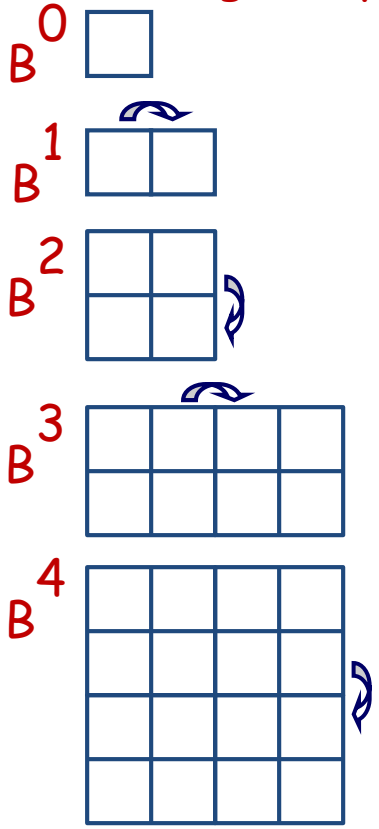
- ✓ Provide a means for:
  - Finding optimum
    - SOP and POS standard forms, and
    - two-level AND/OR and OR/AND circuit implementations

    for functions with small numbers of variables
  - Visualizing concepts related to manipulating Boolean expressions
  - Demonstrating concepts used by computer-aided design programs to simplify large circuits

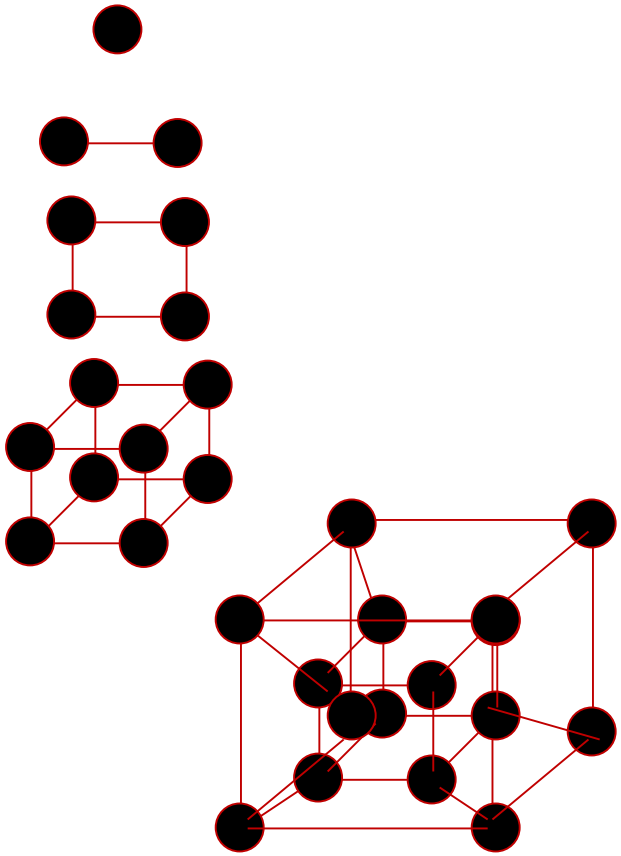
# The Boolean Space $B^n$

- ✓ $B = \{0, 1\}$
- ✓ $B^2 = \{0,1\} \times \{0,1\} = \{00, 01, 10, 11\}$

Karnaugh Maps:




Boolean Cubes:



# Two Variable Maps

- ✓ A 2-variable Karnaugh Map:
  - Note that minterm $m_0$ and minterm $m_1$ are "adjacent" and differ in the value of the variable y
  - Similarly, minterm $m_0$ and minterm $m_2$ differ in the x variable.
  - Also, $m_1$ and $m_3$ differ in the x variable as well.
  - Finally, $m_2$ and $m_3$ differ in the value of the variable y

| x \ y | y = 0 | y = 1 |
|---|---|---|
| x = 0 | $m_0 = \bar{x}\,\bar{y}$ | $m_1 = \bar{x}\,y$ |
| x = 1 | $m_2 = x\,\bar{y}$ | $m_3 = x\,y$ |

# K-Map and Truth Tables

- ✓ The K-Map is just a different form of the truth table.
- ✓ Example – Two variable function:
  - We choose a,b,c and d from the set {0,1} to implement a particular function, F(x,y).

Function Table

| Input Values (x,y) | Function Value F(x,y) |
|---|---|
| 0 0 | a |
| 0 1 | b |
| 1 0 | c |
| 1 1 | d |

K-Map

| x \ y | y = 0 | y = 1 |
|---|---|---|
| x = 0 | a | b |
| x = 1 | c | d |

# K-Map Function Representation

- ✓ Example: F(x,y) = x

| F = x | y = 0 | y = 1 |
|---|---|---|
| x = 0 | 0 | 0 |
| x = 1 | 1 | 1 |

- ✓ For function F(x,y), the two adjacent cells containing 1's can be combined using the Minimization Theorem:

$$F(x,y) = x\bar{y} + xy = x$$

# K-Map Function Representation

- Example: $G(x,y) = \overline{x}y + x\overline{y} + xy$

| G=x+y | y = 0 | y = 1 |
|---|---|---|
| x = 0 | 0 | 1 |
| x = 1 | 1 | 1 |

- For G(x,y), two pairs of adjacent cells containing 1's can be combined using the Minimization Theorem:

$$G(x,y) = (x\overline{y} + xy) + (\overline{x}y + xy) = x + y$$

Duplicate xy

# Three Variable Maps

- A three-variable K-map:

| x \ yz | yz=00 | yz=01 | yz=11 | yz=10 |
|---|---|---|---|---|
| x=0 | m0 | m1 | m3 | m2 |
| x=1 | m4 | m5 | m7 | m6 |

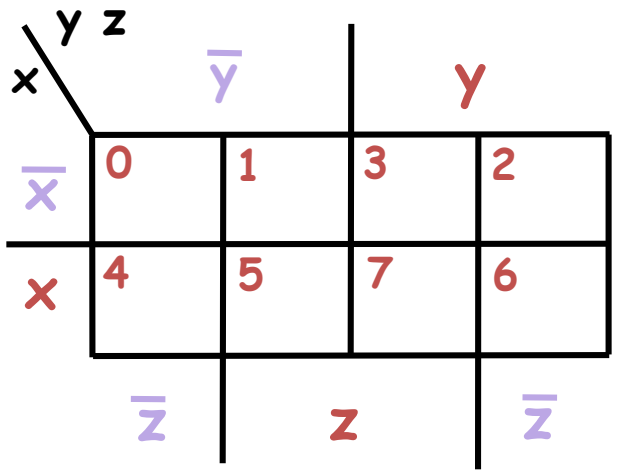
- Where each minterm corresponds to the product terms:

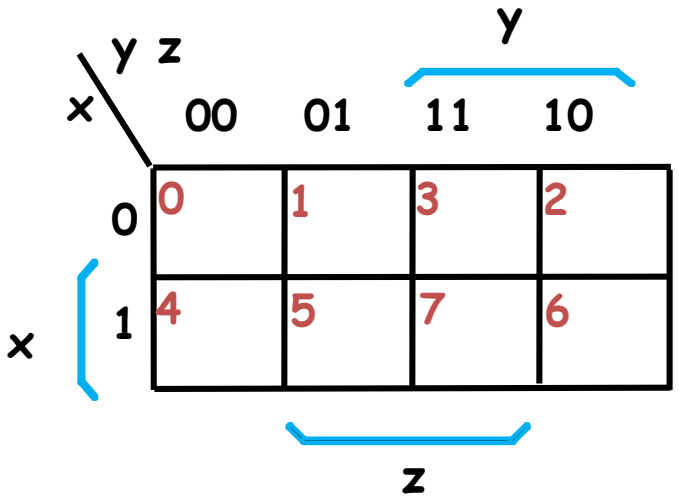
| x \ yz | yz=00 | yz=01 | yz=11 | yz=10 |
|---|---|---|---|---|
| x=0 | $\bar{x}\,\bar{y}\,\bar{z}$ | $\bar{x}\,\bar{y}\,z$ | $\bar{x}\,y\,z$ | $\bar{x}\,y\,\bar{z}$ |
| x=1 | $x\,\bar{y}\,\bar{z}$ | $x\,\bar{y}\,z$ | $x\,y\,z$ | $x\,y\,\bar{z}$ |

- Note that if the binary value for an index differs in one bit position, the minterms are adjacent on the K-Map

# Alternative Map Labeling

- ✓ Map use largely involves:
  - Entering values into the map, and
  - Reading off product terms from the map.
- ✓ Alternate labelings are useful:

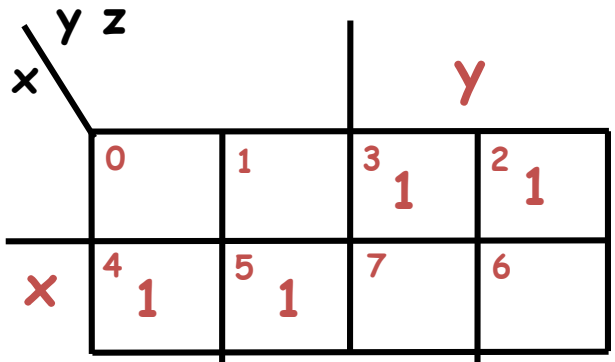


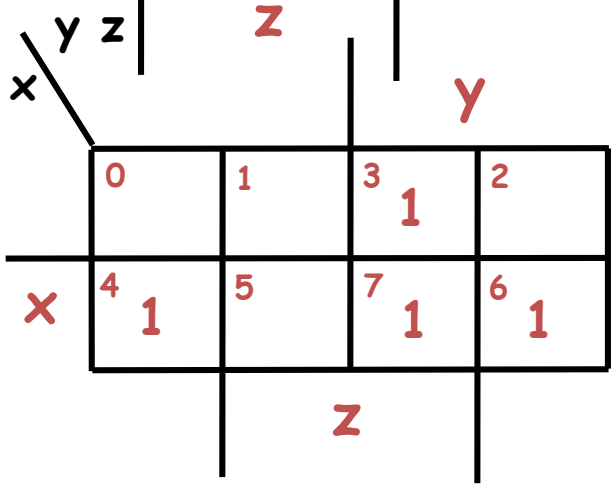
# Example Functions

- ✓ By convention, we represent the minterms of F by a "1" in the map and leave the minterms of blank $\overline{F}$
- ✓ Example:

$$F(x, y, z) = \sum_m (2, 3, 4, 5)$$




- ✓ Example:
- ✓ $F(x, y, z) = \sum_m (3, 4, 6, 7)$
- ✓ Learn the locations of the 8 indices based on the variable order shown (x, most significant and z, least significant) on the map boundaries

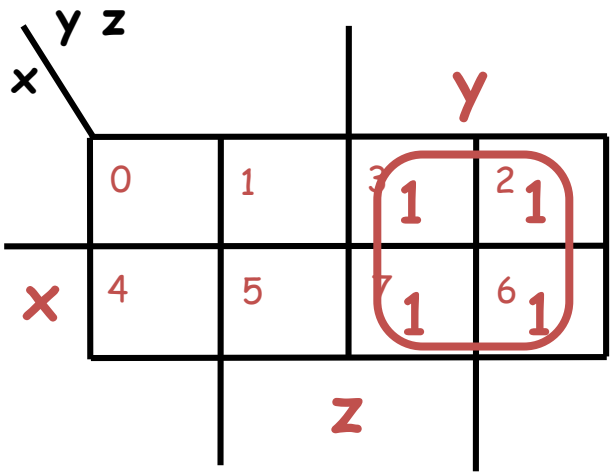


# Combining Squares

- ✓ By combining squares, we reduce number of literals in a product term, reducing the literal cost, thereby reducing the other two cost criteria
- ✓ On a 3-variable K-Map:
  - One square represents a minterm with three variables
  - Two adjacent squares represent a cube that is product term with two variables
  - Four "adjacent" terms represent a cube that is product term with one variable
  - Eight "adjacent" terms is the function of all ones (no variables) is a tautology $f^1=1$.

# Example: Combining Squares

- ✓ Example: Let
- ✓ $F(x, y, z) = \sum_m (2, 3, 6, 7)$




- ✓ Applying the Minimization Theorem three times:

$$F(x, y, z) = \bar{x}\, y\, z + x\, y\, z + \bar{x}\, y\, \bar{z} + x\, y\, \bar{z}$$
$$= yz + y\,\bar{z}$$
$$= y$$

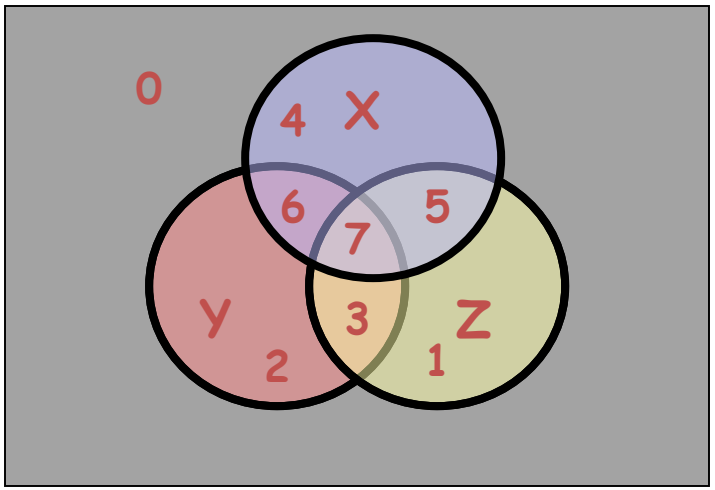
- ✓ Thus the four terms that form a $2 \times 2$ square correspond to the term "y".

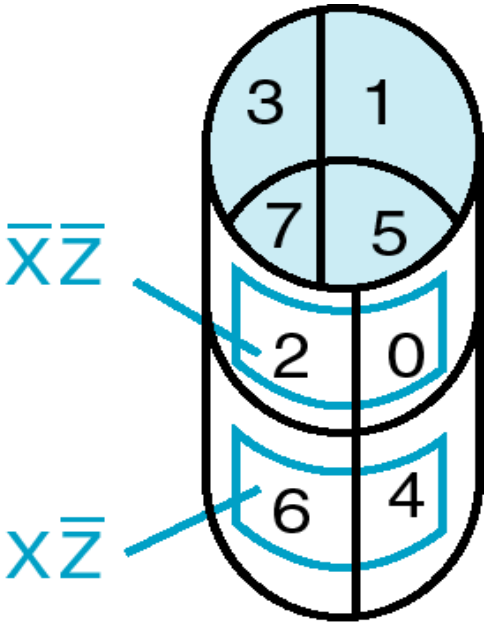
# Three-Variable Maps

- ✓ Reduced literal product terms for SOP standard forms correspond to cubes i.e. to rectangles on the K-maps containing cell counts that are powers of 2.
- ✓ Rectangles of 2 cells represent 2 adjacent minterms; of 4 cells represent 4 minterms that form a "pairwise adjacent" ring.
- ✓ Rectangles can contain non-adjacent cells as illustrated by the "pairwise adjacent" ring above.

# Three-Variable Maps

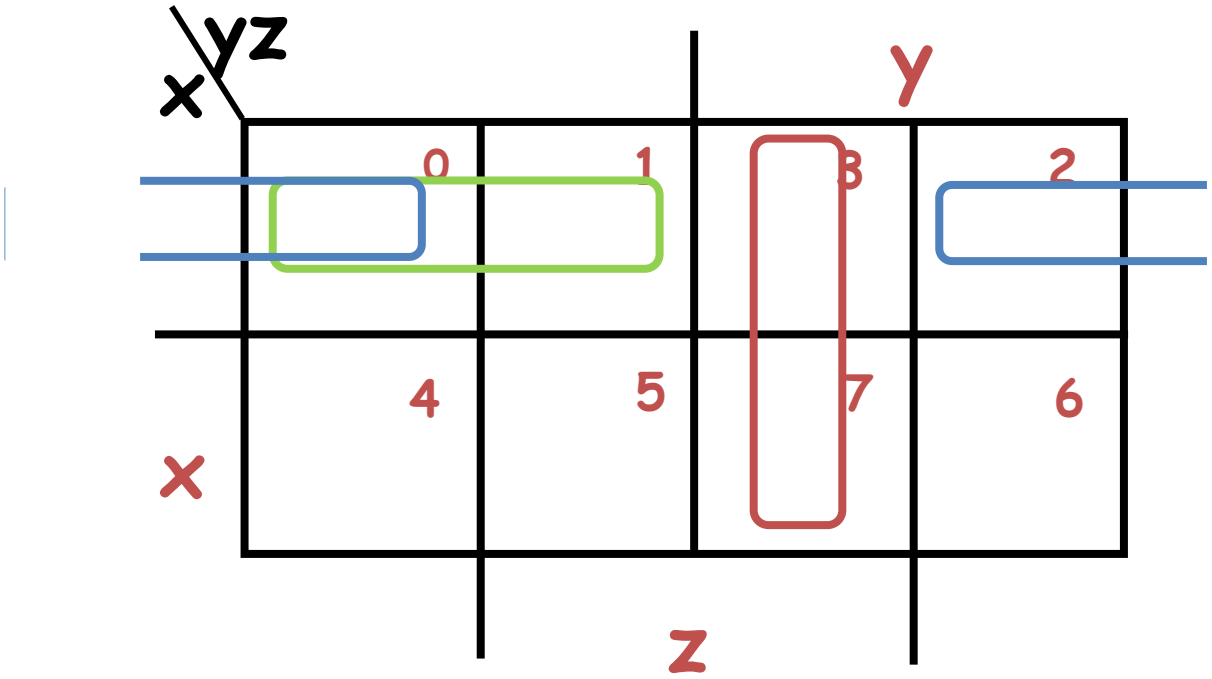
- ✓ Topological warps of 3-variable K-maps that show *all* adjacencies:
  - Venn Diagram
  - Cylinder



# Three-Variable Maps

- ✓ Example Shapes of 2-cell Rectangles:

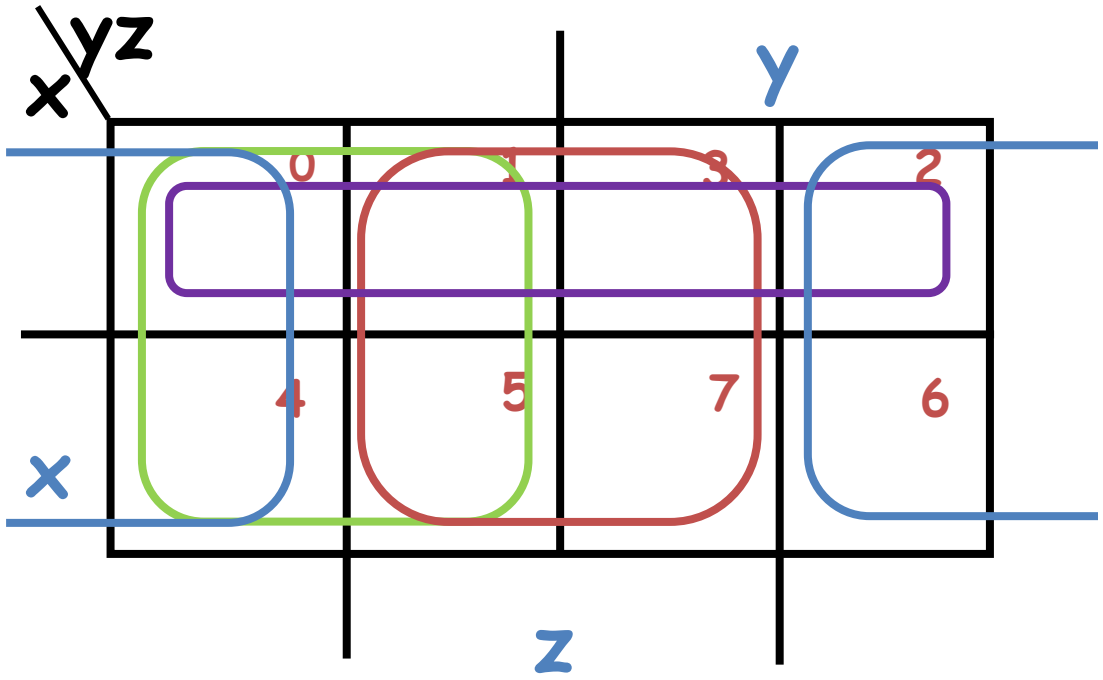



- ✓ Read off the product terms for the rectangles shown

x'z'  x'y'  yz

# Three-Variable Maps

- ✓ Example Shapes of 4-cell Rectangles:

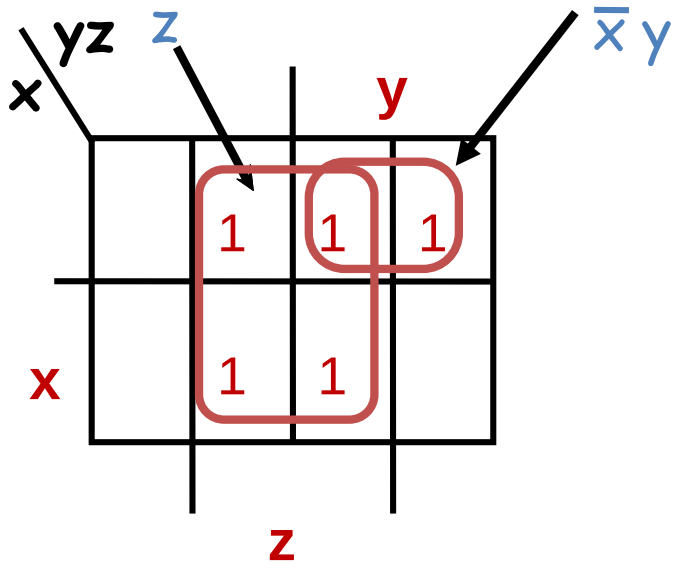



- ✓ Read off the product terms for the rectangles shown

z′ z x′ y′

# Three Variable Maps

- ✓ K-Maps can be used to simplify Boolean functions by a systematic methods. Terms are selected to cover the "1s"in the map.
- ✓ Example: Simplify $F(x, y, z) = \sum_m (1, 2, 3, 5, 7)$



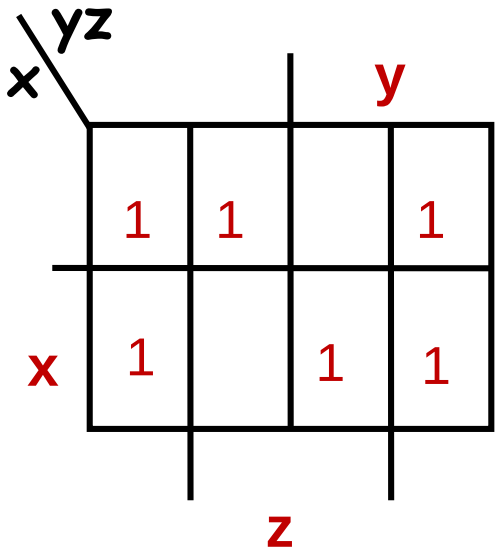

$F(x, y, z) = z + \bar{x}y$

# Three-Variable Map Simplification

- ✓ Use a K-map to find an optimum SOP equation for

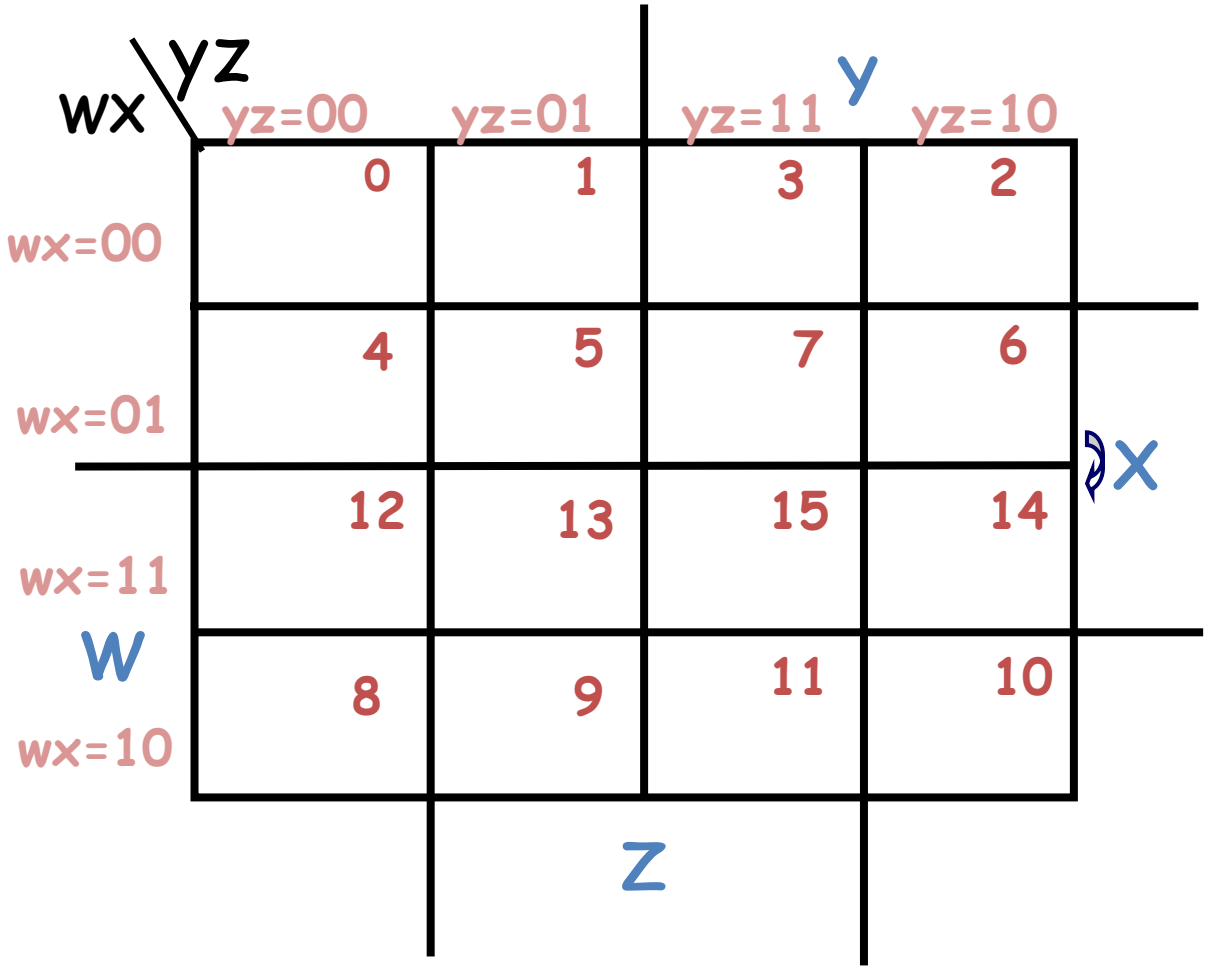
$$F(x, y, z) = \sum_m (0, 1, 2, 4, 6, 7)$$

$$F = \bar{z} + \bar{x}\,\bar{y} + x\,y$$



# Four Variable Maps

✓ Map and location of minterms:

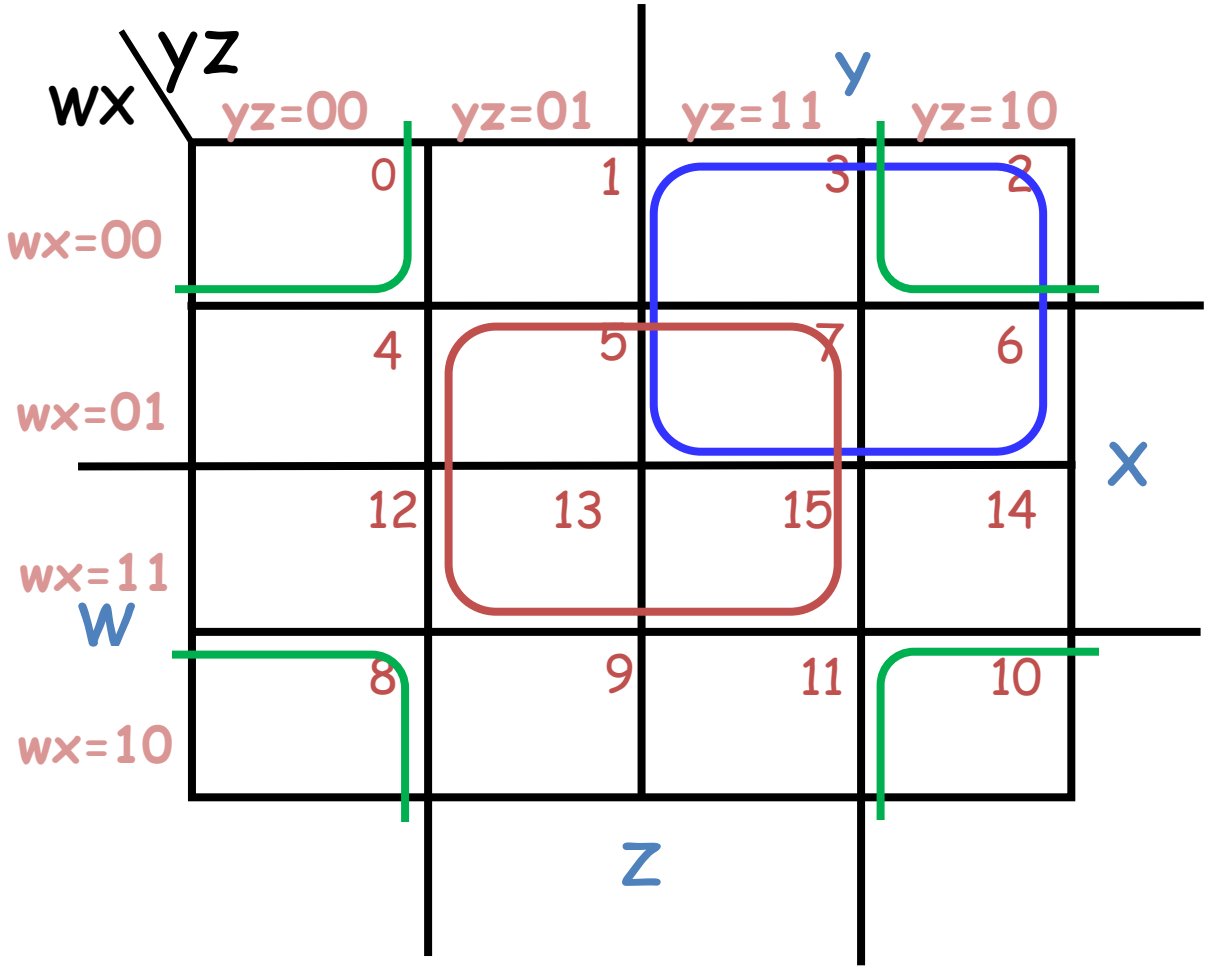


# Four Variable Terms

- ✓ Four variable maps can have rectangles corresponding to:
  - A single 1 = 4 variables, (i.e. Minterm)
  - Two 1s = 3 variables,
  - Four 1s = 2 variables
  - Eight 1s = 1 variable,
  - Sixteen 1s = zero variables (i.e. Constant "1")

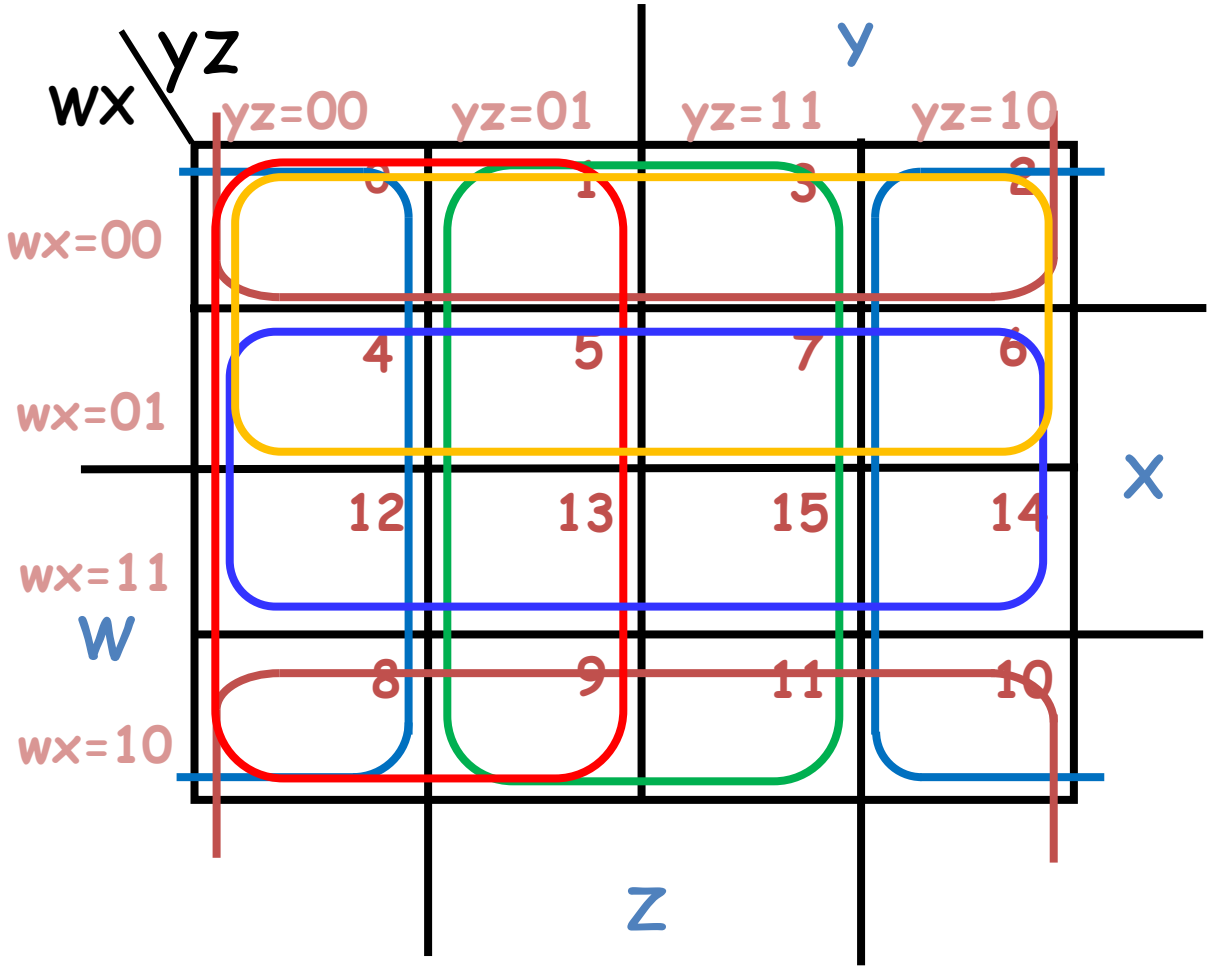
# Four-Variable Maps

- ✓ Example Shapes of Rectangles:



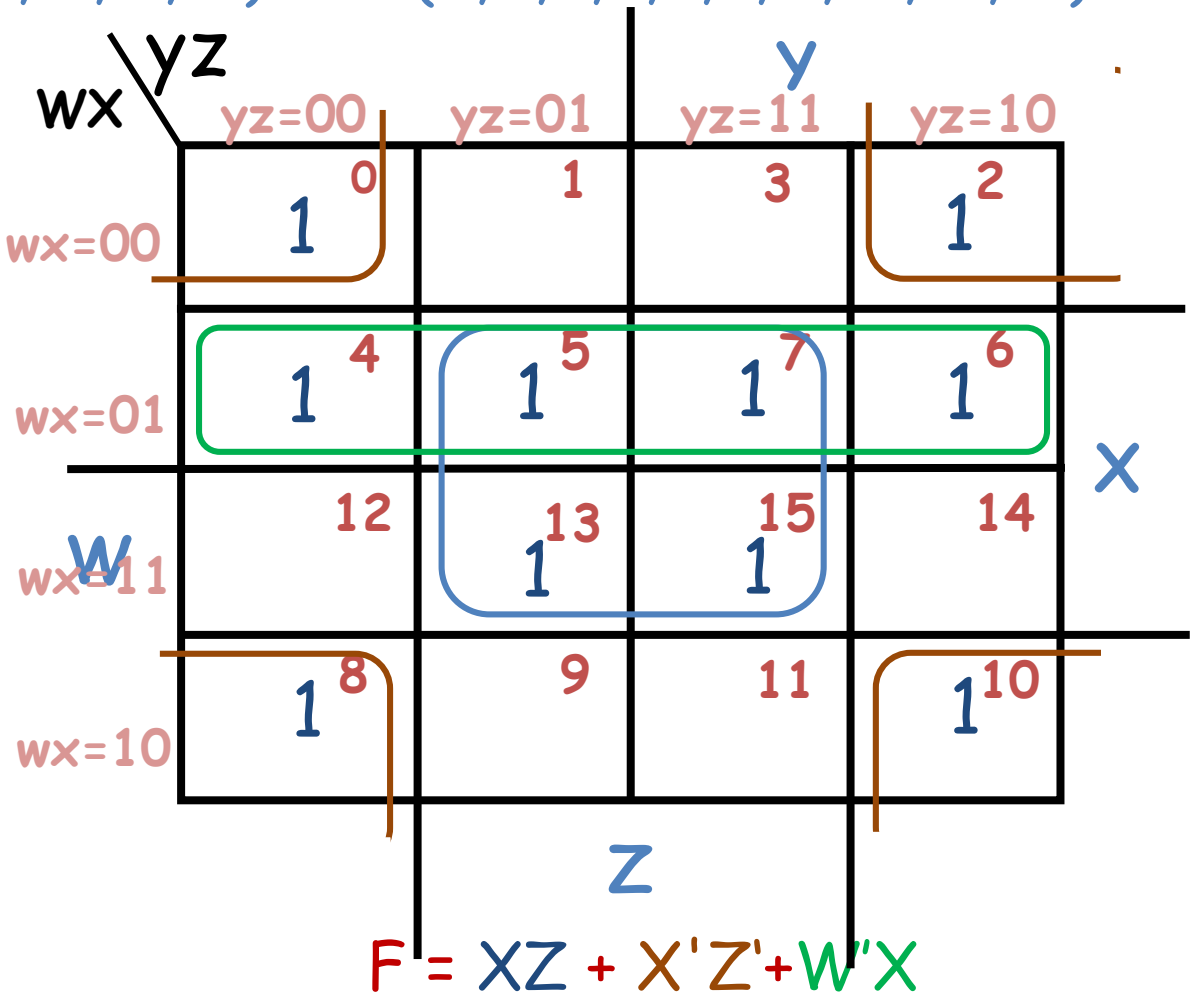
# Four-Variable Maps

- ✓ Example Shapes of Rectangles:



# Four-Variable Map Simplification

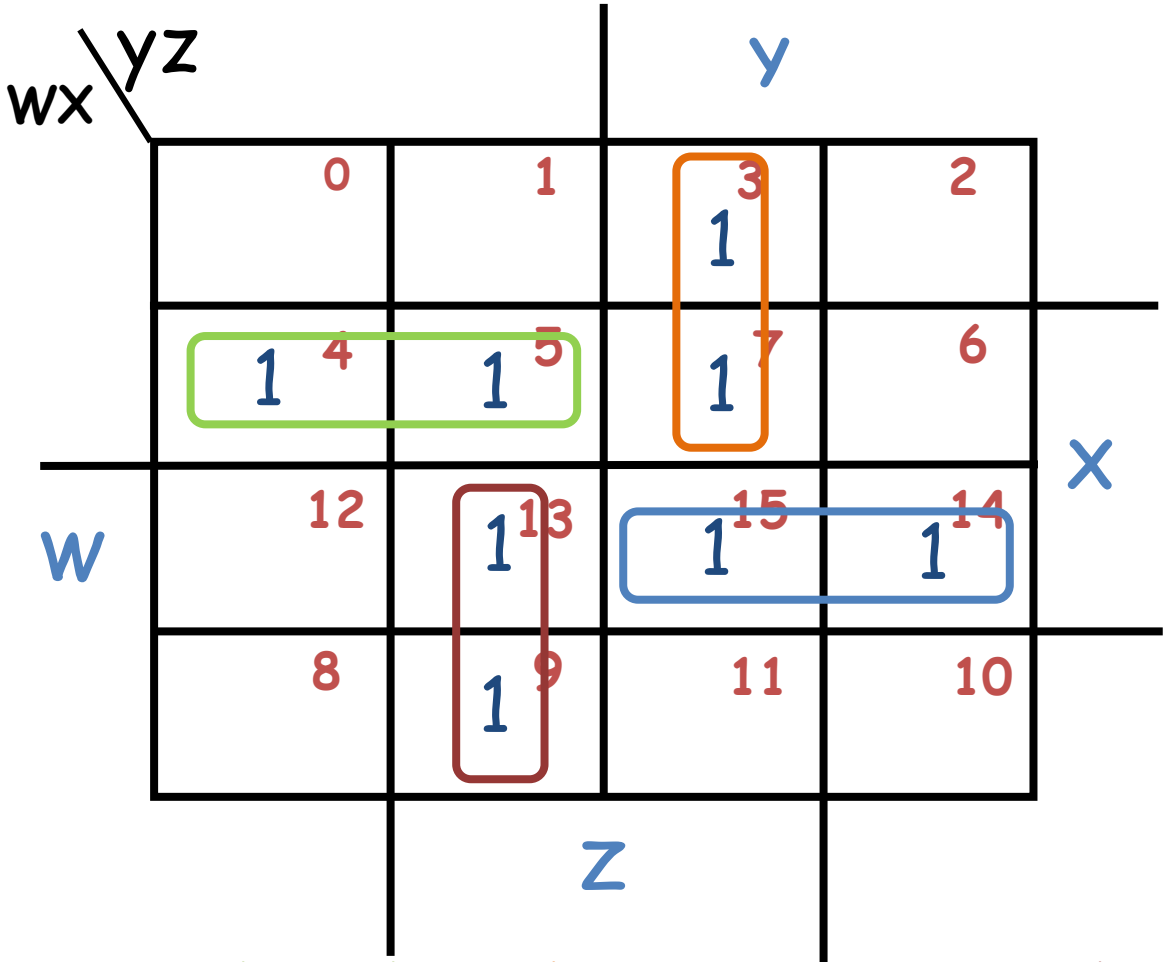
✓ $F(W, X, Y, Z) = \sum_m (0,2,4,5,6,7,8,10,13,15)$




$F = XZ + X'Z' + W'X$

# Four-Variable Map Simplification

$F(W,X,Y,Z) = \sum_m(3,4,5,7,9,13,14,15)$



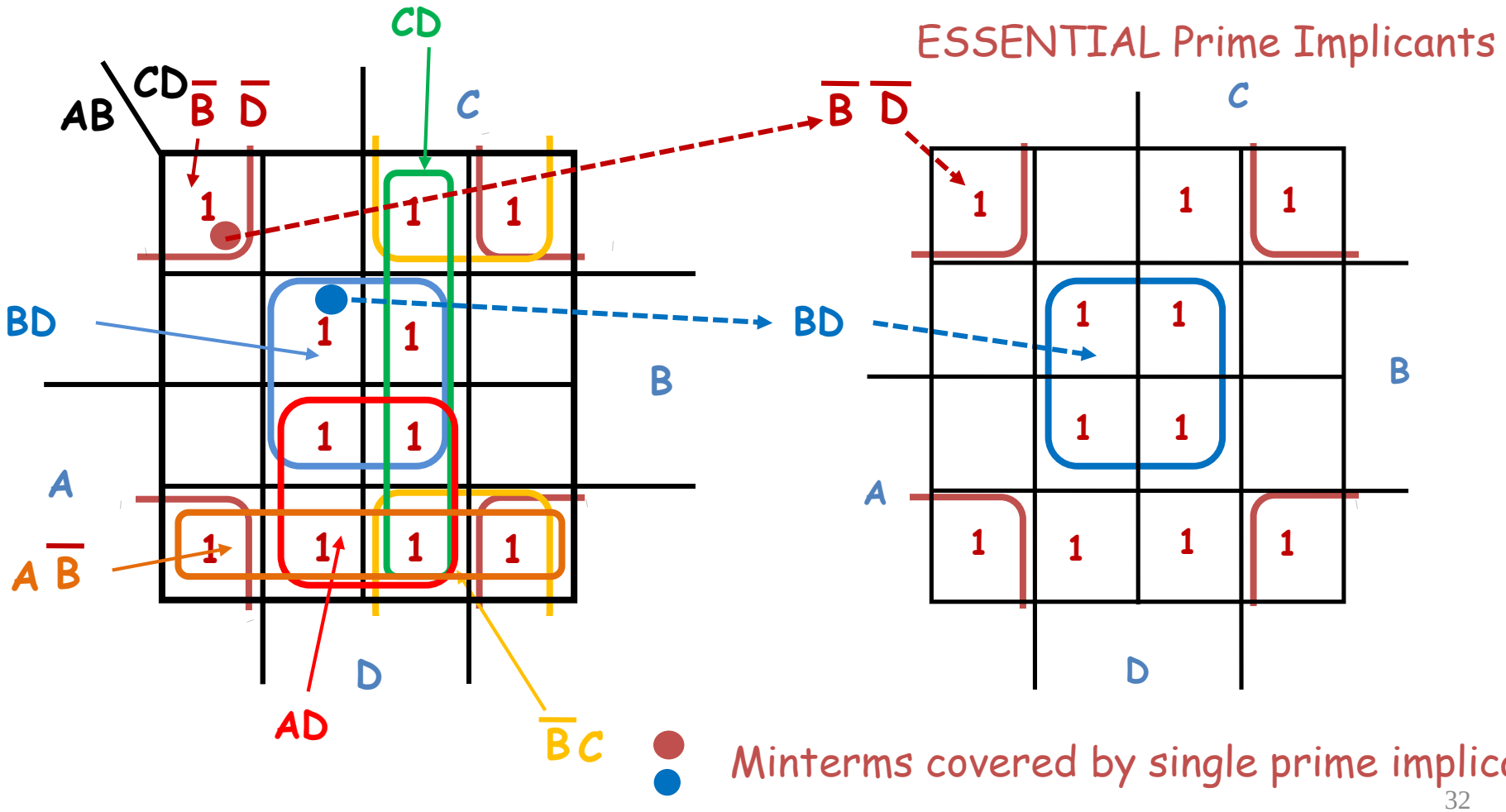

$F = W' X Y' + W' Y Z + WXY + WY'Z$

# Systematic Simplification

- ✓ A Prime Implicant is a cube i.e. a product term obtained by combining the maximum possible number of adjacent squares in the map into a rectangle with the number of squares a power of 2.
- ✓ A prime implicant is called an Essential Prime Implicant if it is the only prime implicant that covers (includes) one or more minterms.
- ✓ Prime Implicants and Essential Prime Implicants can be determined by inspection of a K-Map.
- ✓ A set of prime implicants *"covers all minterms"* if, for each minterm of the function, at least one prime implicant in the set of prime implicants includes the minterm.

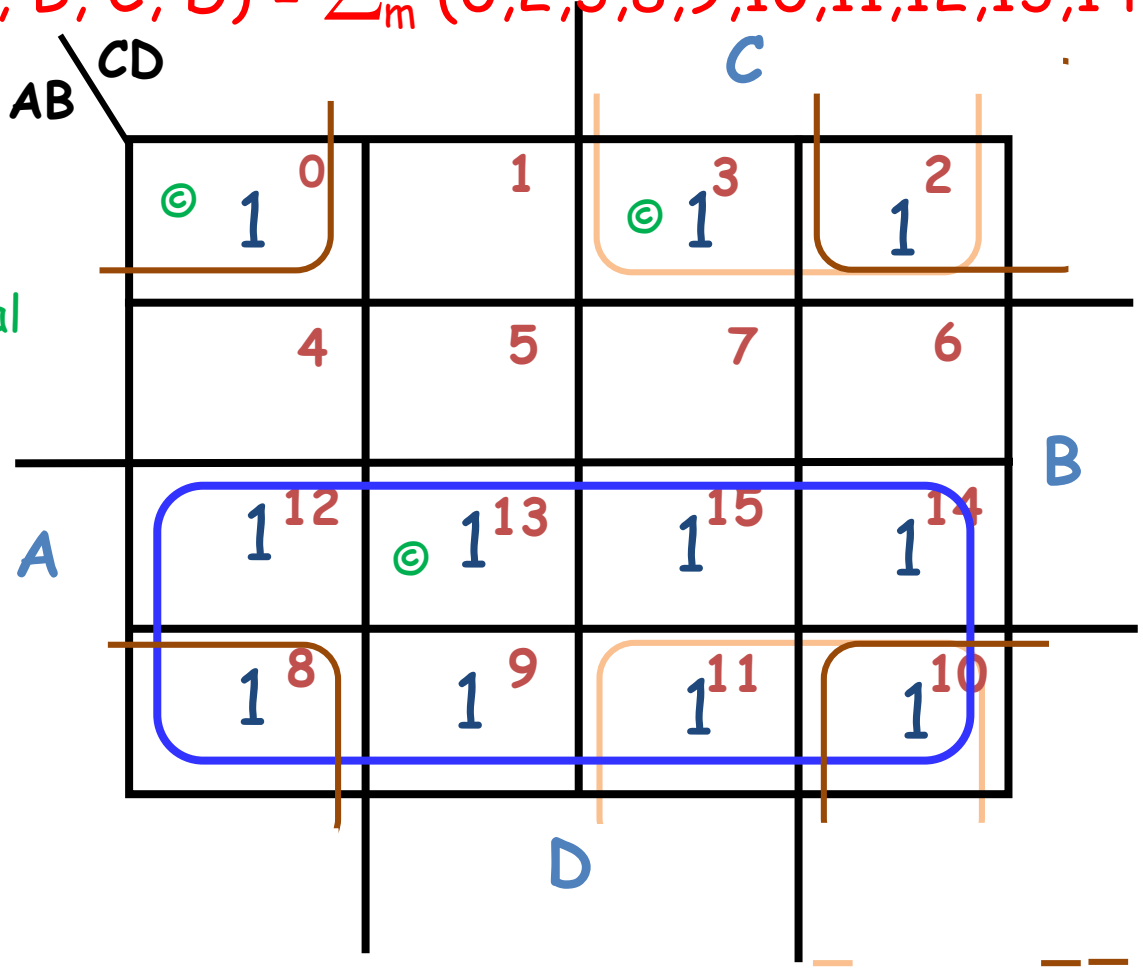
# Example of Prime

✓ Find ALL Prime Implicants



# Prime Implicant Practice

- ✓ Find all prime implicants for:
- ✓ F (A, B, C, D) = $\sum_m$ (0,2,3,8,9,10,11,12,13,14,15)



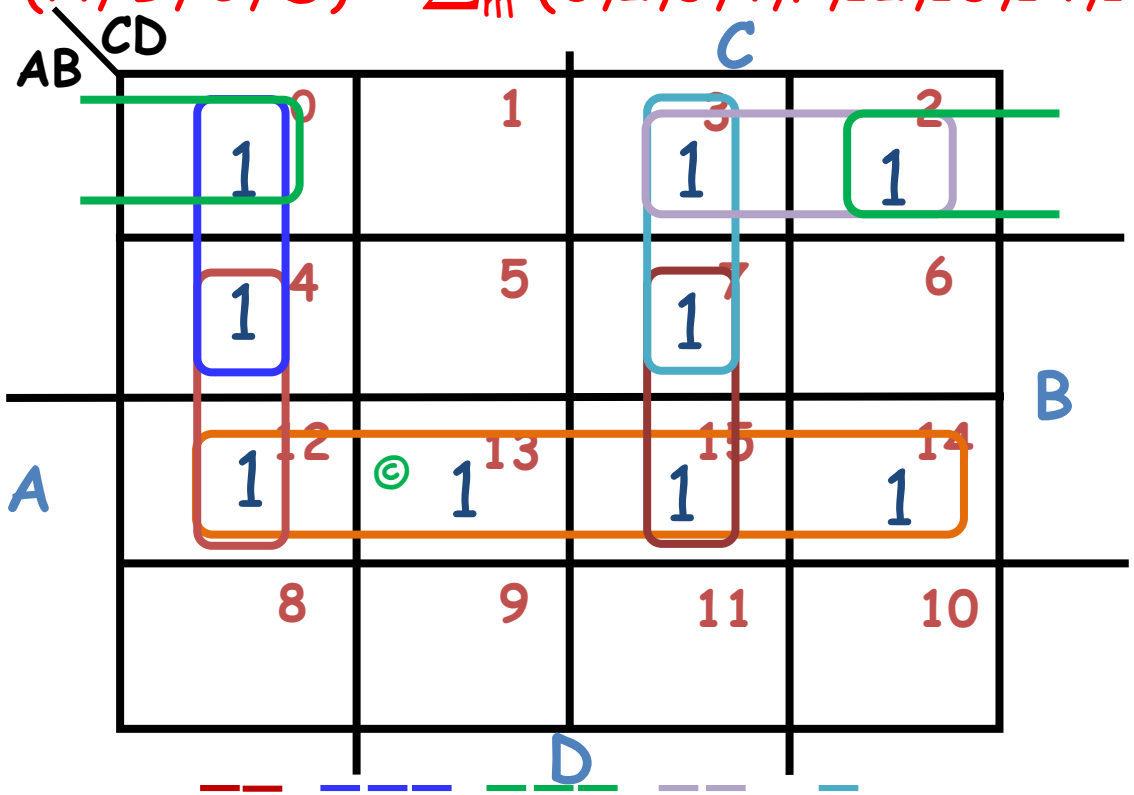

© = essential

Prime implicants are: A, $\bar{B}C$, and $\bar{B}\bar{D}$

# Another Example

✓ Find all prime implicants for:

$$F(A, B, C, D) = \sum_m (0,2,3,4,7,12,13,14,15)$$



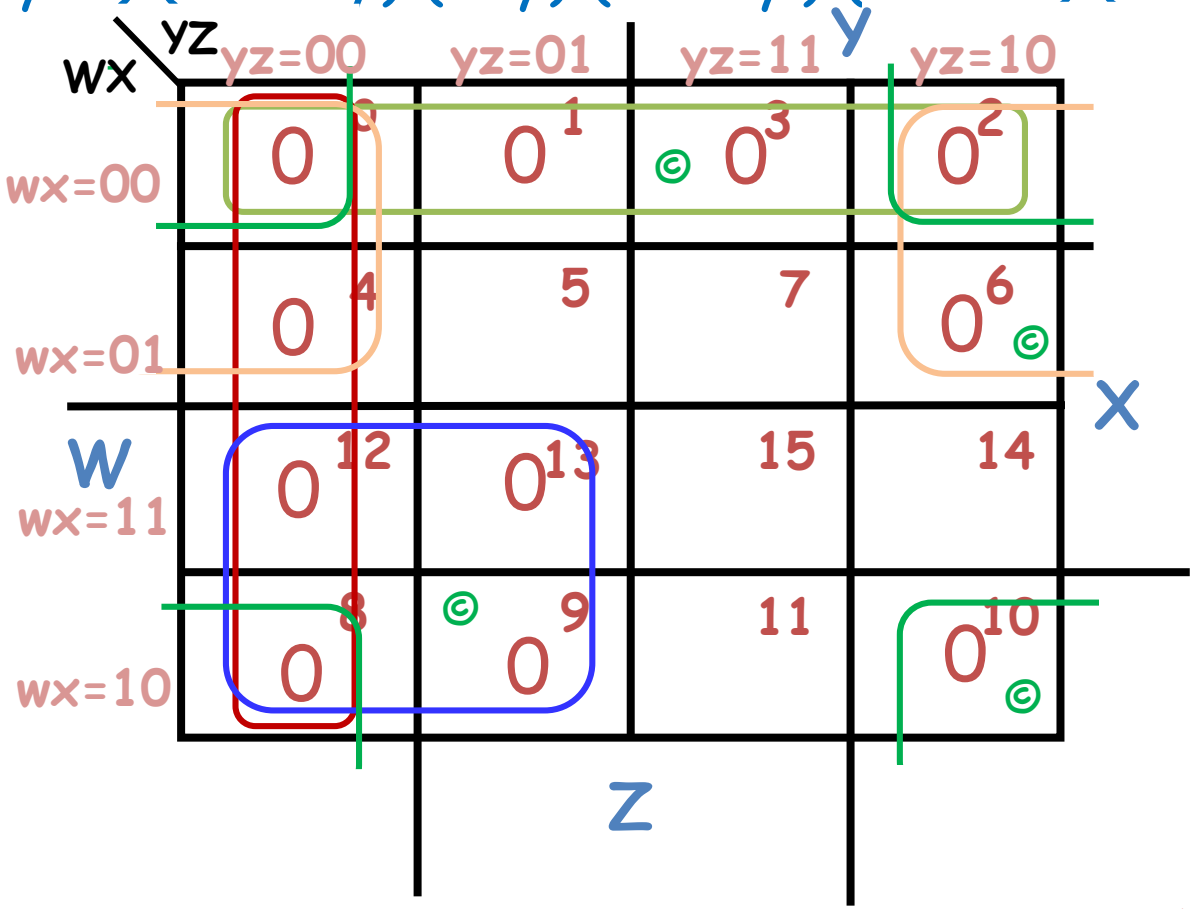

$AB, B\bar{C}\bar{D}, \bar{A}\bar{C}\bar{D}, \bar{A}\bar{B}\bar{D}, \bar{A}\bar{B}C, \bar{A}CD, BCD$

$\sum_m = AB + \bar{A}\bar{C}\bar{D} + \bar{A}CD + \bar{A}\bar{B}C \quad \sum_c = \sum_m + \bar{A}\bar{B}\bar{D} + BCD + B\bar{C}\bar{D}$

# K-Maps, implicates

✓ Find all prime implicates for:

F=(w+y+z) (w'+x'+y) (x+y) (w+x+y') (w+x'+z) (w'+x+z)

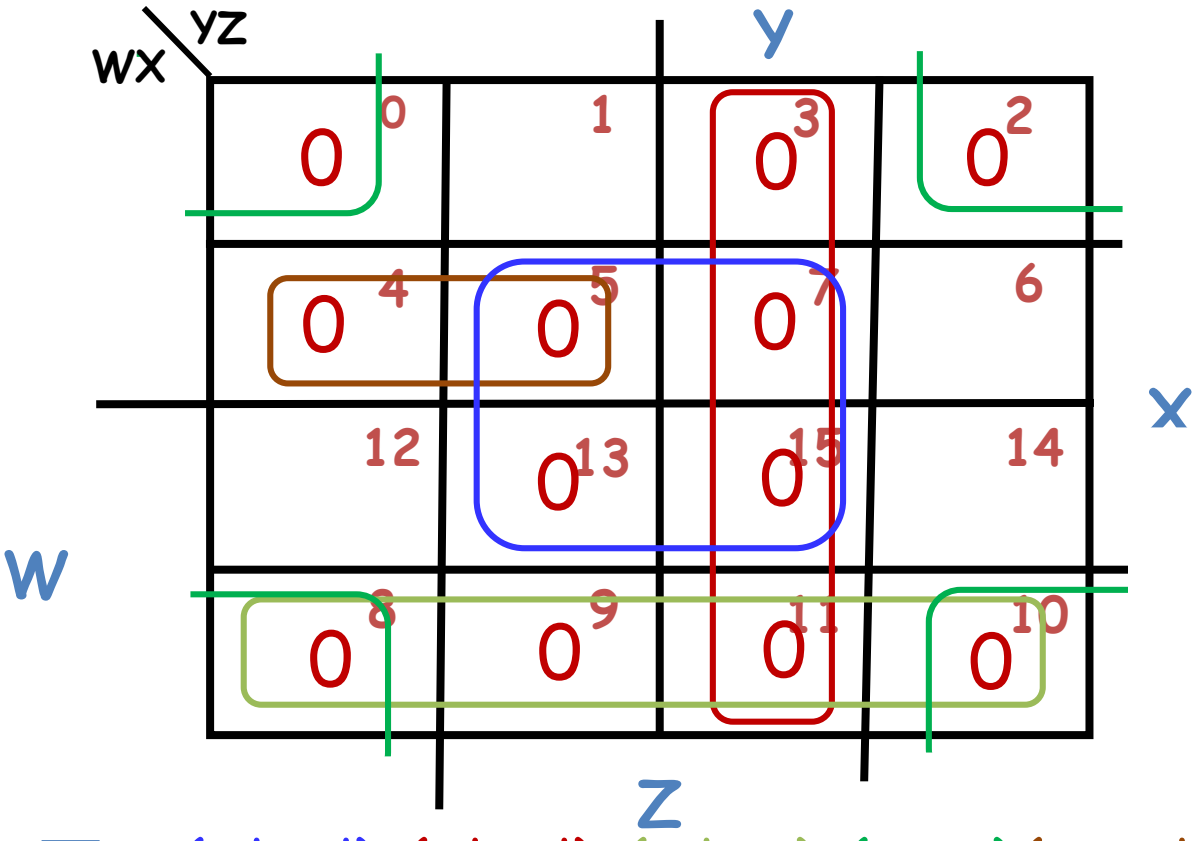



$\Pi_m = (w+x)(w'+y)(w+z)(x+z)$ $\Pi_c = \Pi_m(y+z)$

# K-Maps, implicates

✓ Find all prime implicates for:

F=(w+y+z)(w+x'+y)(w+y'+z')(x+y'+z)(w'+x'+z')(w'+x+y)(x+y'+z')

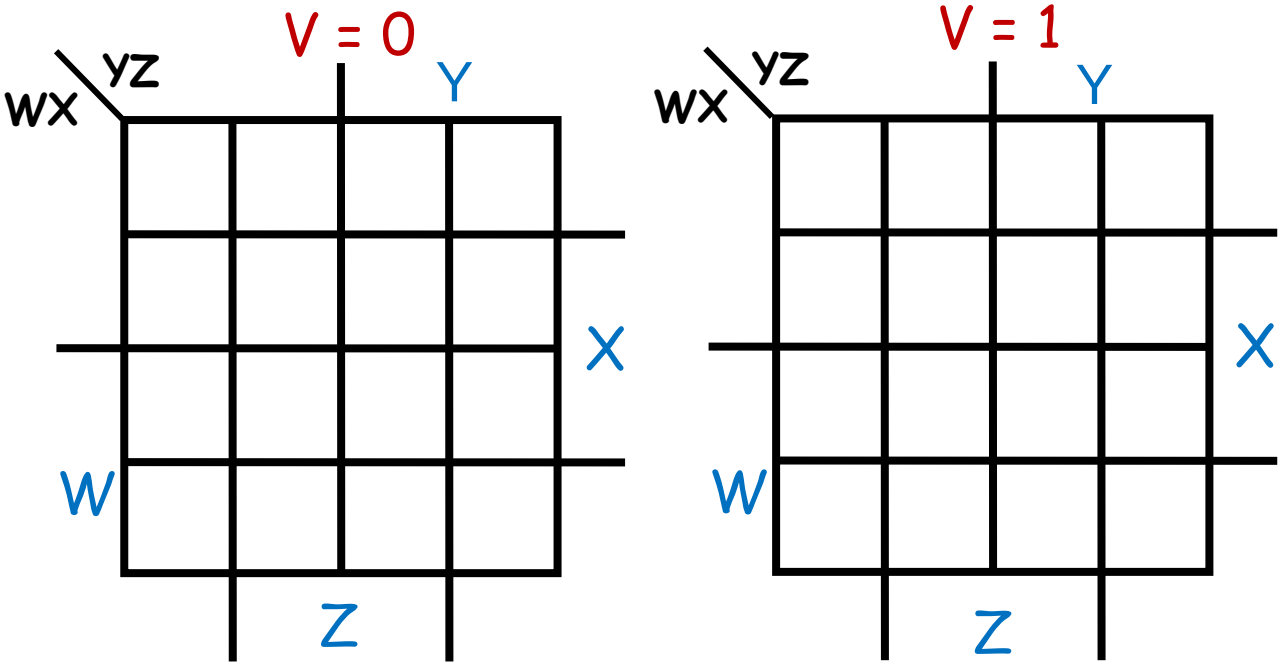



$\Pi_m$=(x'+z') (y'+z') (w'+x) (x+z)(w+x'+y)

$\Pi_c=\Pi_m$(w'+z')(w+y+z)(x+y')

# Five Variable or More K-Maps

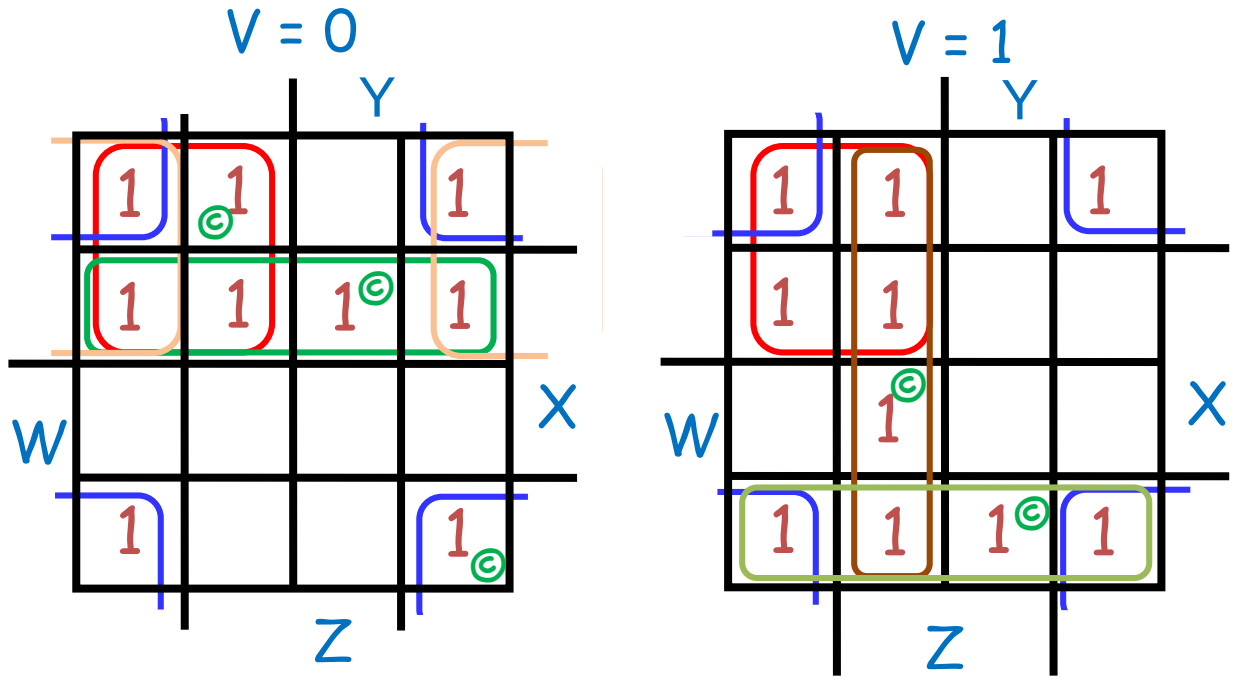
- ✓ For five variable problems, we use two adjacent K-maps. It becomes harder to visualize adjacent minterms for selecting PIs. You can extend the problem to six variables by using four K-Maps.



# Five Variable K-Maps

- ✓ Find all prime implicants for:

$$G(v,w,x,y,z)=\sum_m(0,1,2,4,5,6,7,8,10,16,17,18,20,21,24,25,26,27,29)$$




$\sum_m$= w'y'+x'z'+v'w'x+vy'z+vwx' $\sum_c = \sum_m$+v'w'z'

# Don't Cares in K-Maps

- ✓ Sometimes a function table or map contains entries for which it is known that:
  - the input values for the minterm will never occur or
  - the output value for the minterm is not used
- ✓ In these cases, the output value need not be defined
- ✓ Instead, the output value is defined as a **don't care**
- ✓ By placing "don't cares" ( an "**x**" entry) in the function table or map, the cost of the logic circuit may be lowered.
- ✓ Example: A logic function having the binary codes for the BCD digits as its inputs. Only the codes for 0 through 9 are used. The six codes, 1010 through 1111 never occur, so the output values for these codes are **x** to represent "don't cares."

# Incompletely Specified Functions

- ✓ $F = (f, d, r) : B^n \rightarrow \{0, 1, \mathbf{x}\}$
  where **x** represents "don't care".

  - f = onset function - $f(x)=1 \leftrightarrow F(x)=1$
  - r = offset function - $r(x)=1 \leftrightarrow F(x)=0$
  - d = don't care function - $d(x)=1 \leftrightarrow F(x)=*$

  (f,d,r) forms a partition of $B^n$. i.e.
  - $f + d + r = B^n$
  - $fd = fr = dr = \varnothing$ (pairwise disjoint)

# Example:Logic Minimization

- ✓ Consider $F(a,b,c)=(f,d,r)$, where $f=\{\bar{a}b\bar{c}, a\bar{b}c, abc\}$ and $d=\{a\bar{b}\bar{c}, ab\bar{c}\}$, and the sequence of covers illustrated below:

- On
- Off
- Don't care

$F^1 = \bar{a}b\bar{c} + a\bar{b}c + abc$

**Expand $abc \rightarrow a$**

$F^2 = a + \bar{a}b\bar{c} + a\bar{b}c$

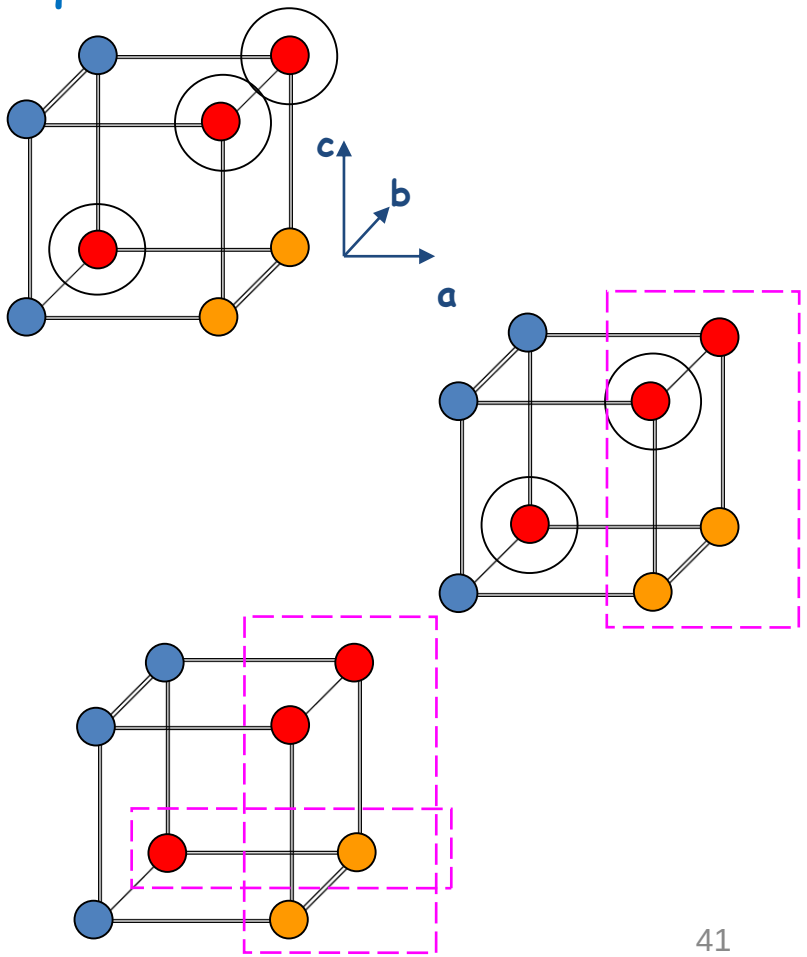
$a\bar{b}c$ is redundant

a is prime

$F^3 = a + \bar{a}b\bar{c}$

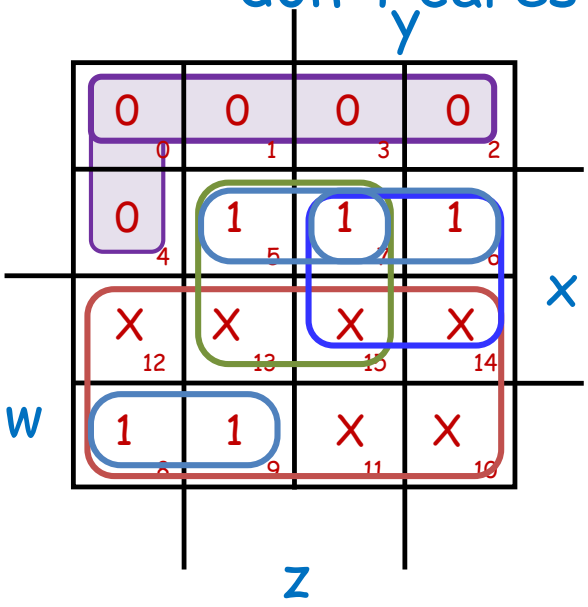
**Expand $\bar{a}b\bar{c} \rightarrow b\bar{c}$**

$F^4 = a + b\bar{c}$



# Example: BCD "5 or More"

- ✓ In the map below gives a function F1(w,x,y,z) which is defined as "5 or more" over BCD inputs. With the don't cares used for the 6 non-BCD combinations:




$$F1(w, x, y, z) = \overline{w}\,x\,z + \overline{w}\,x\,y + w\,\overline{x}\,\overline{y}$$

$$F2(w,x,y,z) = w + x\,z + x\,y$$

$$G = 7$$

This is much lower in cost than F1 where the "don't cares" were treated as "0s" having G = 12

For this particular function, cost G for the POS solution for F1(w,x,y,z) is not changed by using the don't cares.

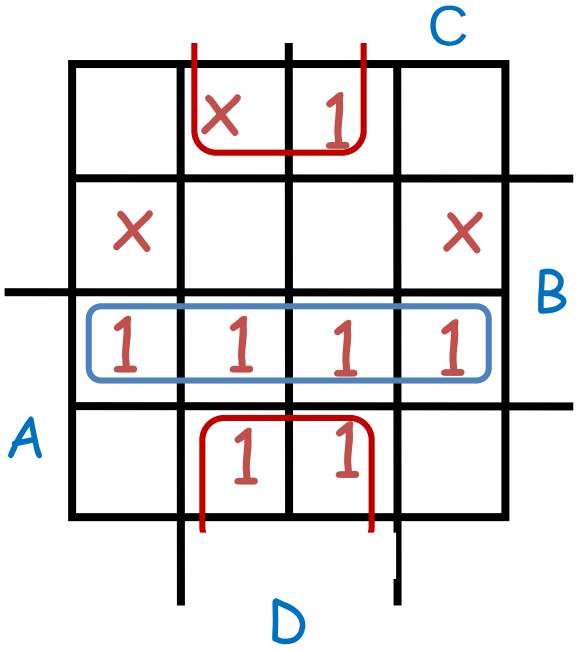
$$F3(w,x,y,z) = (w + x)(w + y + z) \qquad G = 7$$

# Product of Sums Example

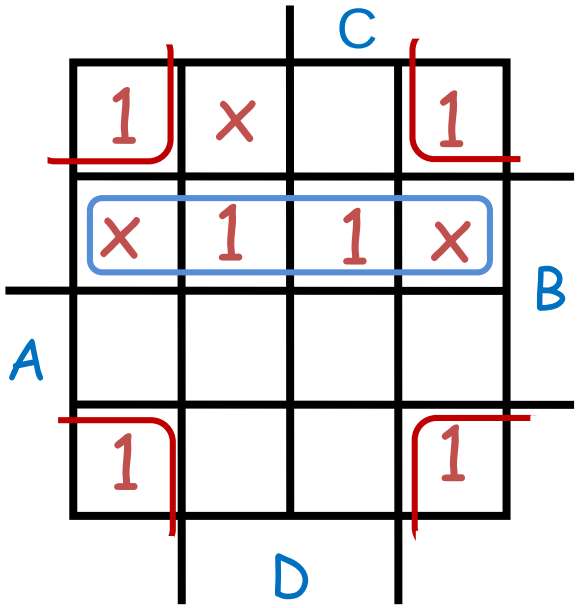
- ✓ Find the optimum POS solution:

$F(A,B,C,D)=\sum_m(3,9,11,12,13,14,15)+\sum_d(1,4,6)$

- Hint: Use F' and complement it to get the result.




$F = A\,B + B'\,D$

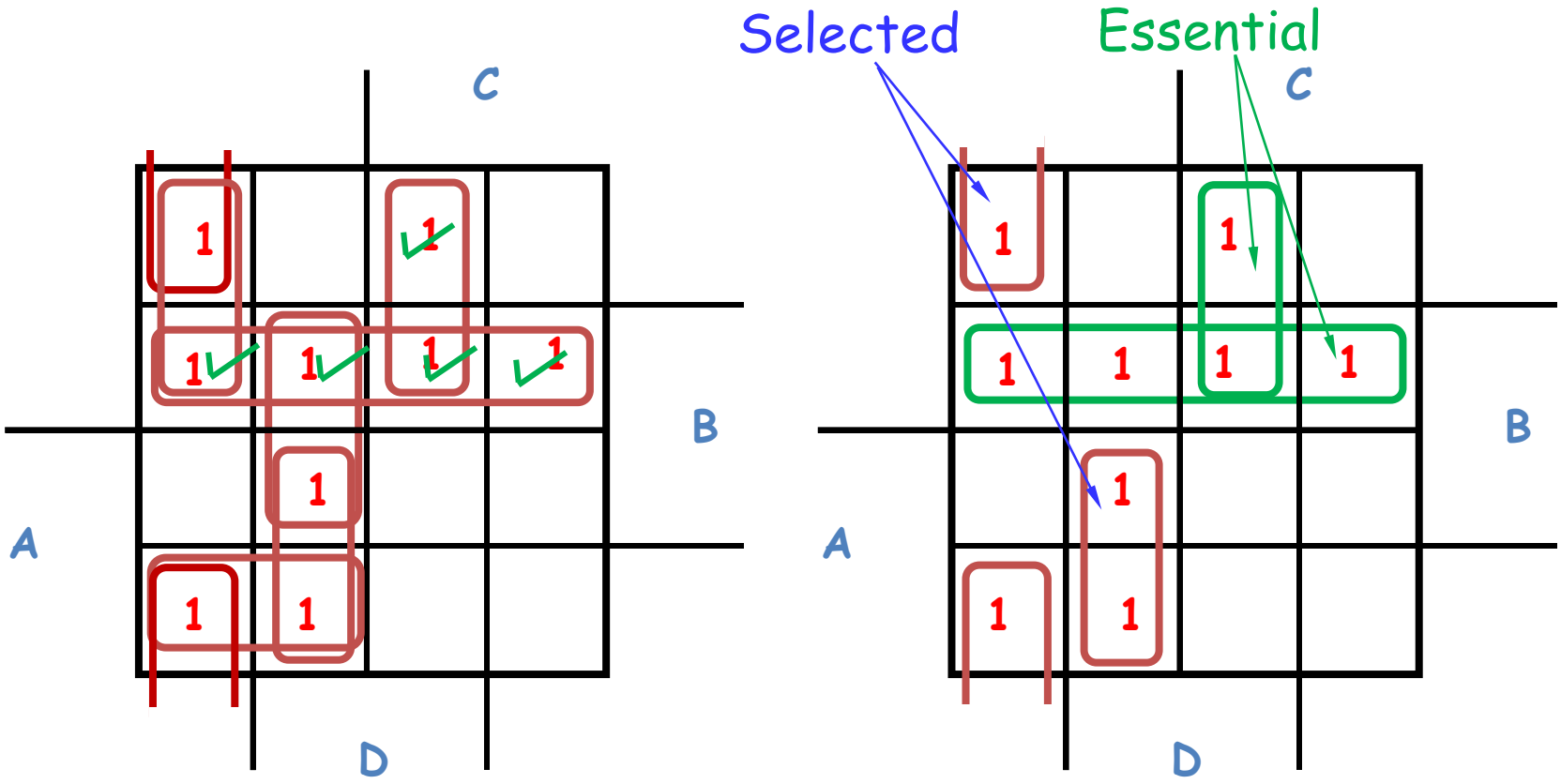



$F' = B'\,D' + A'\,B$

$F = (B + D)(A + B')$

# Selection Rule Example

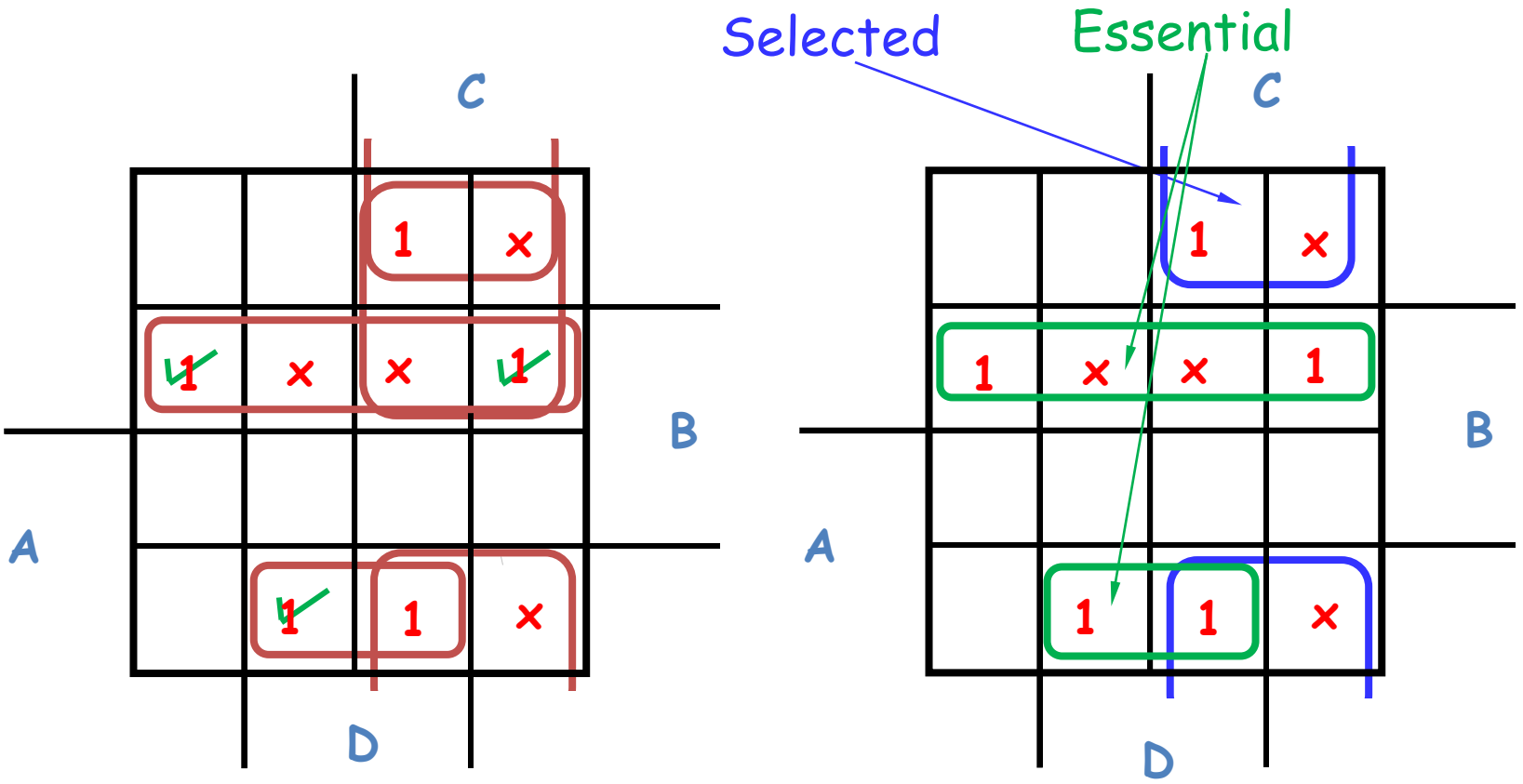
✓ Simplify F(A, B, C, D) given on the K-map.




✓ Minterms covered by essential prime implicants

# Selection Rule Example with Don't Cares

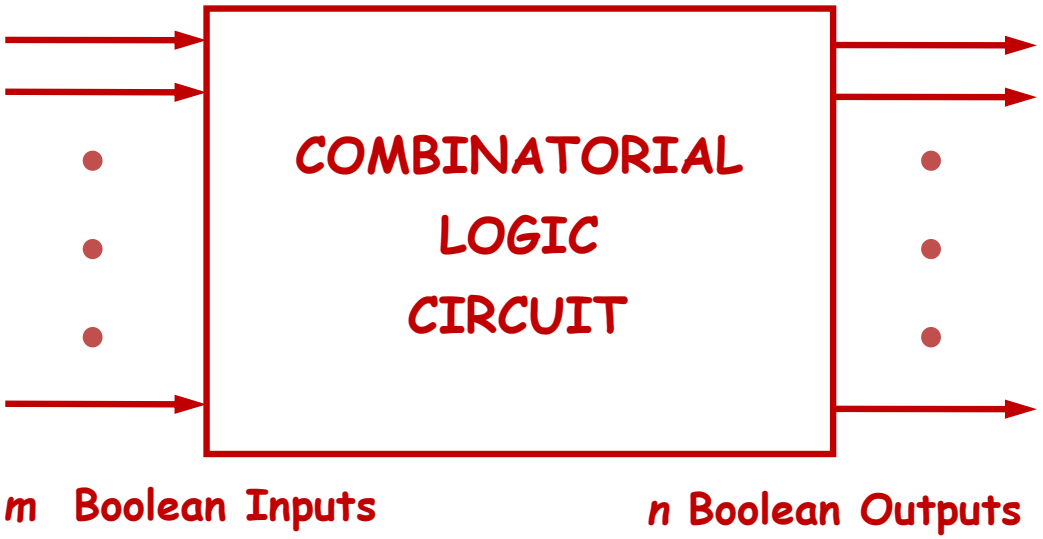
✓ Simplify F(A, B, C, D) given on the K-map.




✓ Minterms covered by essential prime implicants

# Combinational Circuits

- ✓ A combinational logic circuit has:
  - A set of *m* Boolean inputs,
  - A set of *n* Boolean outputs, and
  - *n* switching functions, each mapping the $2^m$ input combinations to an output such that the current output depends only on the current input values
- ✓ A block diagram:



# Design Procedure

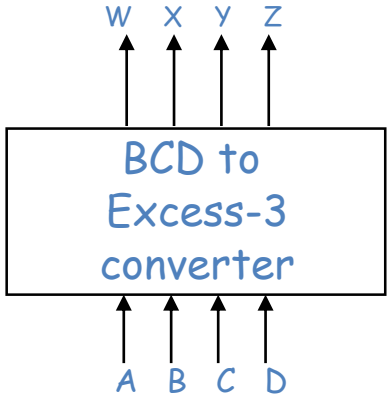
1. Specification
   - Write a specification for the circuit
2. Formulation
   - Derive a truth table or initial Boolean equations that define the required relationships between the inputs and outputs
3. Optimization
   - Apply 2-level and multiple-level optimization
   - Draw a logic diagram for the resulting circuit using ANDs, ORs, and inverters
4. Technology Mapping
   - Map the logic diagram to the implementation technology selected
5. Verification
   - Verify the correctness of the final design

# Design Example BCD to Excess-3 code converter

1. Specification
   - BCD to Excess-3 code converter
     - Transforms BCD code for the decimal digits to Excess-3 code for the decimal digits
   - BCD code words for digits 0 through 9
     - 4-bit patterns 0000 to 1001, respectively
   - Excess-3 code words for digits 0 through 9
     - 4-bit patterns consisting of 3 (binary 0011) added to each BCD code word
   - Implementation:
     - multiple-level circuit
     - NAND gates (including inverters)

# Specification of BCD-to-Excess3

- ✓ Inputs: a BCD input, A,B,C,D with A as the most significant bit and D as the least significant bit.
- ✓ Outputs: an Excess-3 output W,X,Y,Z that corresponds to the BCD input.
- ✓ Internal operation – circuit to do the conversion in combinational logic.



# Formulation of BCD-to-Excess-3

- ✓ Excess-3 code is easily formed by adding a binary 3 to the binary or BCD for the digit.
- ✓ There are 16 possible inputs for both BCD and Excess-3.
- ✓ It can be assumed that only valid BCD inputs will appear so the six combinations not used can be treated as don't cares.

# BCD-to-Excess-3 Code converter

| Decimal Digit | Input BCD | Output Excess-3 |
|---|---|---|
| 0 | 0 0 0 0 | 0 0 1 1 |
| 1 | 0 0 0 1 | 0 1 0 0 |
| 2 | 0 0 1 0 | 0 1 0 1 |
| 3 | 0 0 1 1 | 0 1 1 0 |
| 4 | 0 1 0 0 | 0 1 1 1 |
| 5 | 0 1 0 1 | 1 0 0 0 |
| 6 | 0 1 1 0 | 1 0 0 1 |
| 7 | 0 1 1 1 | 1 0 1 0 |
| 8 | 1 0 0 0 | 1 0 1 1 |
| 9 | 1 0 0 1 | 1 1 0 0 |

# Expressions for W X Y Z

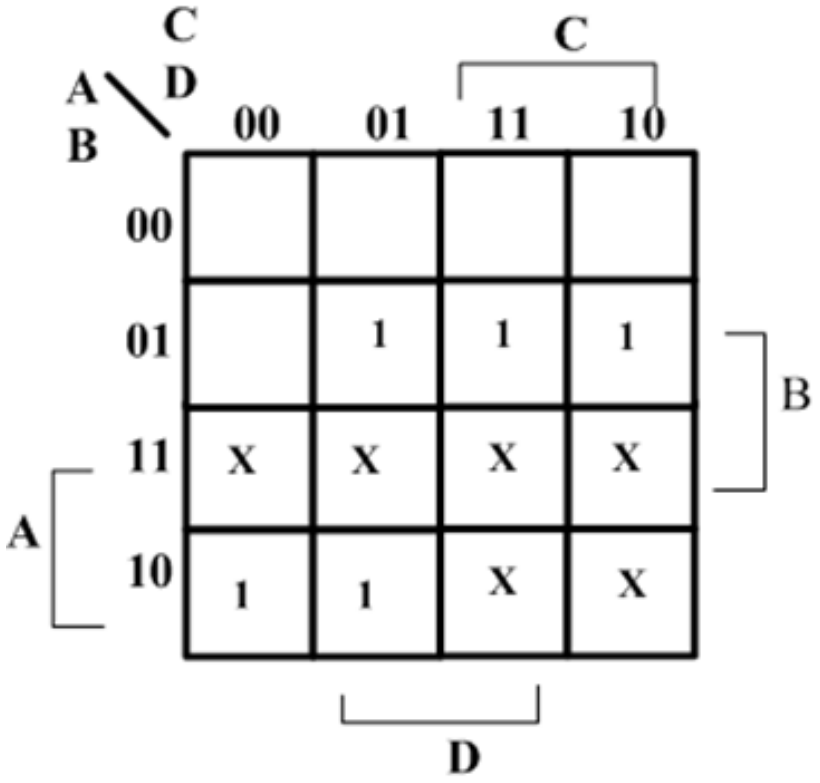
- ✓ $W(A,B,C,D) = \Sigma m(5,6,7,8,9) + d(10,11,12,13,14,15)$
- ✓ $X(A,B,C,D) = \Sigma m(1,2,3,4,9) + d(10,11,12,13,14,15)$
- ✓ $Y(A,B,C,D) = \Sigma m(0,3,4,7,8) + d(10,11,12,13,14,15)$
- ✓ $Z(A,B,C,D) = \Sigma m(0,2,4,6,8) + d(10,11,12,13,14,15)$

| Decimal Digit | Input BCD | Output Excess-3 |
|---|---|---|
| 0 | 0 0 0 0 | 0 0 1 1 |
| 1 | 0 0 0 1 | 0 1 0 0 |
| 2 | 0 0 1 0 | 0 1 0 1 |
| 3 | 0 0 1 1 | 0 1 1 0 |
| 4 | 0 1 0 0 | 0 1 1 1 |
| 5 | 0 1 0 1 | 1 0 0 0 |
| 6 | 0 1 1 0 | 1 0 0 1 |
| 7 | 0 1 1 1 | 1 0 1 0 |
| 8 | 1 0 0 0 | 1 0 1 1 |
| 9 | 1 0 0 1 | 1 1 0 0 |

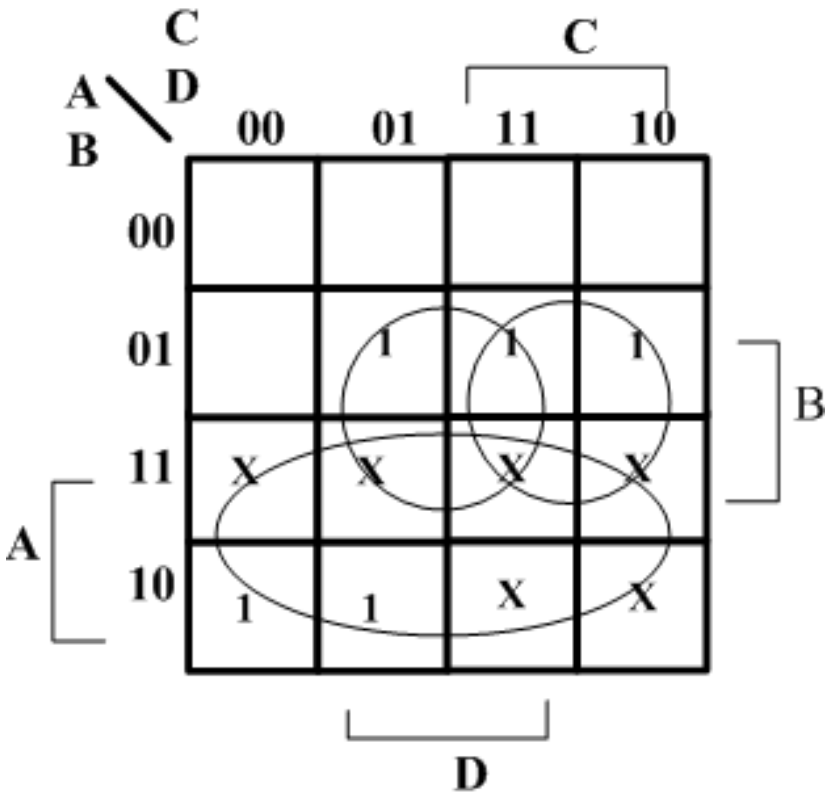
# Optimization – BCD-to-Excess-3

✓ Lay out K-maps for each output, W X Y Z

**K-map for W** $\Sigma_m(5,6,7,8,9)$




**K-map for W**

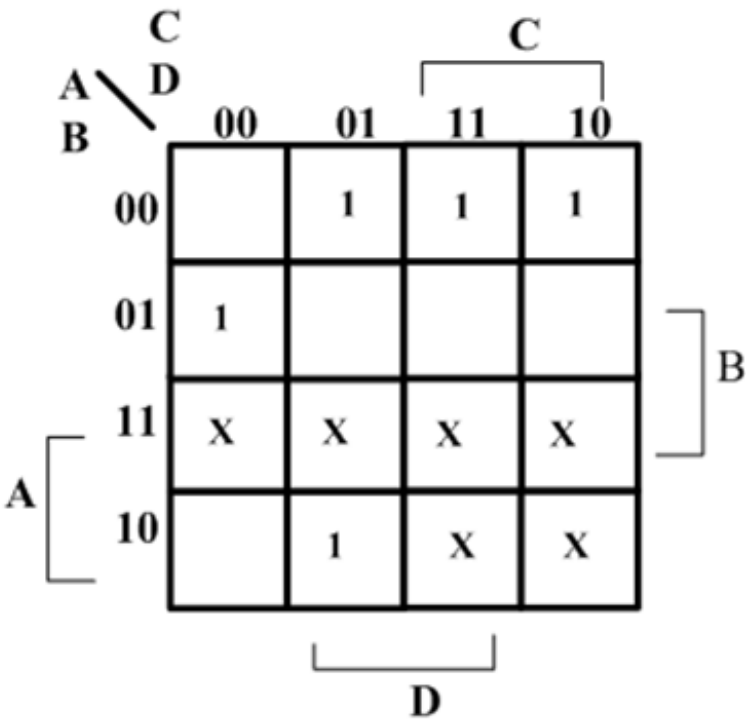



$W = A + BC + BD$

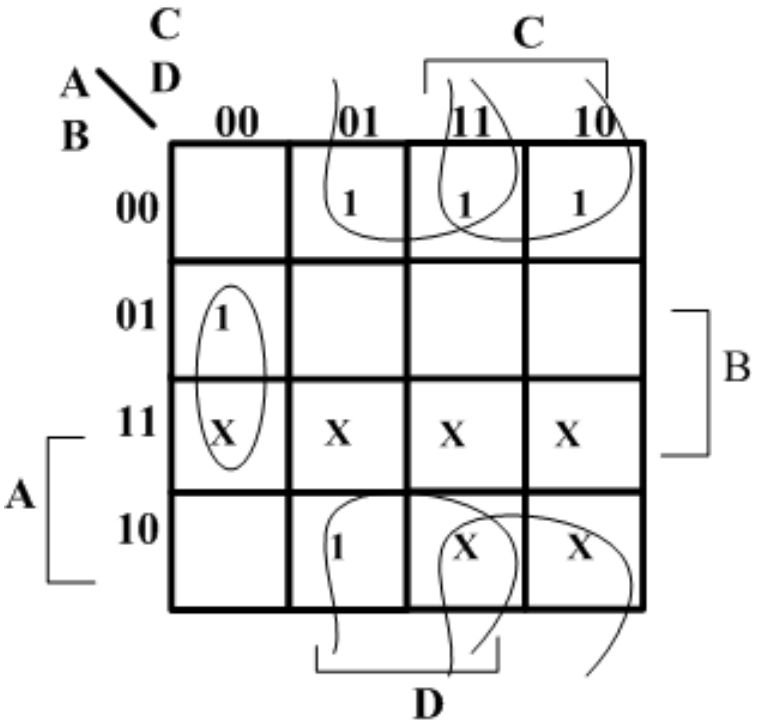
# Optimization – BCD-to-Excess-3

✓ Lay out K-maps for each output, W X Y Z

**K-map for X** $\Sigma_m(1,2,3,4,9)$



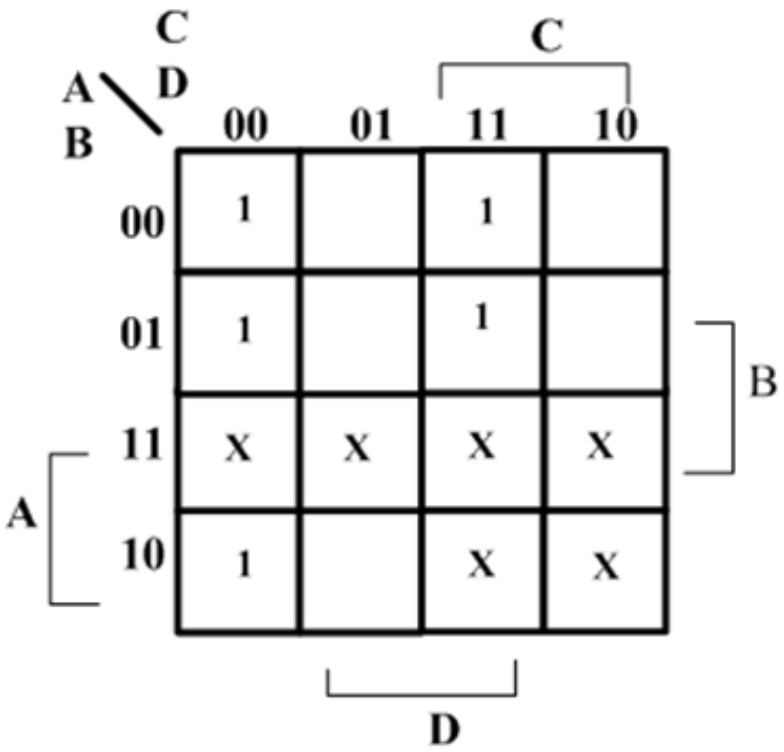

**K-map for X**



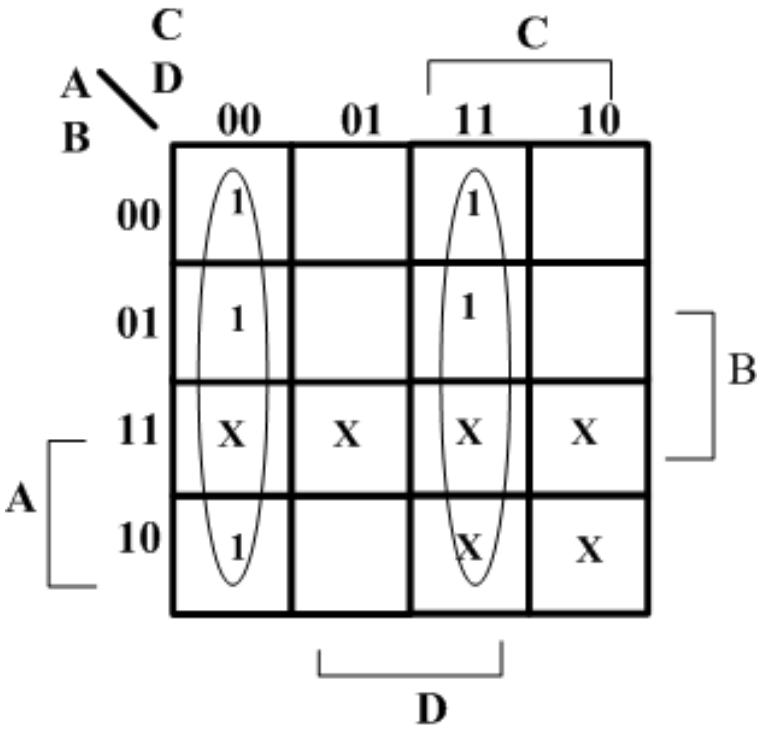

$X = BC'D' + B'C + B'D$

# Optimization – BCD-to-Excess-3

**K-map for Y** $\Sigma_m(0,3,4,7,8)$



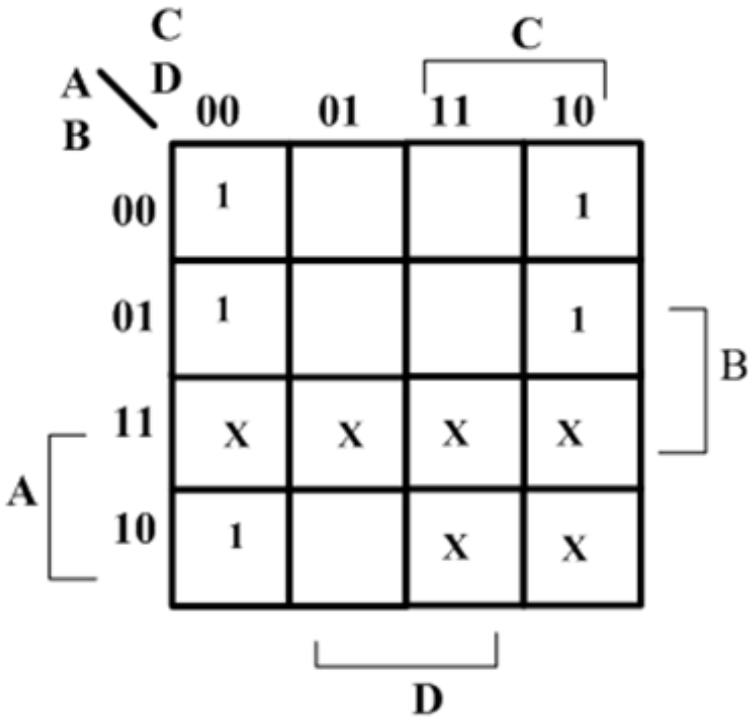

**K-map for Y**



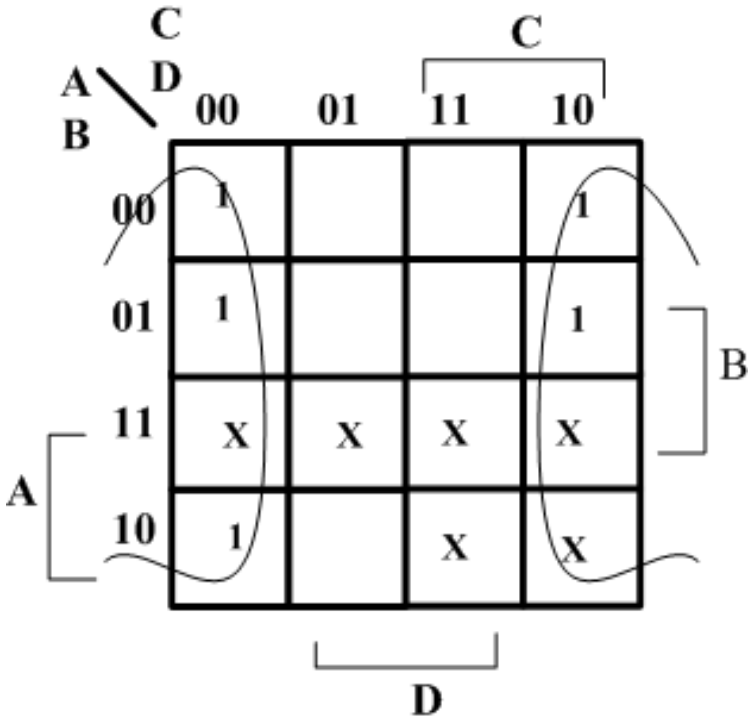

$Y = C'D' + CD$

# Minimize K-Maps

K-map for Z $\Sigma_m(0,2,4,6,8)$




K-map for Z




$Z = D'$

# Two level versus multi-level circuit implementation

- ✓ Have equations
  - W = A + BC + BD
  - X = B'C + B'D + BC'D'
  - Y = CD + C'D'
  - Z = D'
- ✓ Call T= C+D, then T' = (C+D)' = C'D'
  - W = A + BT
  - X = B'T + BT'
  - Y = CD + T'
  - Z = D'

# Create the digital circuit

- ✓ Implementing the second set of equations where T=C+D results in a lower gate count.
- ✓ This gate has a fanout of 3