A Growing Self-Organizing Network for Manifold Reconstruction

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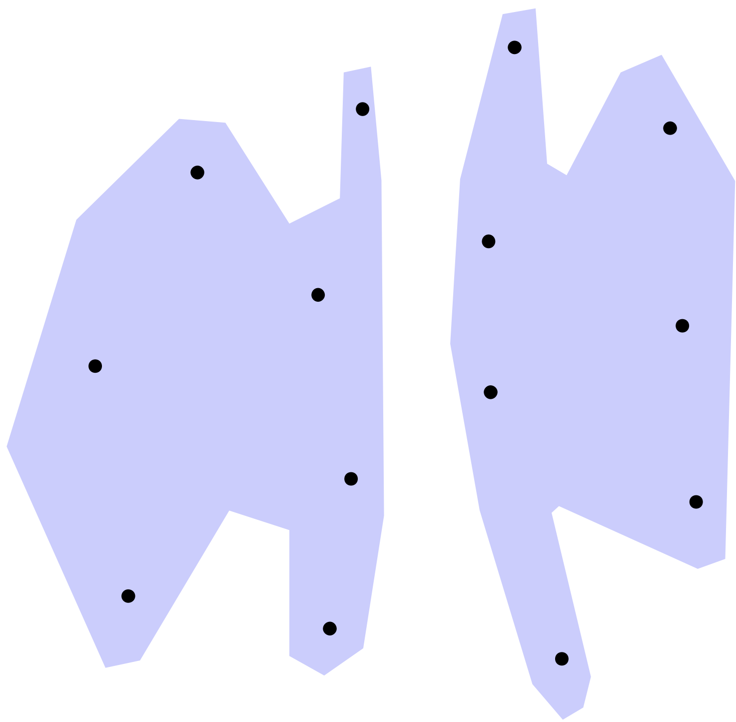
Restricted Delaunay Complex

- **Manifold (a surface embedded in $\mathbf{R}^2$)**

Restricted Delaunay Complex

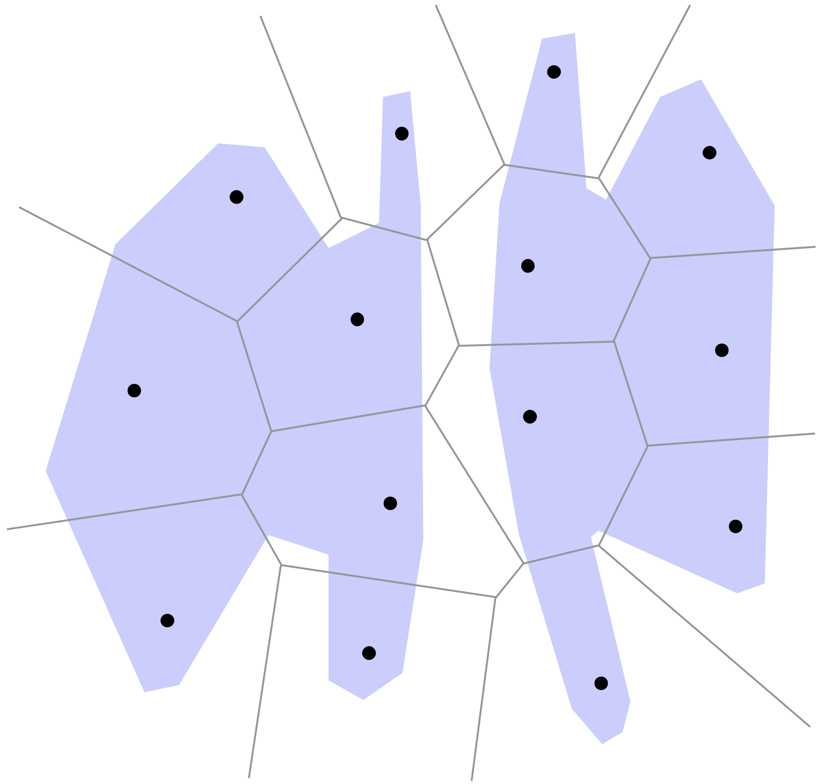
- **Point sample (*landmarks*) of the manifold**



Restricted Delaunay Complex

- **Voronoi complex of the landmarks**

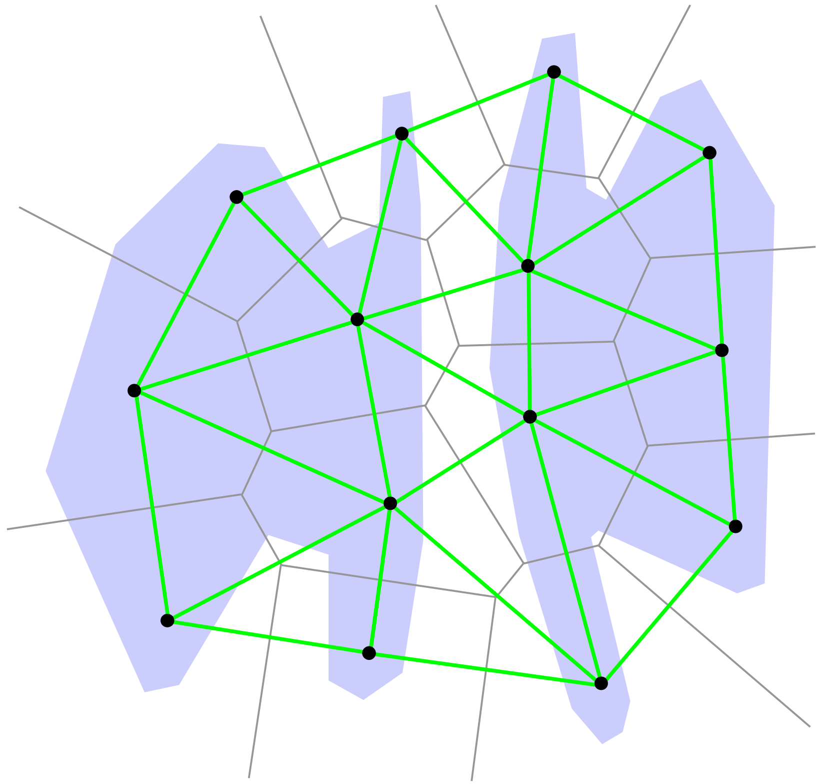
  Each cell contains all points of $\mathbf{R}^2$ being closer to a specific *landmark*



Restricted Delaunay Complex

- **Delaunay graph of the landmarks**

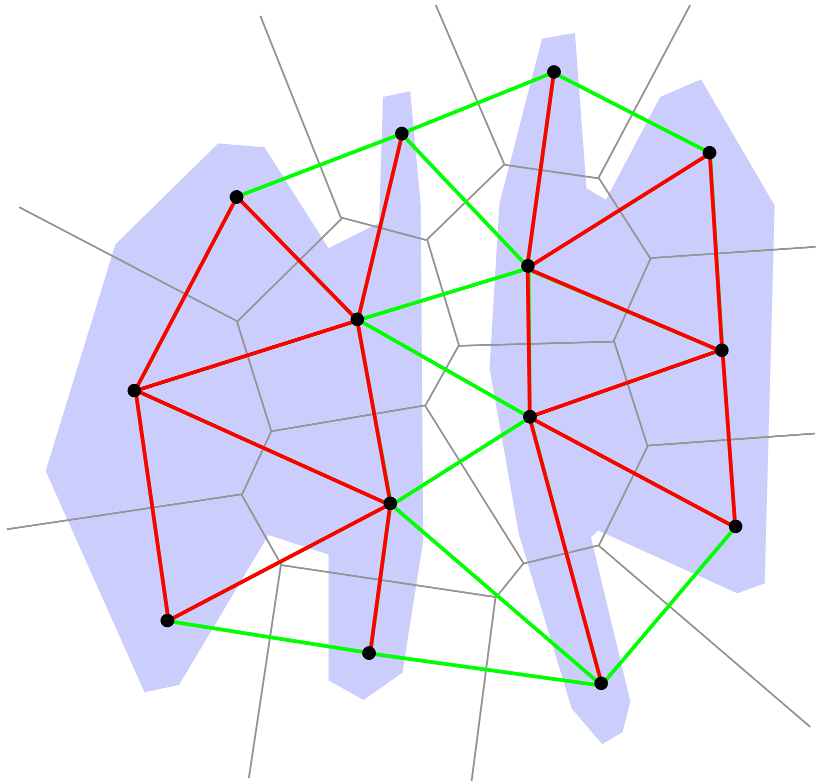
  An edge connects each two landmarks
  whose Voronoi cells have a common *boundary*



Restricted Delaunay Complex

- ***Restricted*** **Delaunay graph of the landmarks**

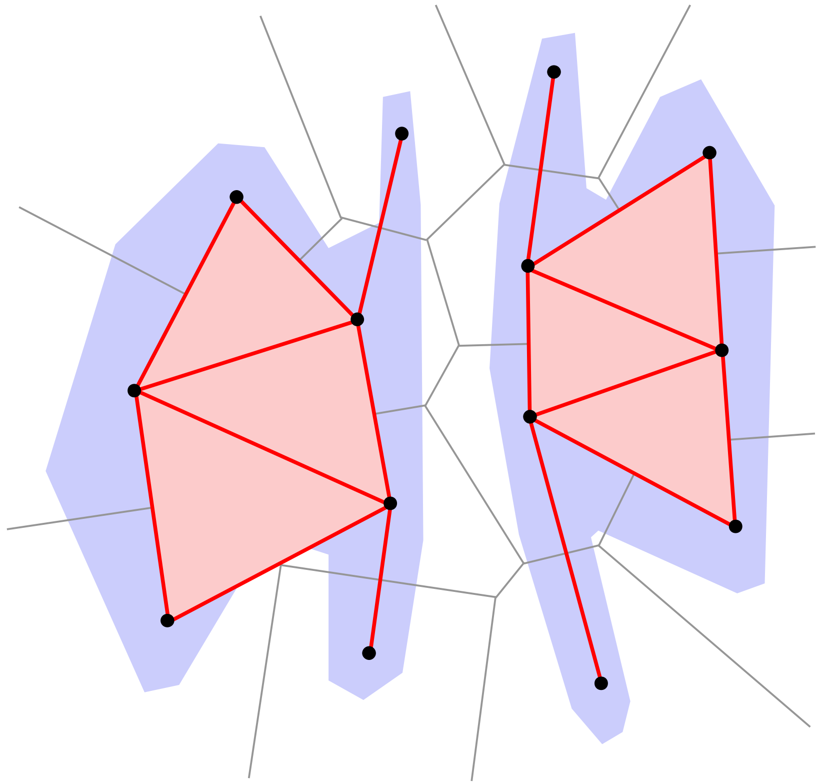
  An edge connects each two landmarks
  whose Voronoi cells have a common *boundary* <u>which intersects the manifold M</u>



Restricted Delaunay Complex

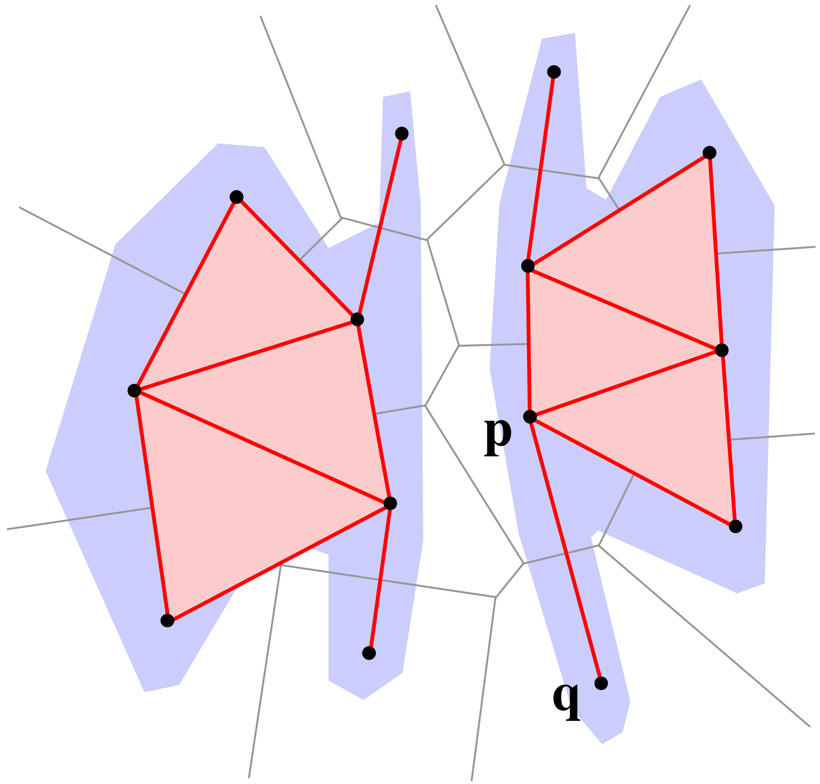
- **Restricted Delaunay complex of the landmarks**

  A $(n-1)$-dimensional n-face corresponds to n landmarks whose Voronoi cells have a common *boundary* which intersects M



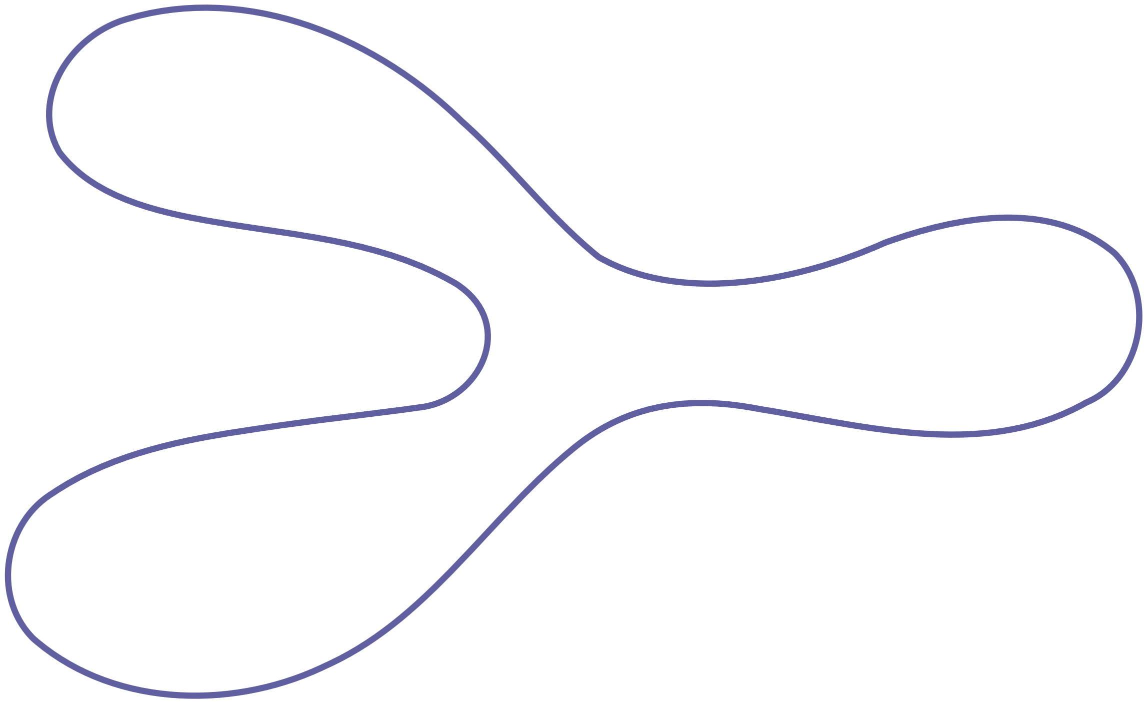
Restricted Delaunay Complex

- **Restricted Delaunay complex of the landmarks**
  - The complex, in general, is *not* homeomorphic to the manifold
    - Here, for instance, the neighborhoods of either $\mathbf{p}$ or $\mathbf{q}$ have no counterparts in M



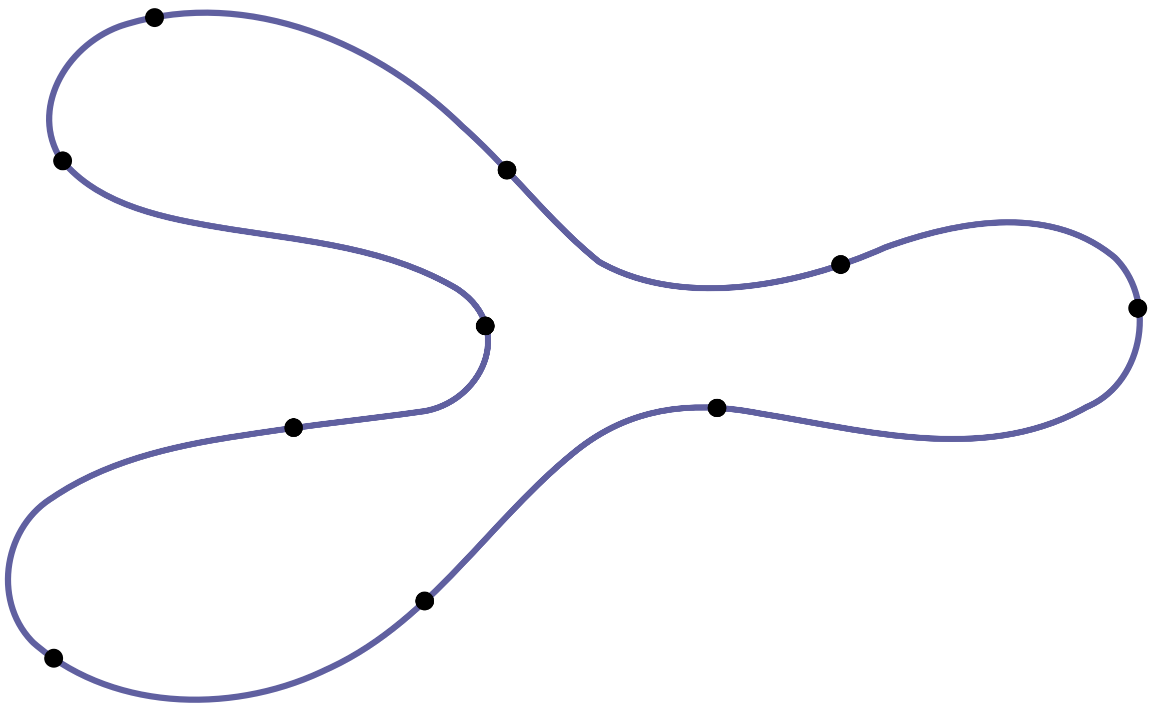
Restricted Delaunay Complex and Homeomorphism

- **Manifold (a curve embedded in $\mathbf{R}^2$)**



Restricted Delaunay Complex and Homeomorphism

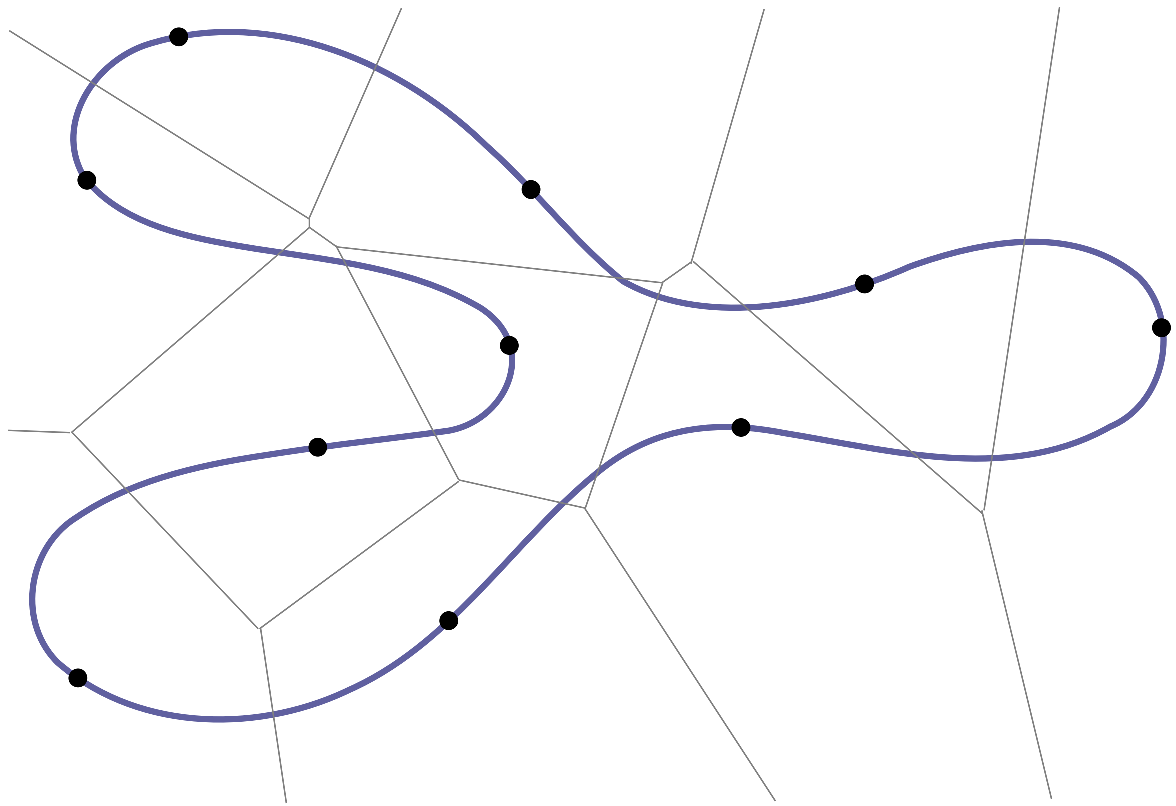
- **A first point sample (*landmarks*) of the manifold**



Restricted Delaunay Complex and Homeomorphism

- **Voronoi complex**

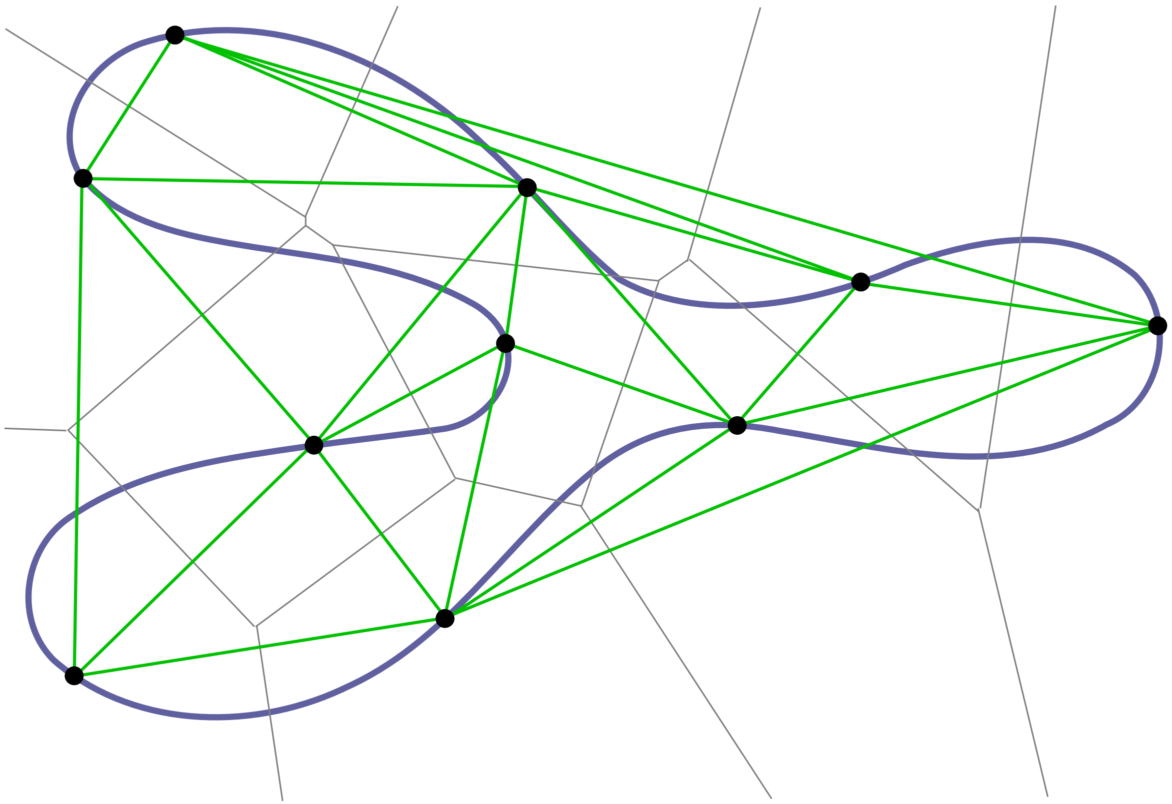
  Each cell contains all points of $\mathbf{R}^2$ being closer to a specific *landmark*



Restricted Delaunay Complex and Homeomorphism

- **Delaunay graph**

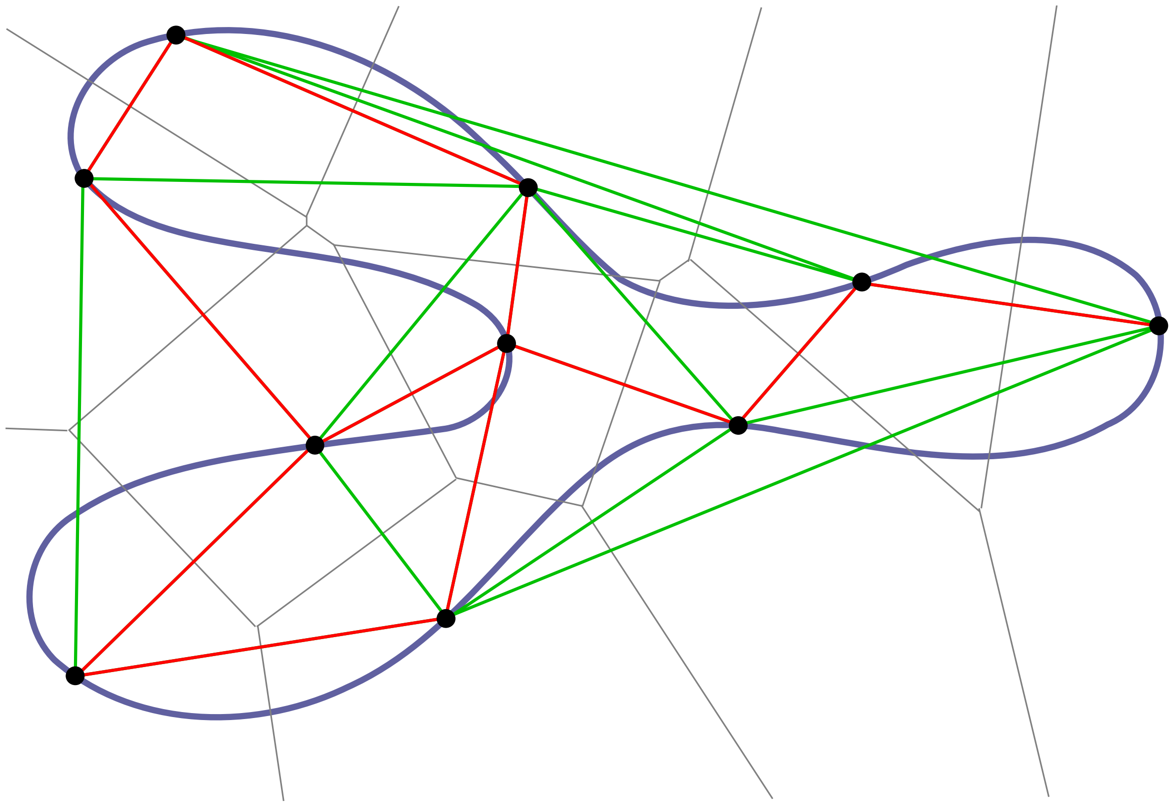
  An edge connects each two landmarks whose Voronoi cells have a common *boundary*



Restricted Delaunay Complex and Homeomorphism

- **Restricted Delaunay graph**

  An edge connects each two landmarks whose Voronoi cells have a common *boundary* which intersects M

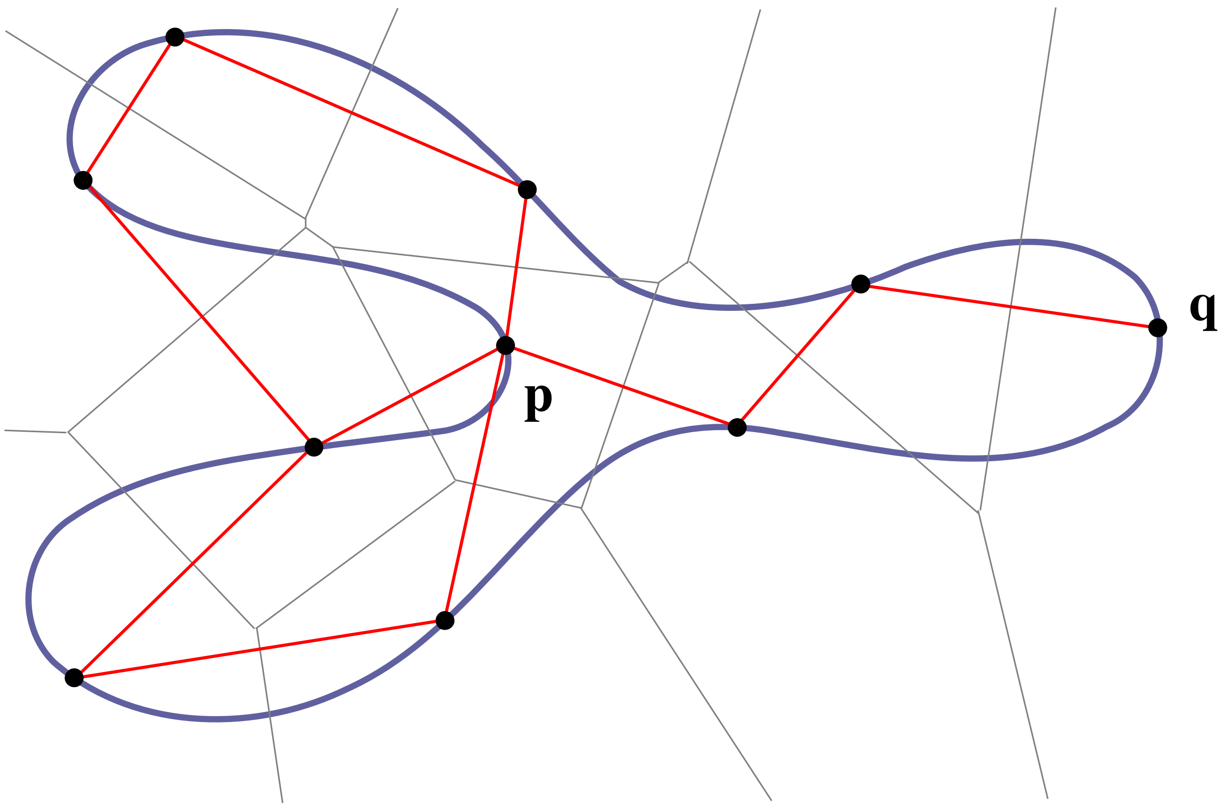


Restricted Delaunay Complex and Homeomorphism

- **Restricted Delaunay graph**

  Once again and in general, the complex is *not* <u>homeomorphic</u> to the manifold

  Here, for instance, the neighborhoods of either **p** or **q** have no counterparts in M

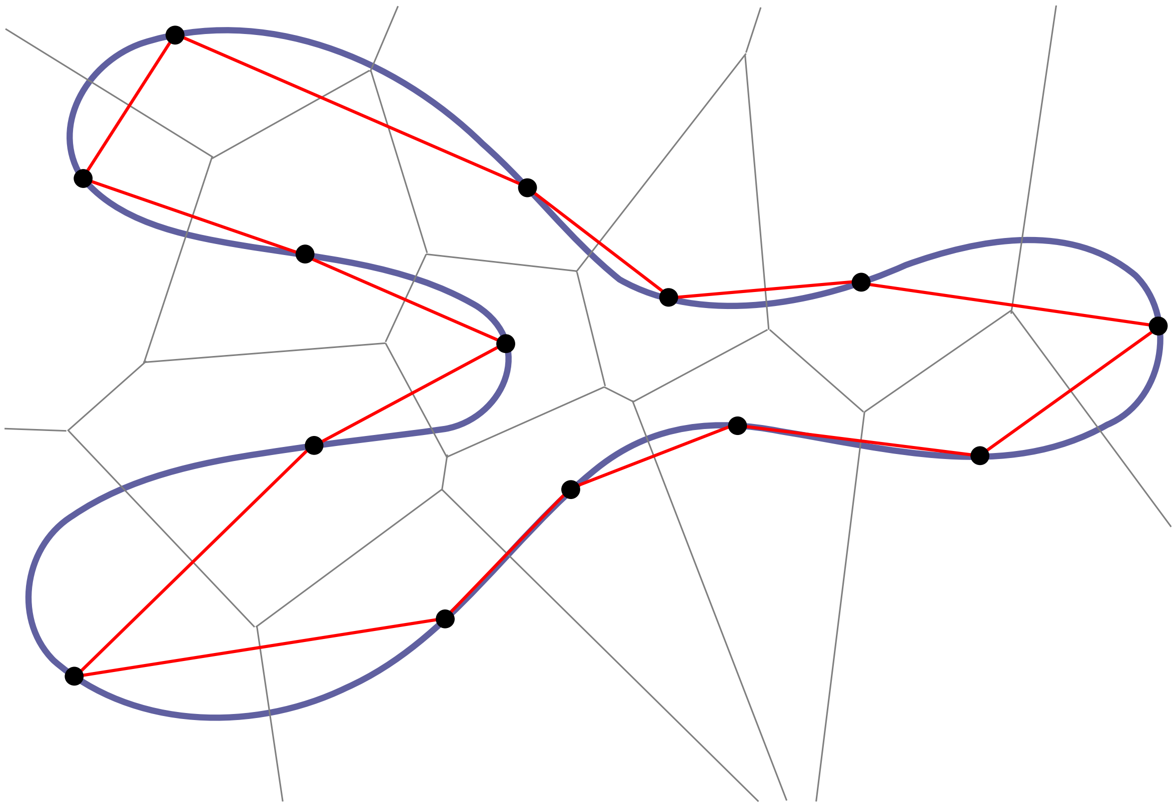


Restricted Delaunay Complex and Homeomorphism

- **Want homeomorphism?**

  Just add more landmarks.

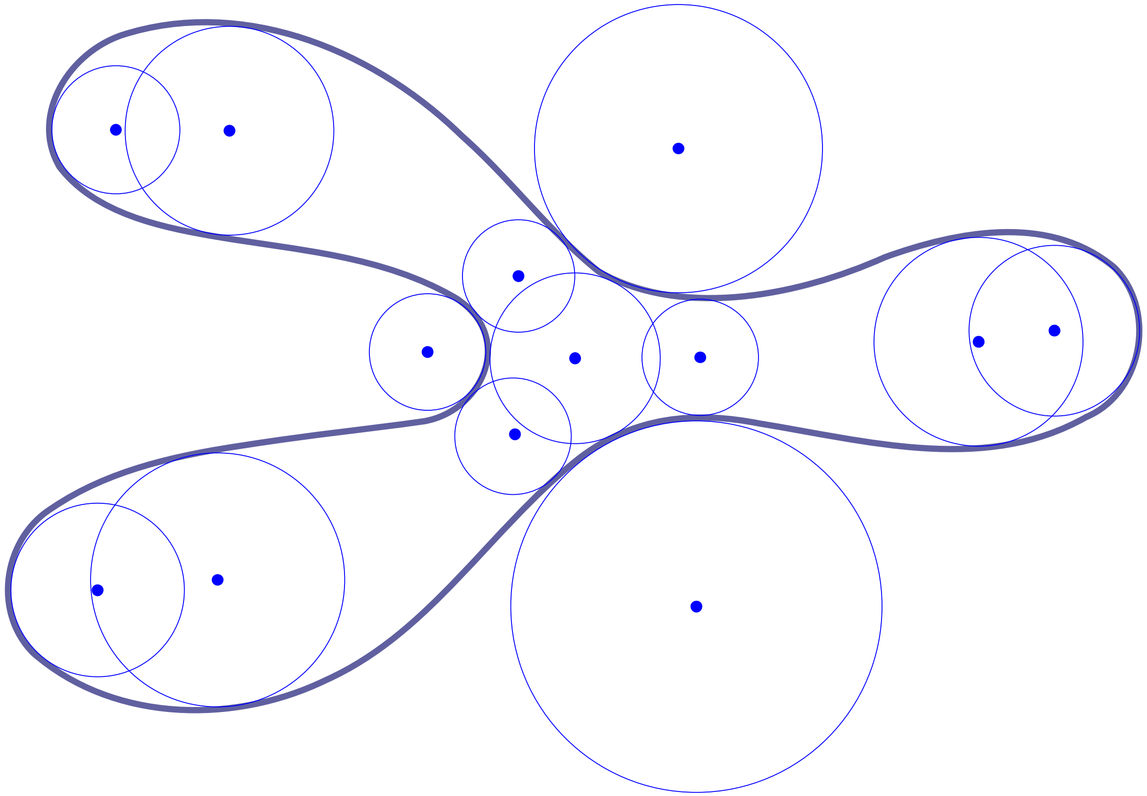
  (There exists a *density threshold*)



ε-sample

- **Medial balls**

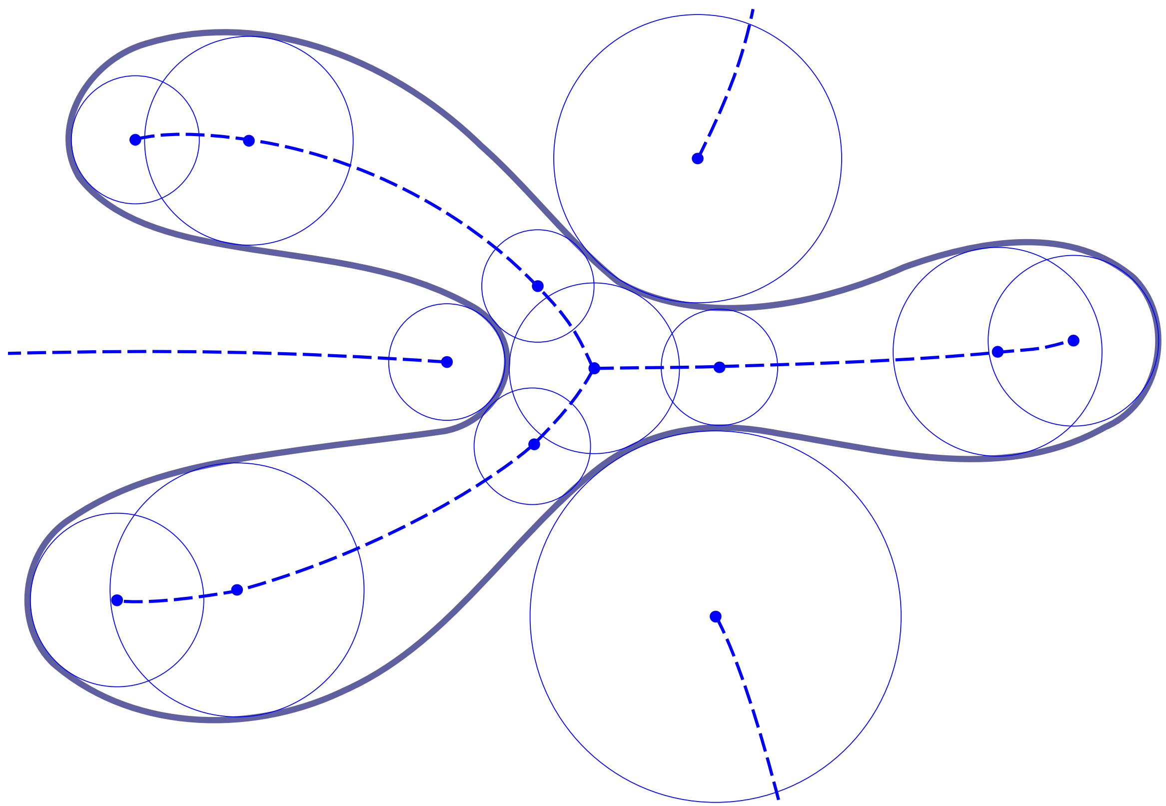
  Maximal balls whose interiors are empty of any points from M



ε-sample

- **Medial axis**

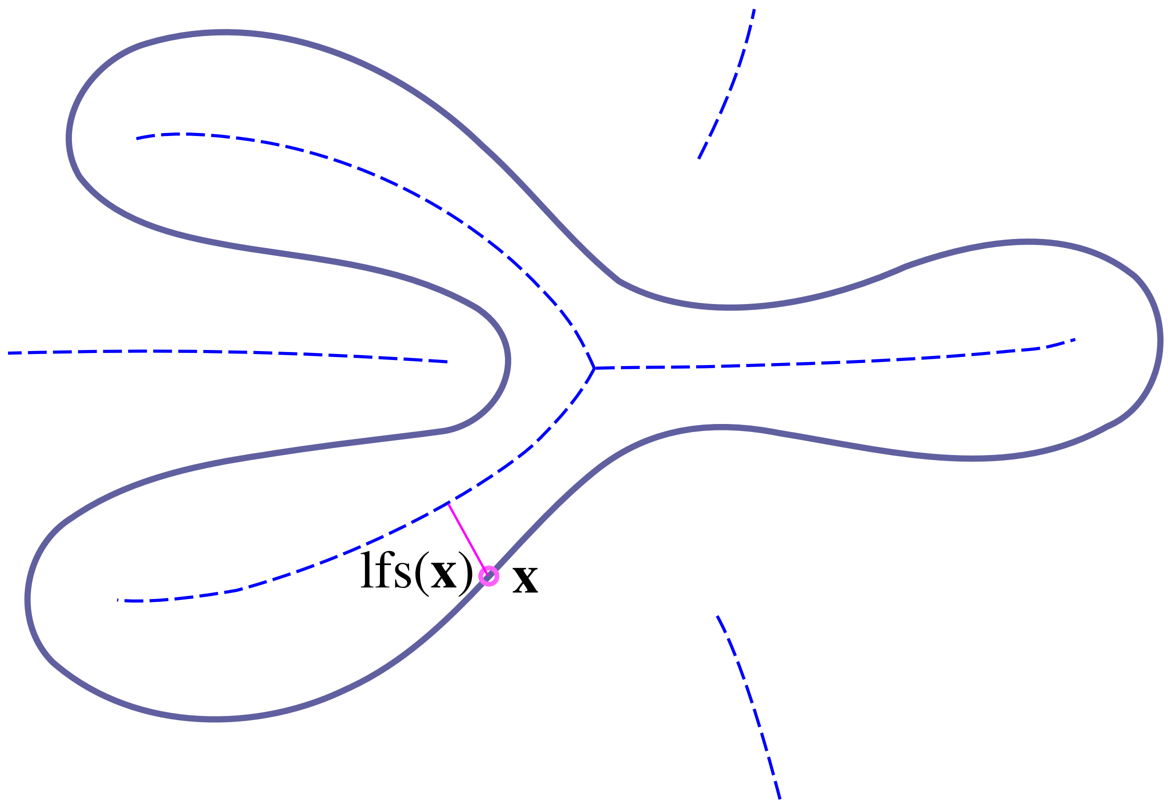
  The closure of the set of points that are centers of maximal balls



ε-sample

- **Local Feature Size**

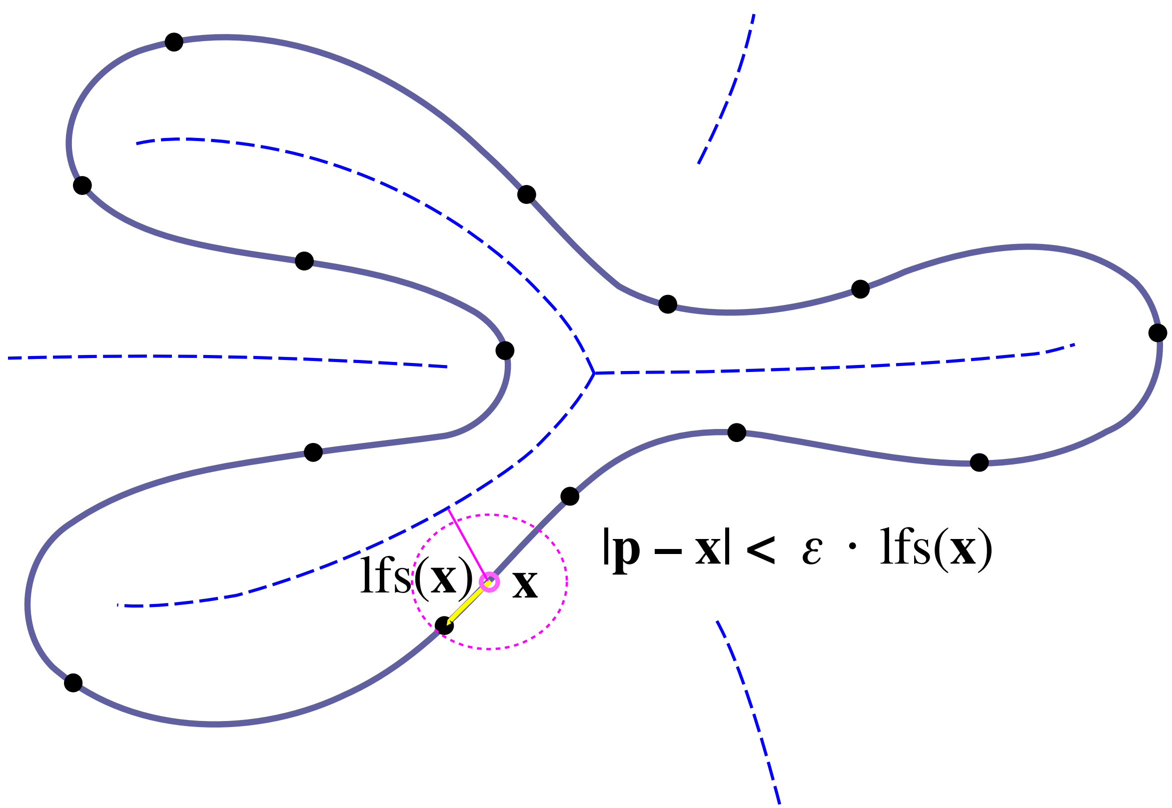
  (at a point **x** on M)
  It is the distance between **x** and the medial axis



ε-sample

- **ε-sample**

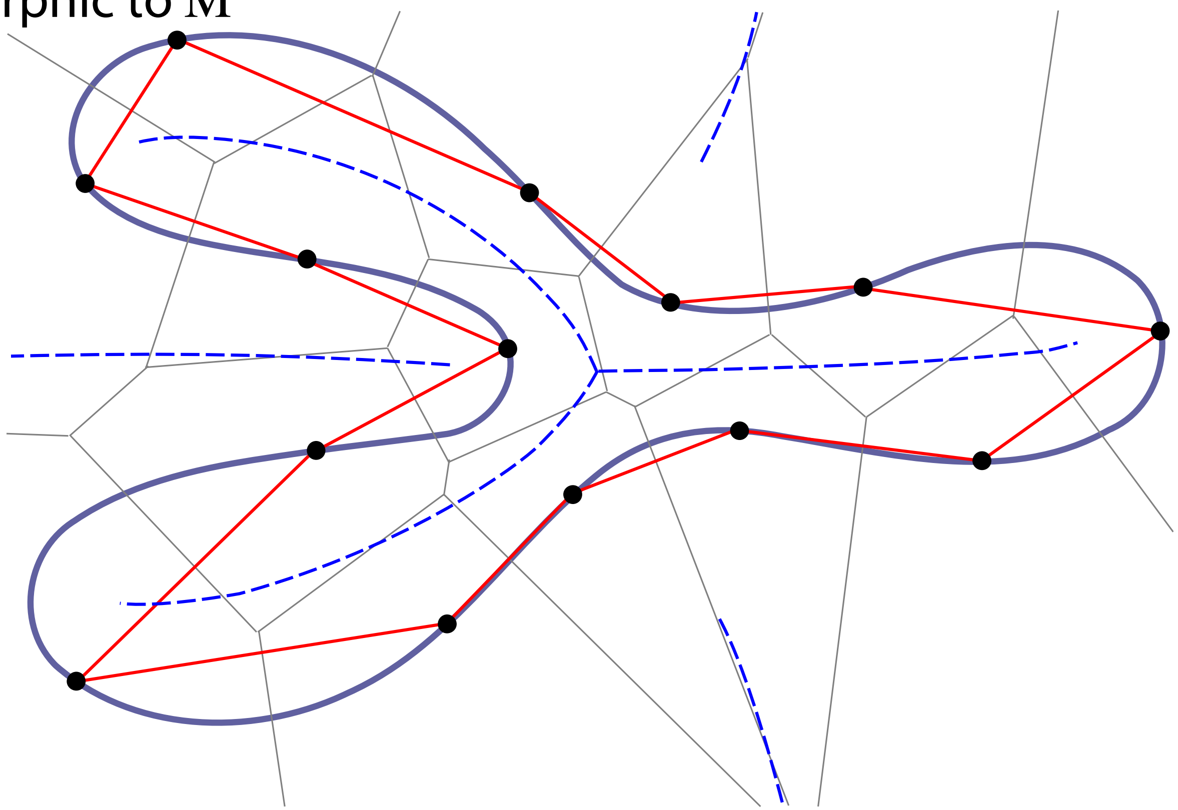
  A set of landmarks such that every point $\mathbf{x}$ on M
  is at most $\varepsilon \cdot \mathrm{lfs}(\mathbf{x})$ away from the closest landmark $\mathbf{p}$



ε-sample

- **ε-sample and homeomorphism**

  [Amenta et al., 2000]
  If M is compact, closed and *smooth*, there exists a positive ε such that the restricted Delaunay complex for any ε-sample of M is homeomorphic to M



ε-sample

- **The restricted Delaunay complex of an ε-sample**

  When M is compact, closed and *smooth* and ε is sufficiently small

  - It is homeomorphic to M
  - The Hausdorff distance to M is $O(\varepsilon^2)$
  - It allows a reliable estimate of curvatures, normals, lengths or areas of M

- **Limitations**

  It works only with manifolds of dimension 1 or 2

  Although the dimension of the ambient space could be any

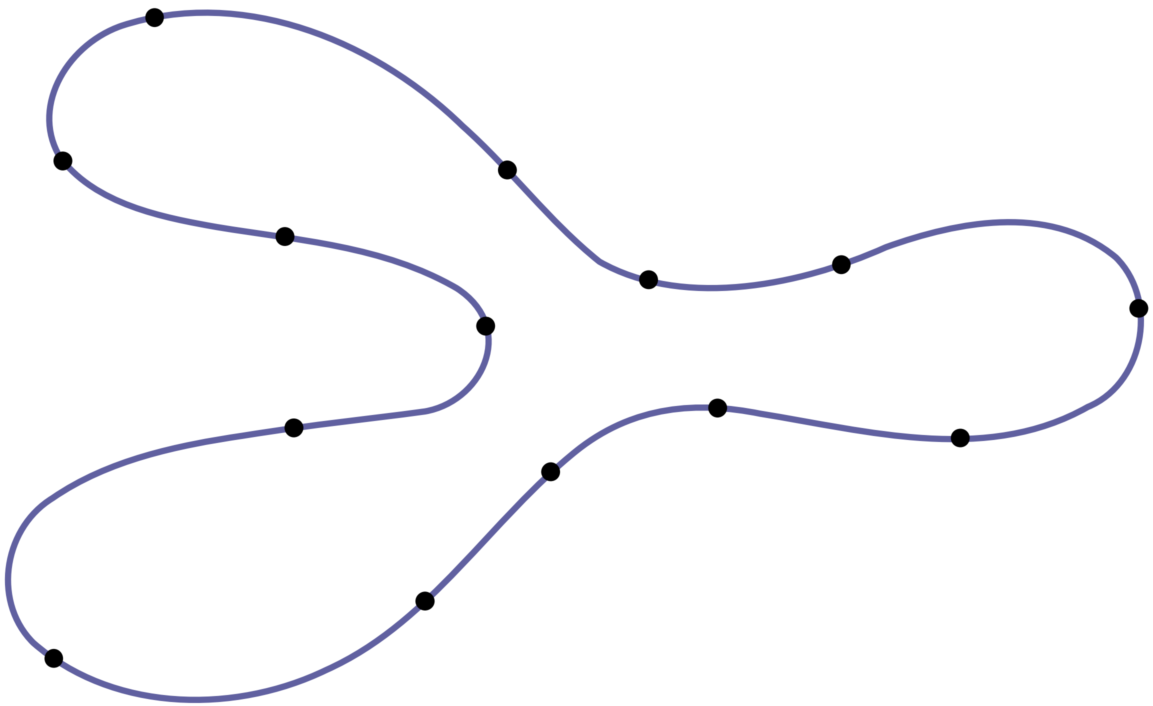
  [Oudot, 2008]
  For manifolds of dimension greater than 2,
  no positive value of ε guarantees
  that an ε-sample has the properties above

A <u>weighted</u> Delaunay complex could bring those properties back (*but this is another story*)

Witness complex

- **How can the restricted Delaunay complex be constructed?**

  (From a given set of landmarks)

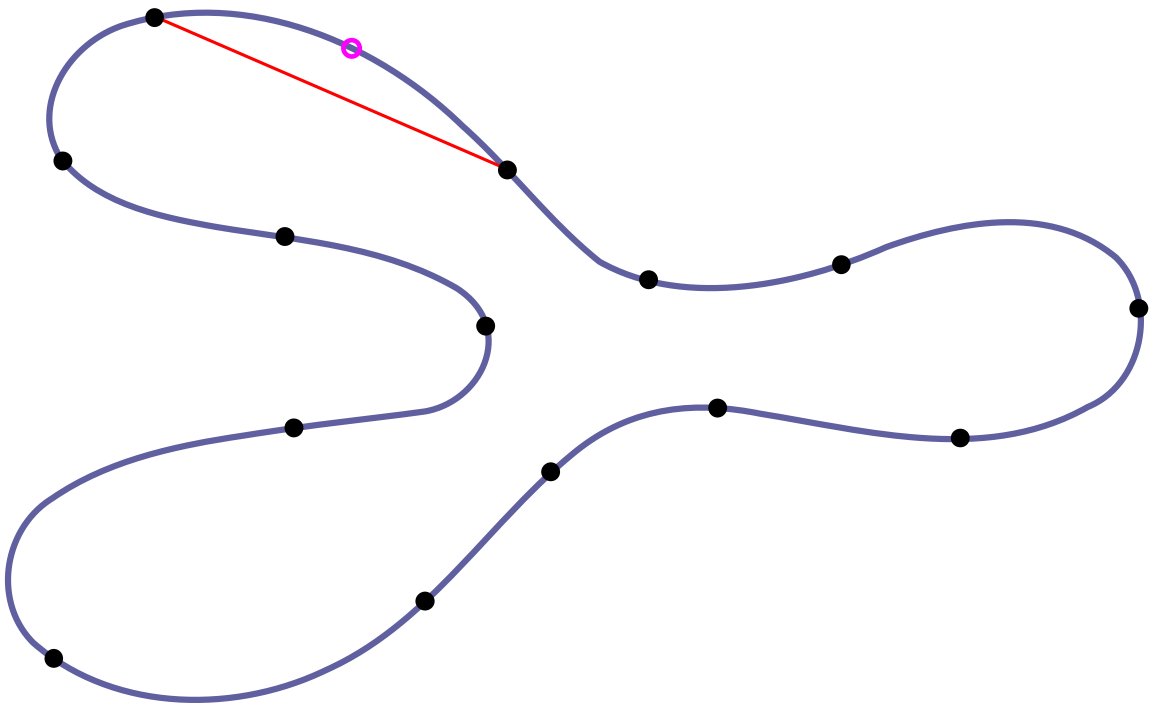


Witness complex

- **Try sampling the manifold at random**

  For each sample, add a connection between the two closest landmarks

  The sampled point is deemed a *witness* for the corresponding connection



Witness complex

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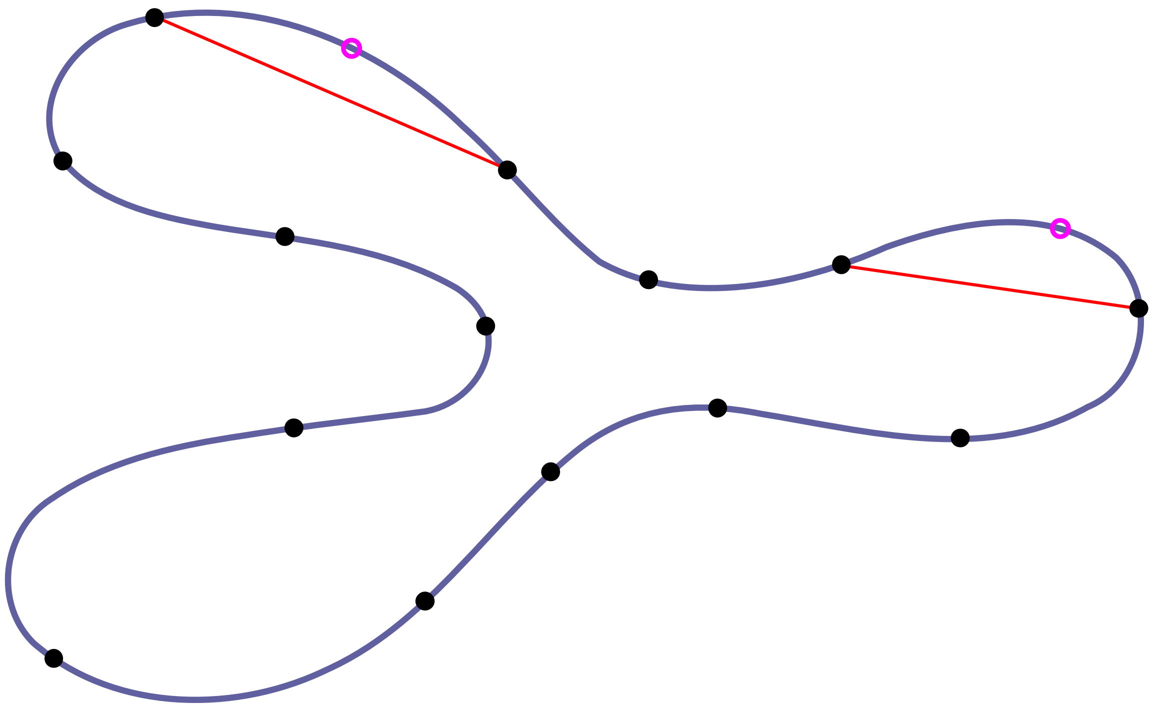

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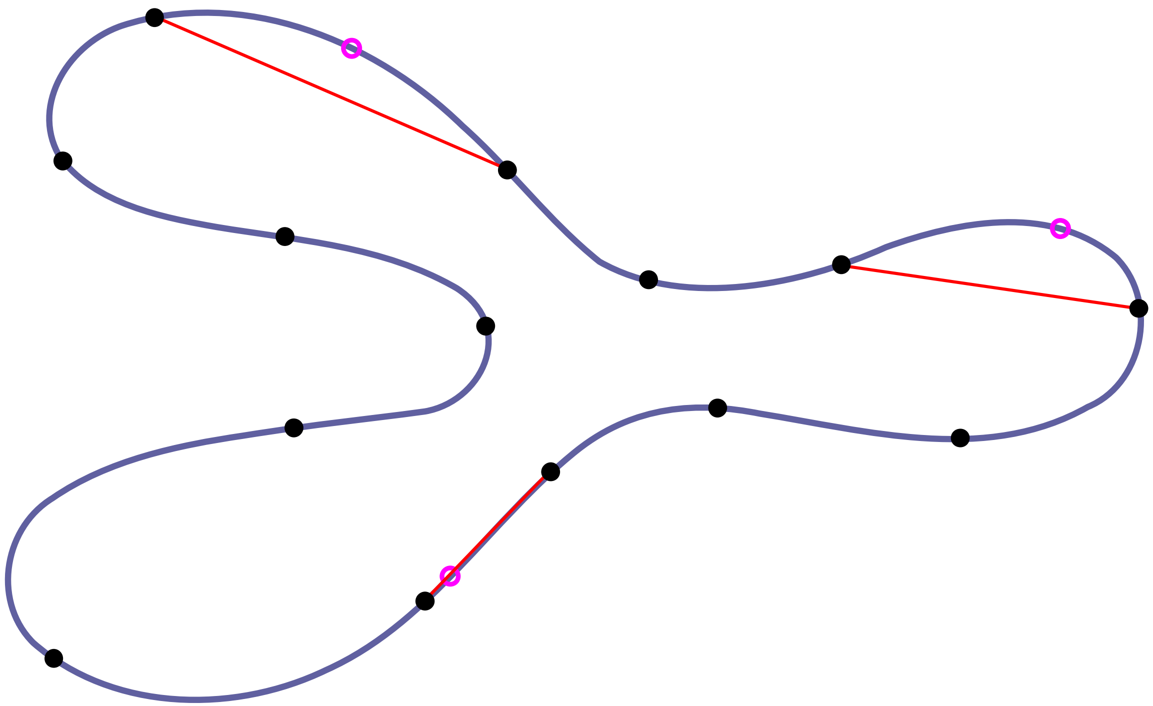

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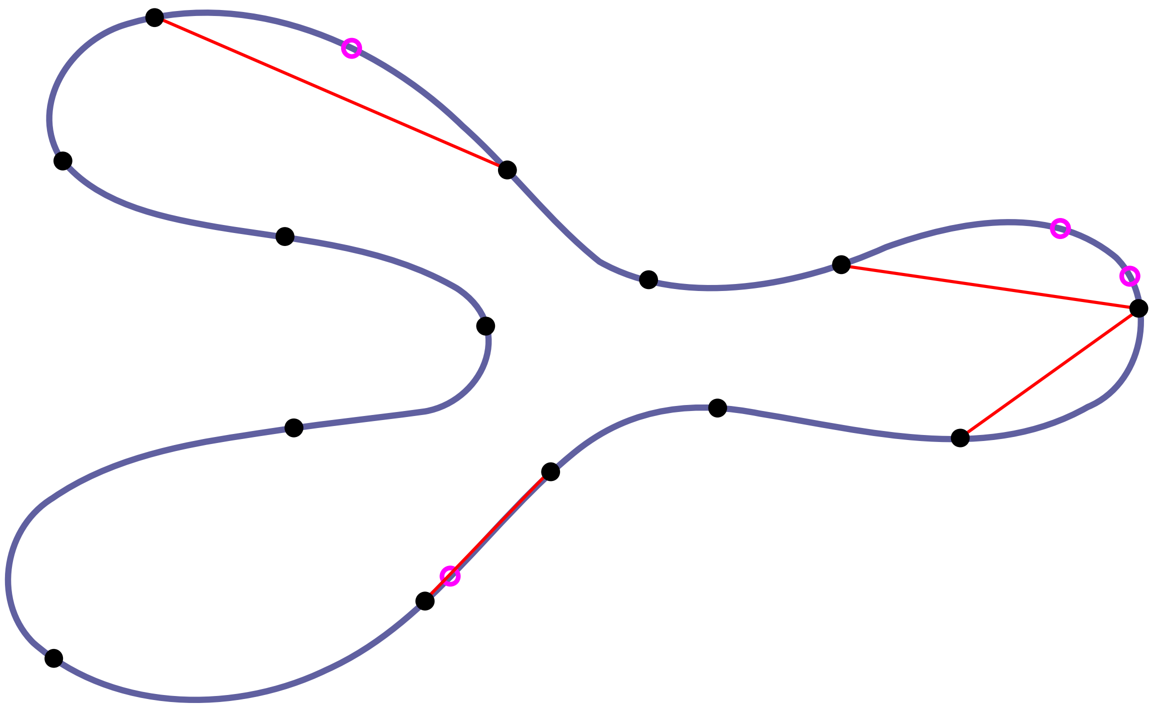
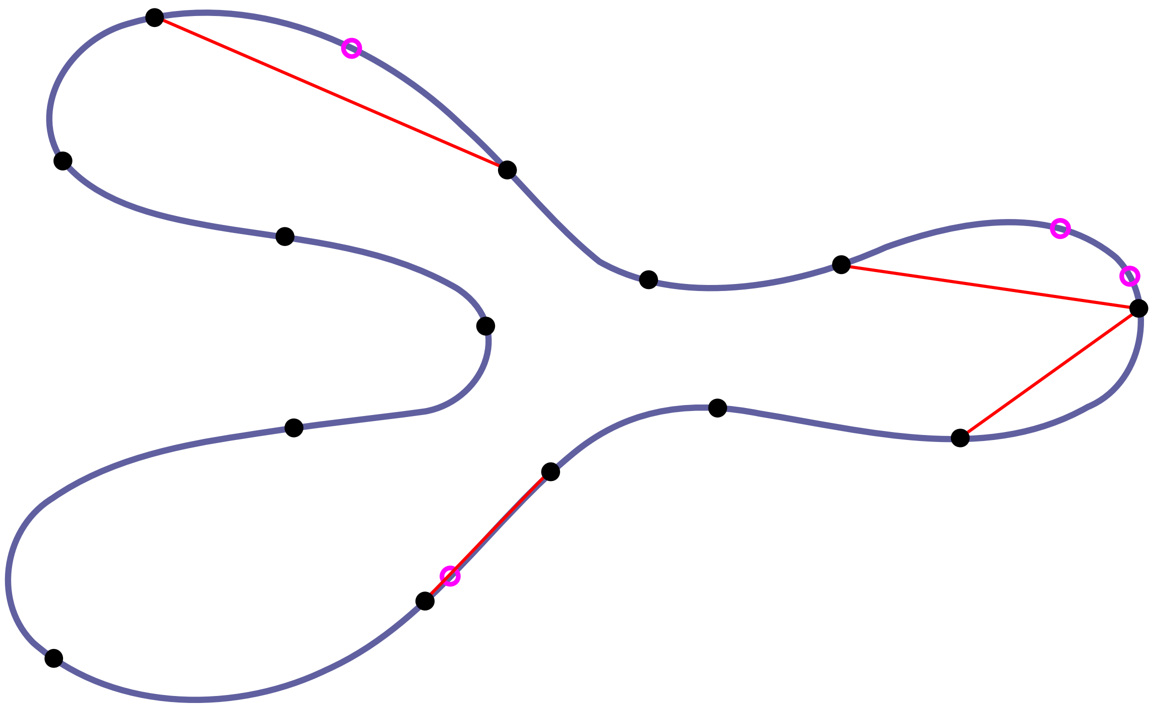


Witness complex

- **Witness complex**

  It is the structure obtained by taking the sampling process to the limit

  i.e. when the whole M has been sampled



Witness complex

- **Witness complex**

  It is the structure obtained by taking the sampling process to the limit

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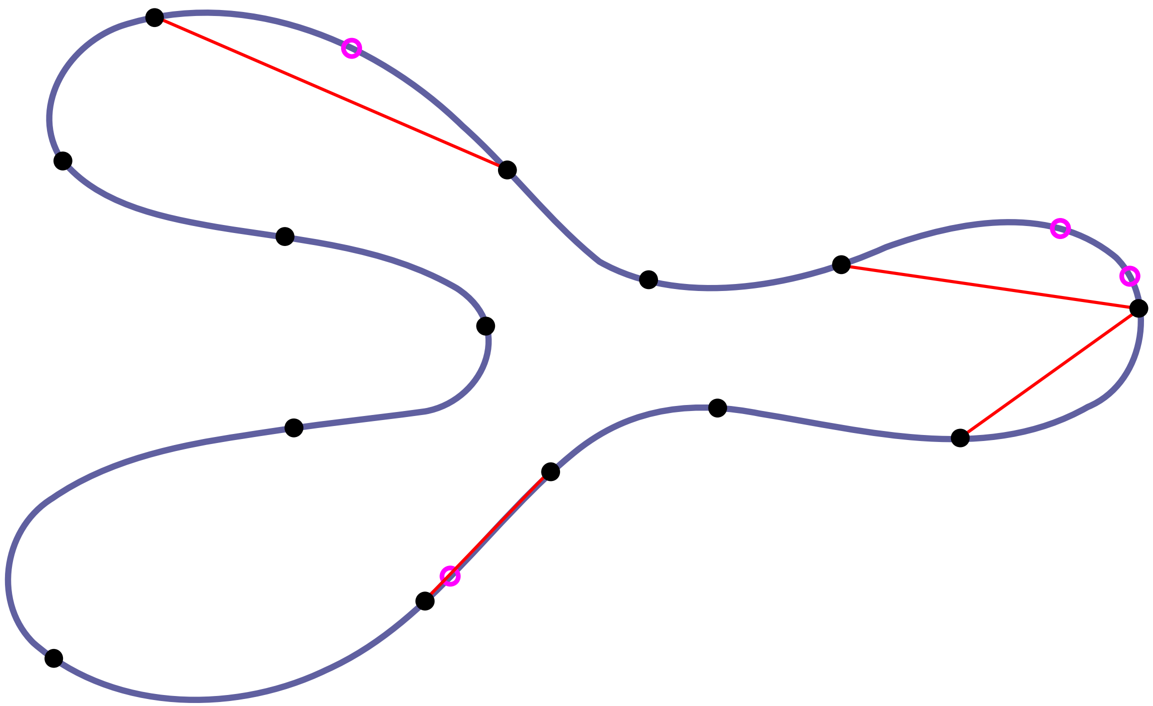
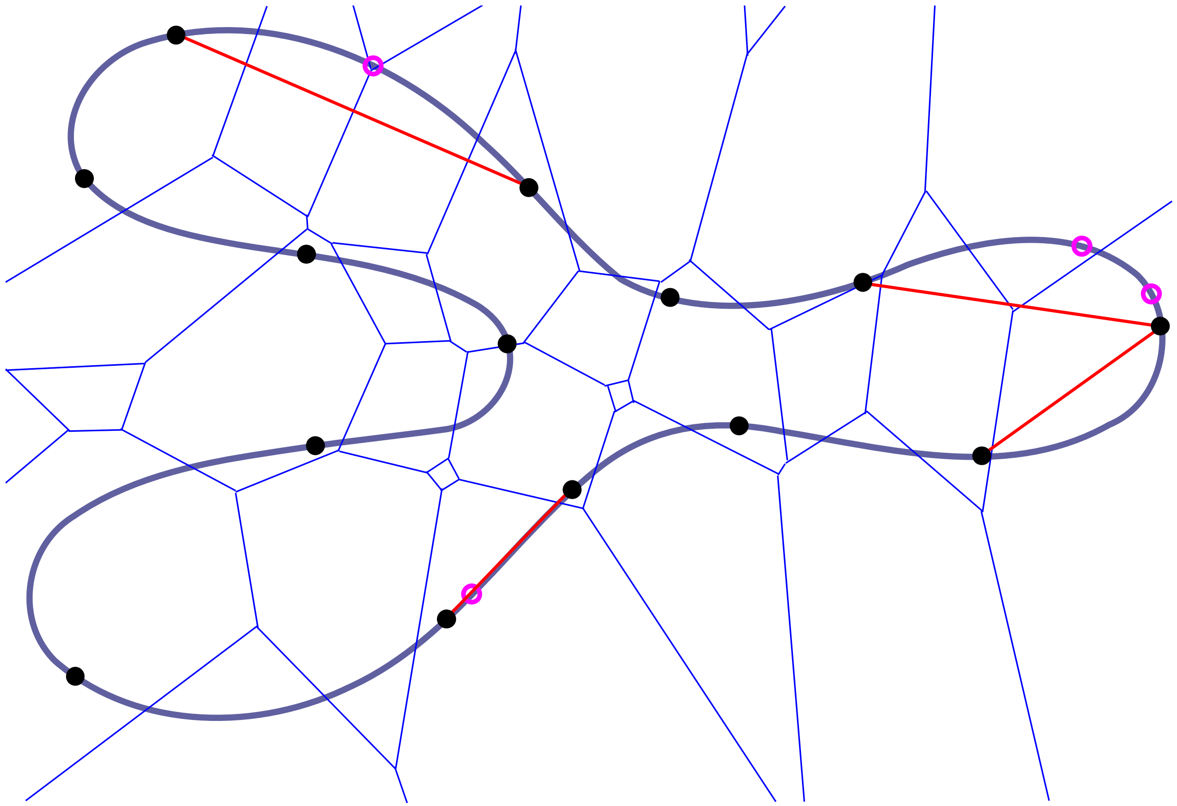
  Will it coincide with the restricted Delaunay complex?



Witness complex

- **Second-order Voronoi complex**

  Each cell contains all points of $\mathbf{R}^2$ being closer to a specific *pair of landmarks*

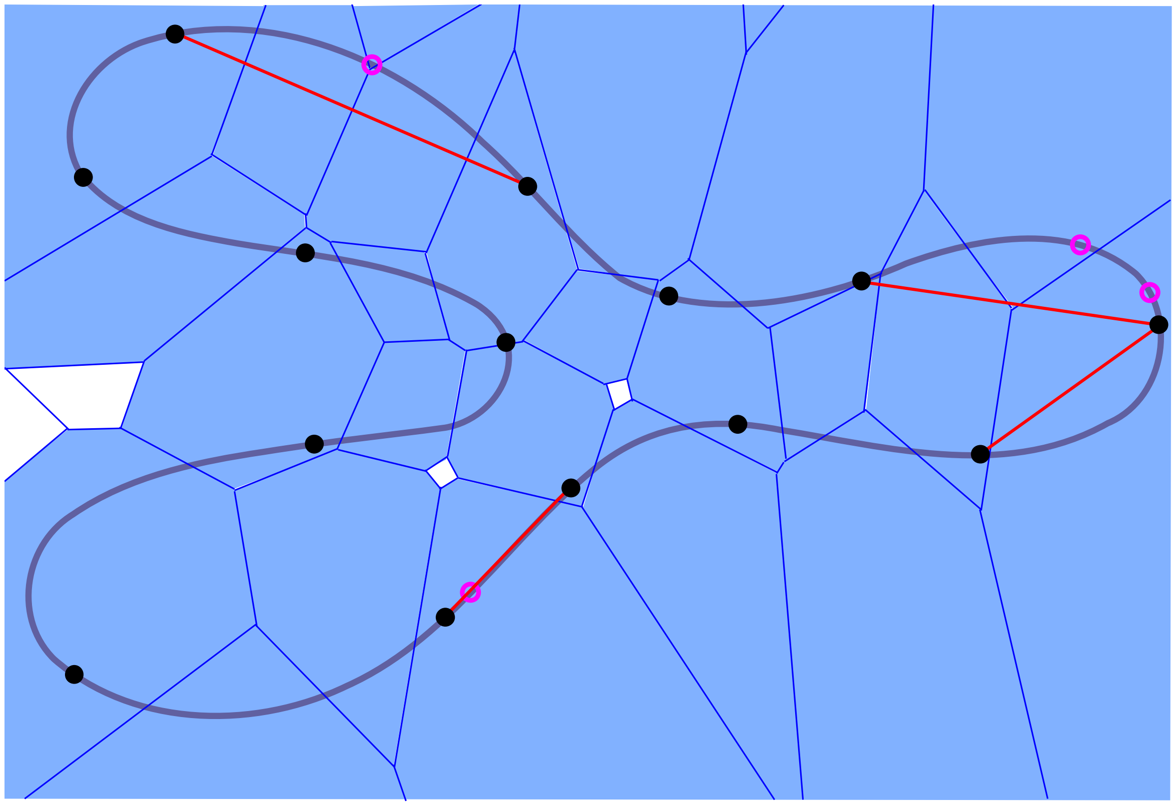


Witness complex

- **Second-order Voronoi complex**

  Each cell contains all points of $\mathbf{R}^2$ being closer to a specific *pair of landmarks*

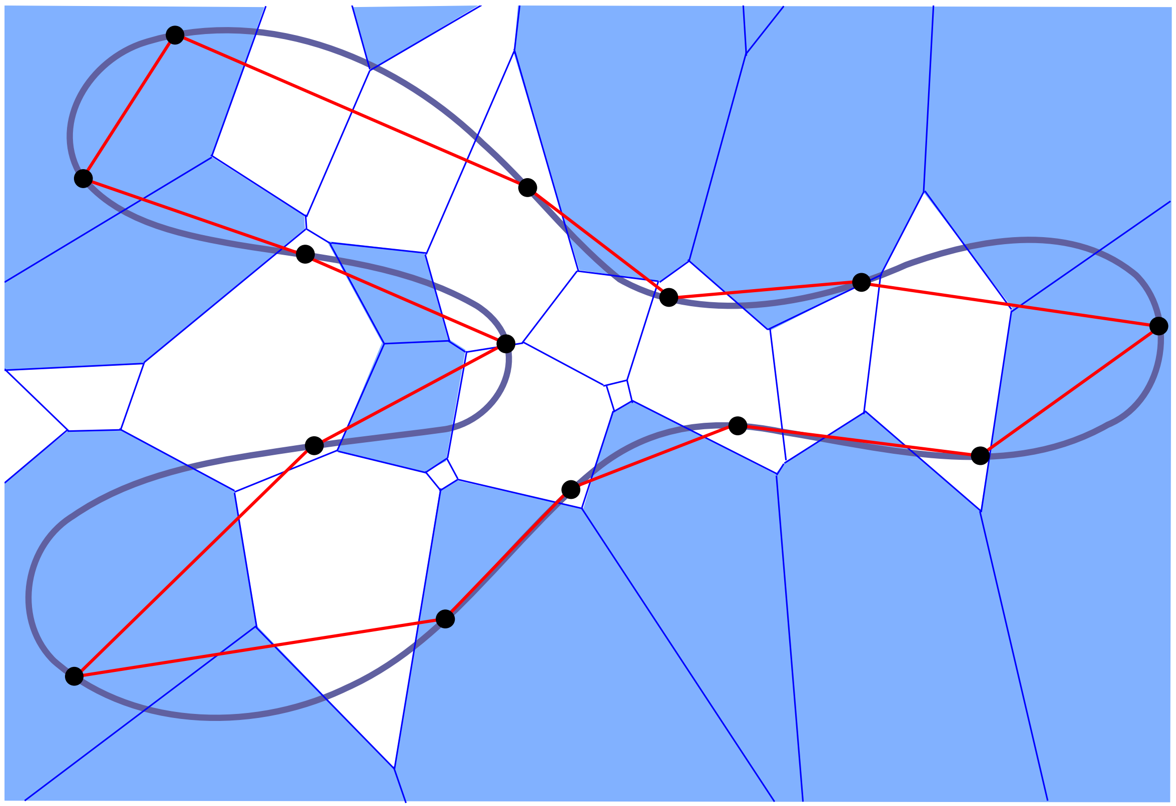
  Therefore, each cell intersecting M contains *witnesses* for one connection



Witness complex

- **Second-order Voronoi complex and witness complex**

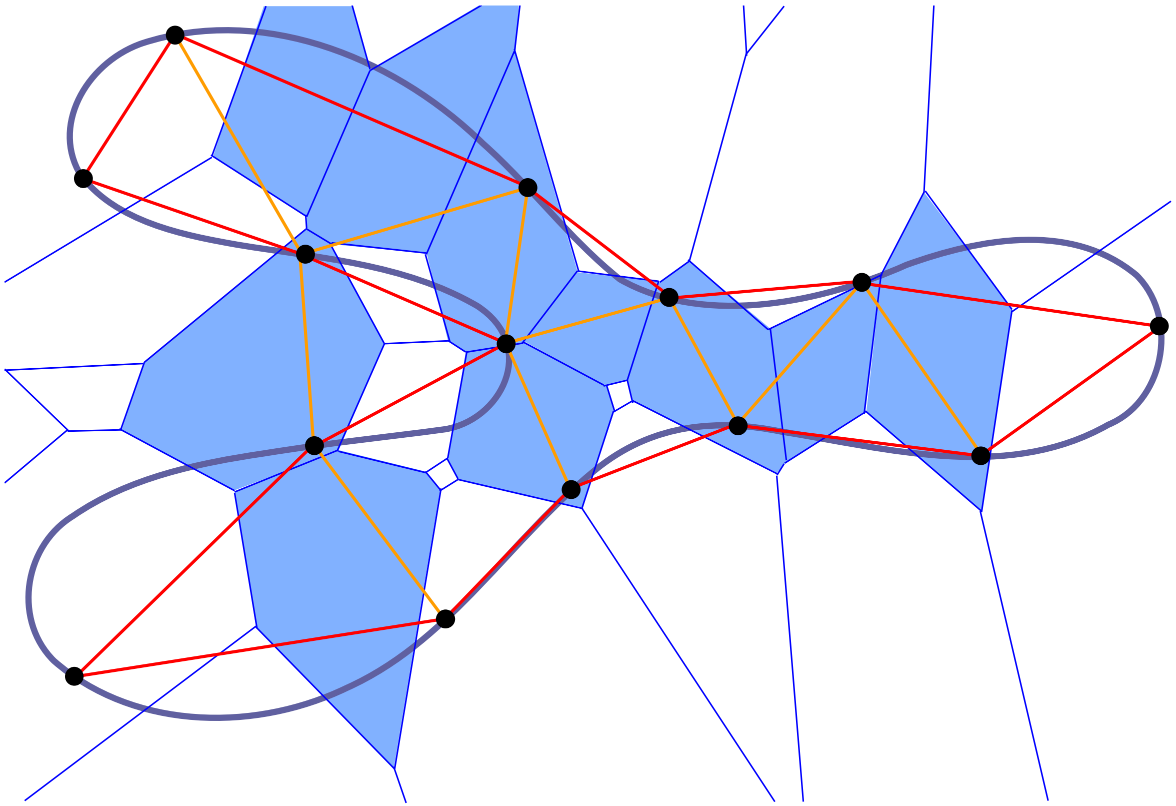
  Certainly, there are witnesses for the restricted Delaunay complex



Witness complex

- **Second-order Voronoi complex and witness complex**

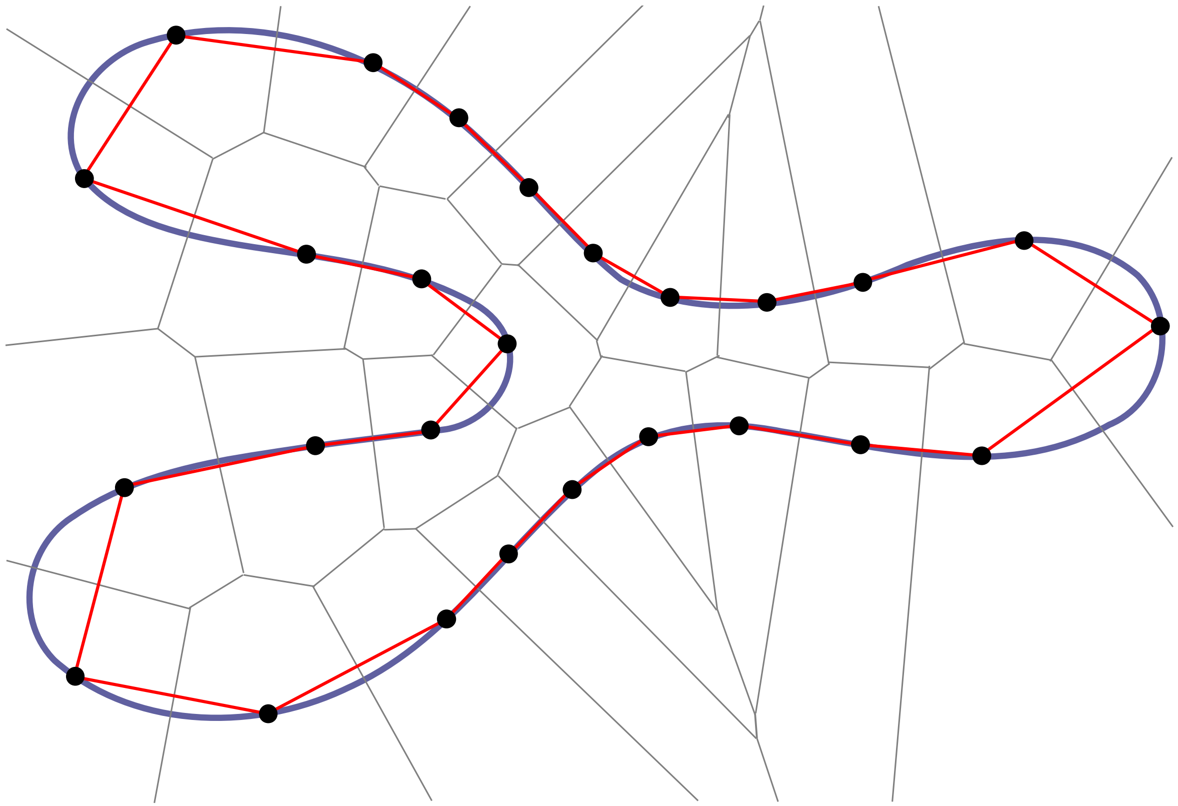
  Certainly, there are witnesses for the restricted Delaunay complex but there will be also witnesses for a few extra connections …



Witness complex

- **Witness complex and the restricted Delaunay complex**

  The solution? Add even more landmarks

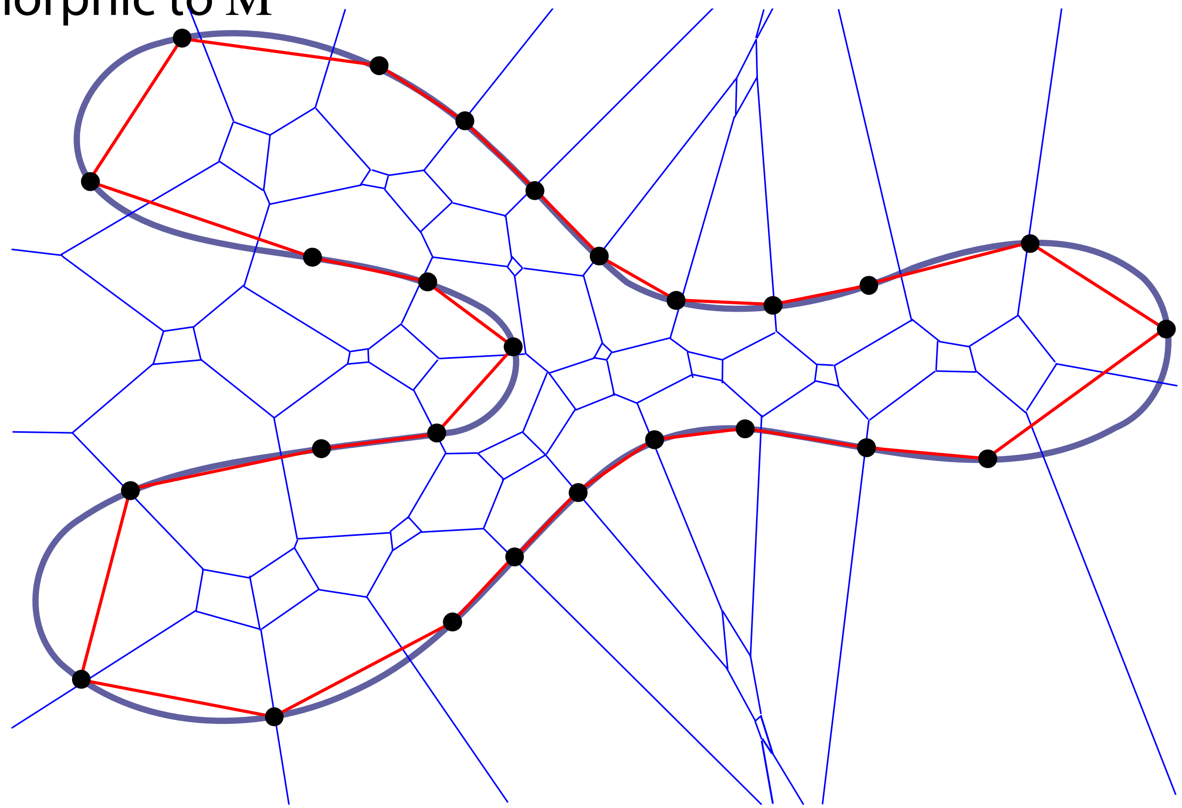


Witness complex

- **Witness complex and the restricted Delaunay complex**

  [Attali et al., 2007]

  There exists a positive ε such that the restricted Delaunay complex for an ε-sample coincides (in the limit) with the witness complex and both are homeomorphic to M



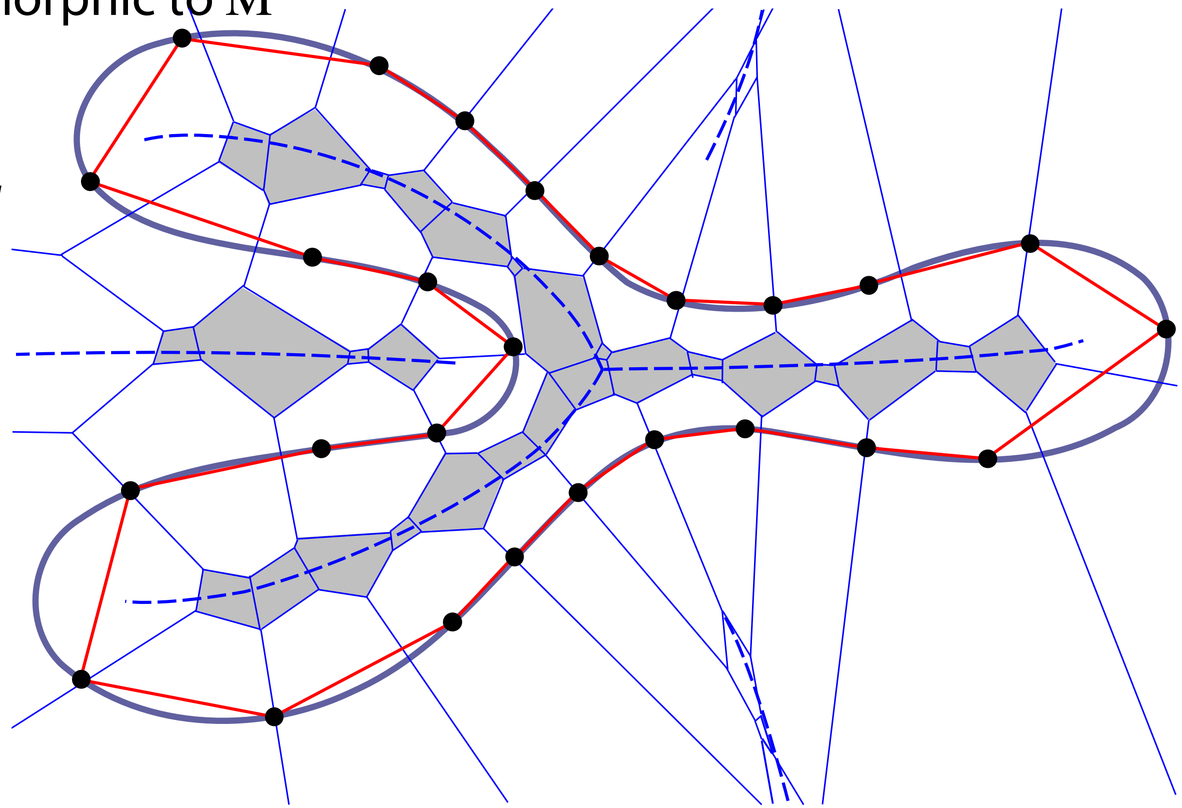
Witness complex

- **Witness complex and the restricted Delaunay complex**

  [Attali et al., 2007]

  There exists a positive ε such that the restricted Delaunay complex for an ε-sample coincides (in the limit) with the witness complex and both are homeomorphic to M

The second-order cells for the "extra" connections tend to aggregate around the medial axis



Self-Organizing Adaptive Map (SOAM)

- **The algorithm**

A set L of *units* (aka *landmarks*), initially containing two units only.

Each unit is associated to a few variables:

1) A *position* $\mathbf{p}$ in the ambient space
2) A *firing counter* f , which decays exponentially with unit activation
3) An *activity radius* r
4) A $state$, which changes dynamically during the process

A set of *connections* C, initially empty

Each connection is established between two units and is associated to one variable:

1) An age

A probability distribution $P(\xi)$, having M as its *support*

Self-Organizing Adaptive Map (SOAM)

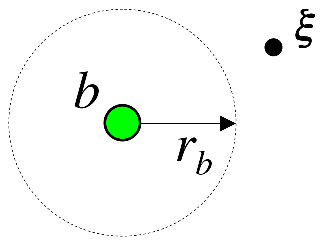
- **The algorithm**

1. Draw a sample ξ from $P(\xi)$
2. Determine the two units b and s whose positions are closest and second-closest to ξ
3. Add the connection (b, s) with $age = 0$ to C, if it is not already present. Otherwise, set its age to 0
4. Unless unit b is in a *stable* state (see below) increase by one the age of all connections involving b.
   Remove all connections whose age exceeds a threshold T_{age}
   Remove all units that became unconneted, due to this

Self-Organizing Adaptive Map (SOAM)

- **The algorithm**

5. If unit b is at least in the *habituated* state and the distance between the input ξ and its position $\mathbf{p}_b$ exceeds its *activity radius* r_b
   - create a new unit n
   - set its position to $\mathbf{x}$
   - remove the connection (b, s)
   - add new connections (b, n) and (n, s)



6. Decrease exponentially the *firing counters* of unit b and of all units connected to it

$$
\begin{aligned}
\Delta f_b &= (\alpha_h \cdot (F - f_b) - 1)/\tau_f \\
\Delta f_{nb} &= (\alpha_h \cdot (F - f_{nb}) - 1)/\tau_{f,n}
\end{aligned}
$$

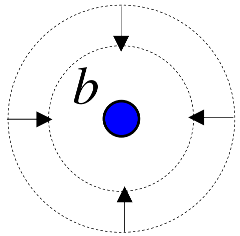
where F is the initial value and the α's and τ's are suitable constants

Self-Organizing Adaptive Map (SOAM)

- **The algorithm**

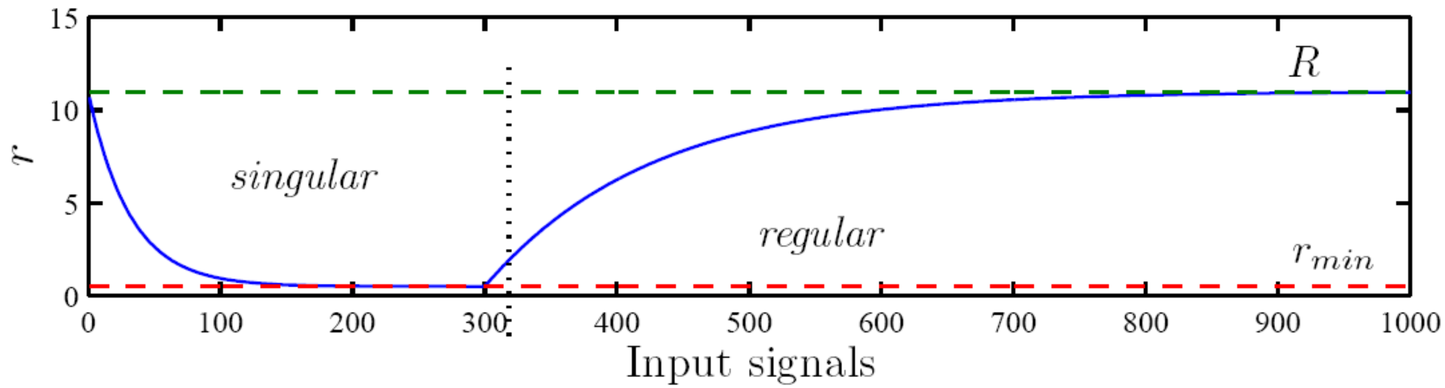
7. Update the state of unit b, according to the value of the *firing counter* f_b and the topology of its *neighborhood* of connected units (see below)
8. If unit b is in a *singular* state, <u>decrease</u> exponentially its *activity radius* r_b

$$\Delta r_b = (\alpha_r \cdot (R - r_b) - 1)/\tau_{r,\,hab}$$



otherwise, if unit b is in a *stable* state <u>increase</u> exponentially r_b

$$\Delta r_b = ((\alpha_r/\tau_{r,\,dis}) \cdot (R - r_b)$$

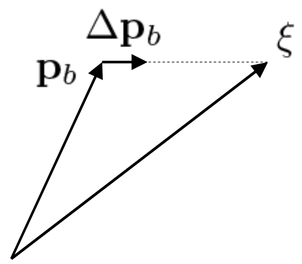


Self-Organizing Adaptive Map (SOAM)

- **The algorithm**

9. Unless unit b is in a *stable* state, adapt its position and those of all connected units

$$\begin{aligned} \Delta \mathbf{p}_b &= \eta_b \cdot f_b \cdot (\xi - \mathbf{p}_b) \\ \Delta \mathbf{p}_{nb} &= \eta_{nb} \cdot f_{nb} \cdot (\xi - \mathbf{p}_{nb}) \end{aligned}$$



otherwise, if unit b is *stable*, adapt only the position of b itself

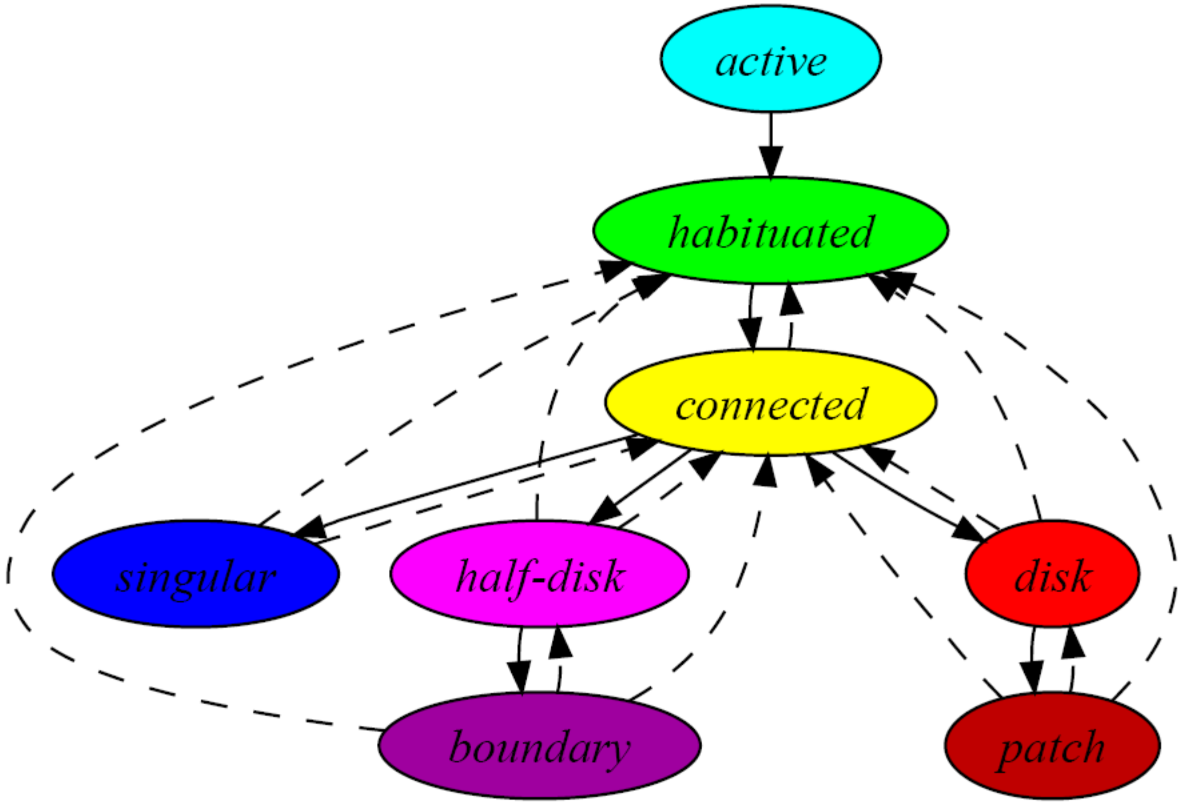
$$\Delta \mathbf{p}_b = \eta_{stable} \cdot f_b \cdot (\xi - \mathbf{p}_b)$$

10. Unless some termination criterion has been met, return to step 1.

Self-Organizing Adaptive Map (SOAM)

- **Unit states and state transitions**

  The full set of states and state transitions



Self-Organizing Adaptive Map (SOAM)

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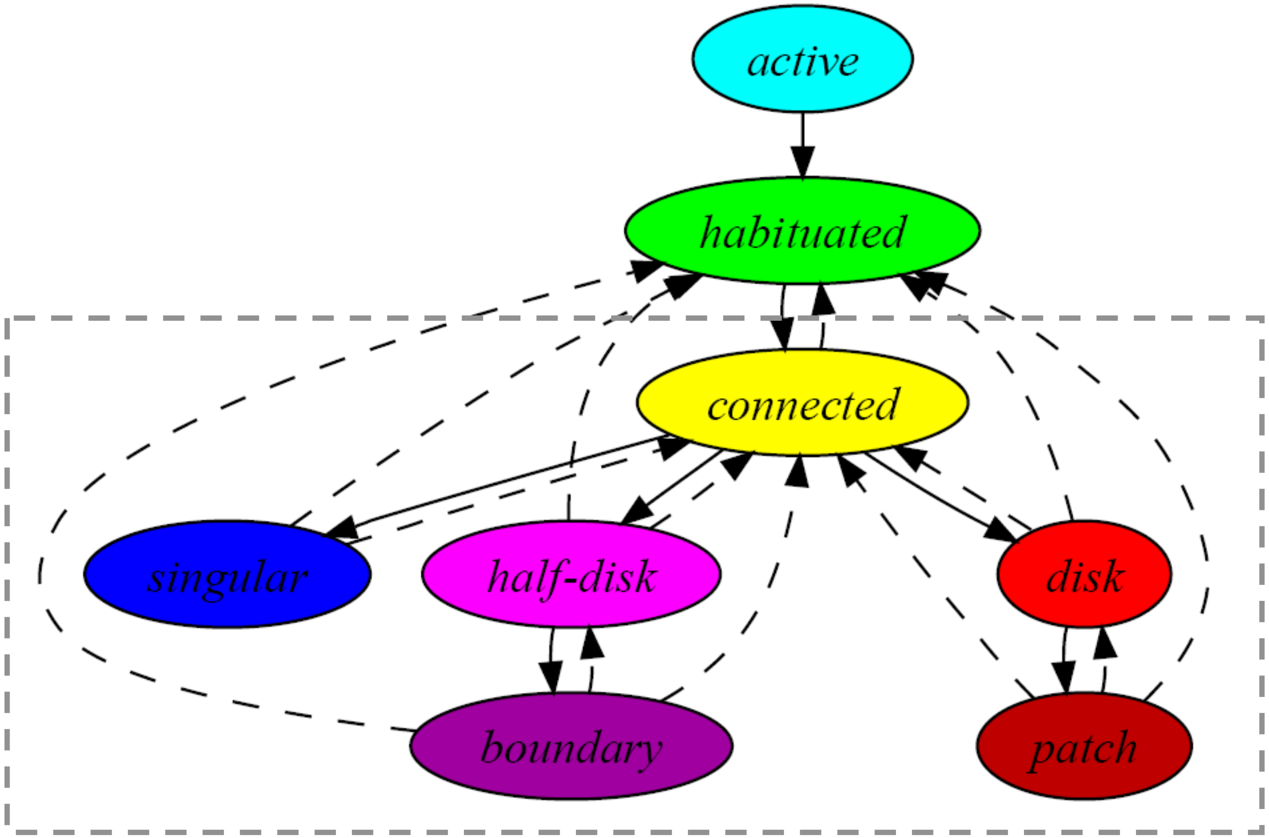
  The full set of states and state transitions

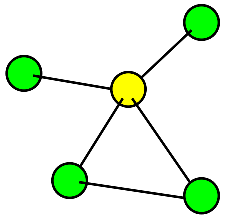


These states are determined by the neighborhood topology

Self-Organizing Adaptive Map (SOAM)

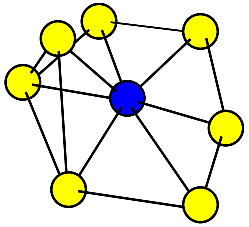
▪ Unit states and neighborhood topology

For *surface* reconstruction



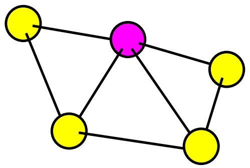
connected

the neighboring units are habituated



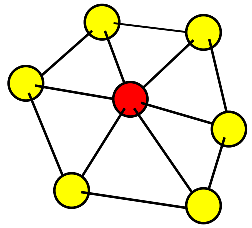
singular

the configuration of connected neighboring units exceeds a disk



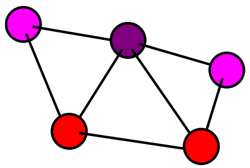
half-disk

formed by connected neighboring units



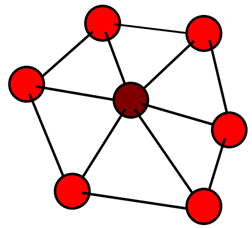
disk

formed by connected neighboring units



boundary

an half-disk formed by regular neighboring units



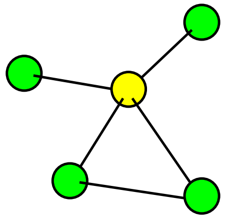
patch

a disk formed by regular neighboring units

Self-Organizing Adaptive Map (SOAM)

▪ Unit states and neighborhood topology

For *surface* reconstruction



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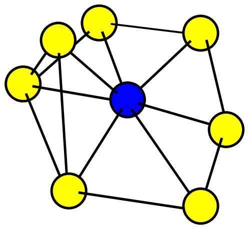

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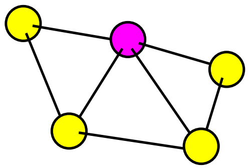
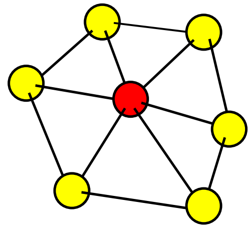


half-disk

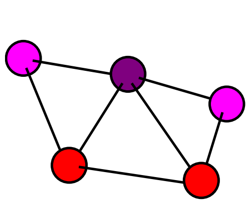
formed by connected neighboring units



disk

formed by connected neighboring units

These states are deemed stable



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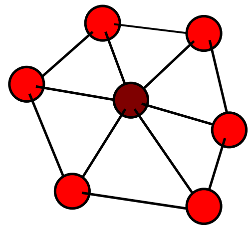
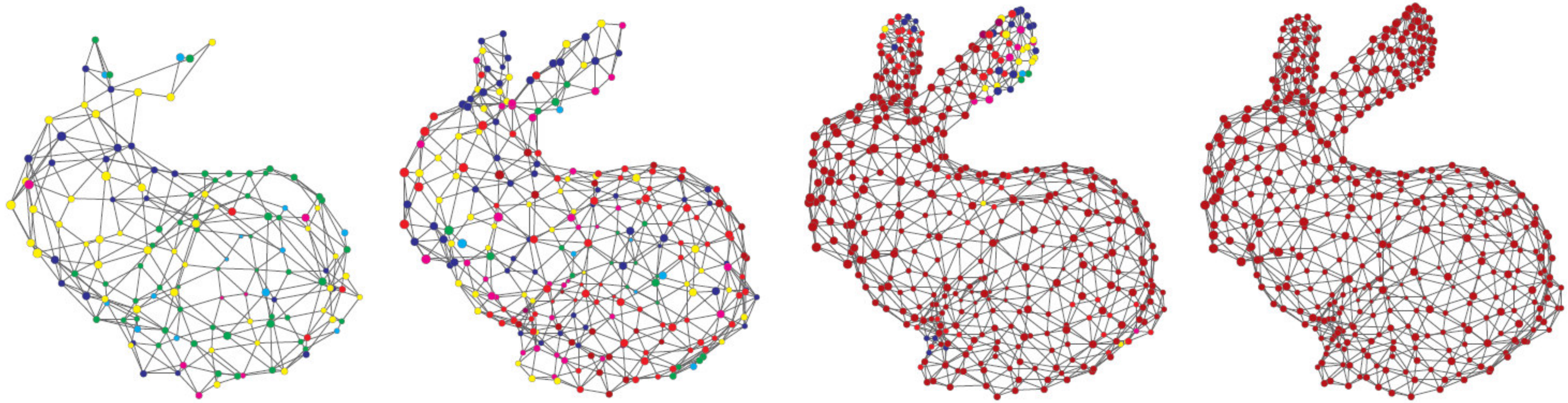


patch

a disk formed by regular neighboring units

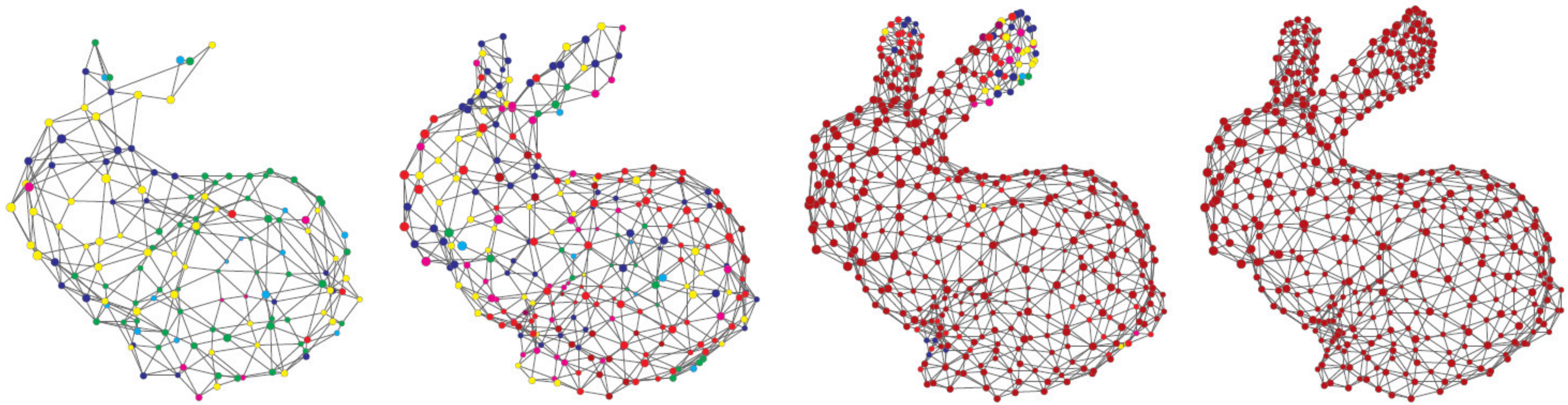
Self-Organizing Adaptive Map (SOAM)

- **SOAM adaptation process**

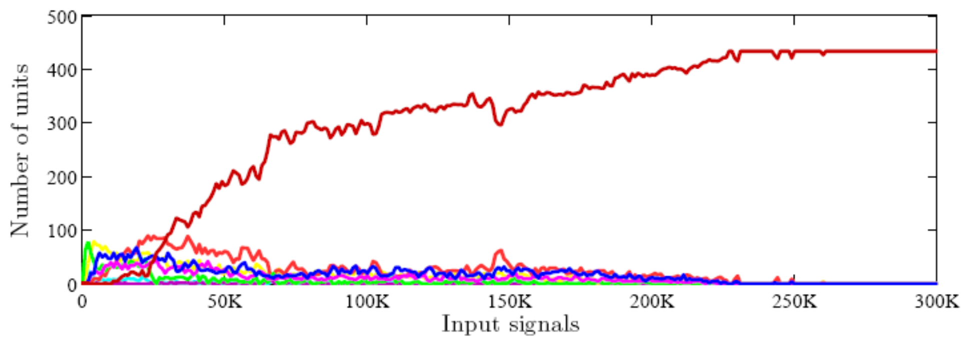


Self-Organizing Adaptive Map (SOAM)

- **SOAM adaptation process**



How the number of units varies with time (i.e. input signals)



Each line describes the number of units in the corresponding state/color

Self-Organizing Adaptive Map (SOAM)

- **SOAM adaptation process**

  Another example, a closed surface with genus 22

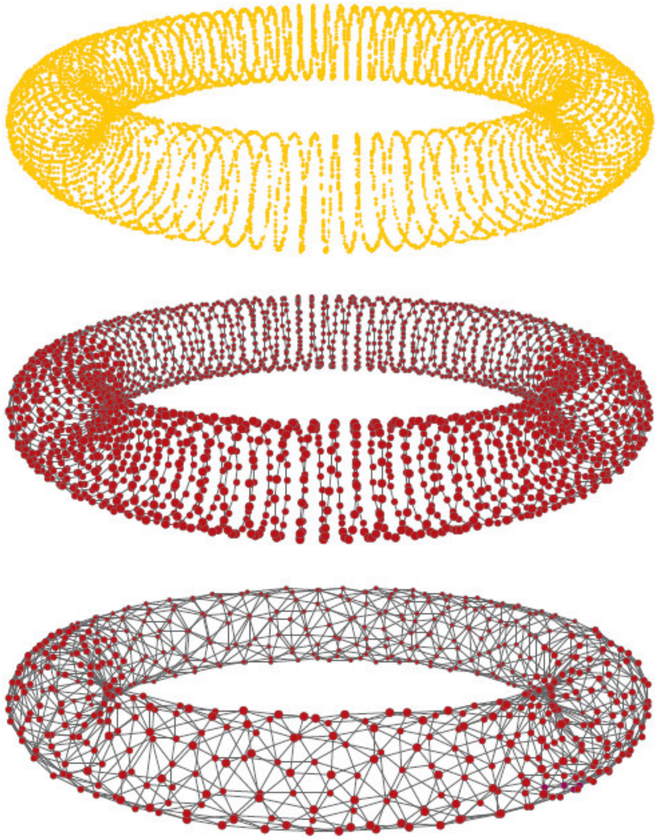
The same network interpreted as a mesh

Self-Organizing Adaptive Map (SOAM)

- **SOAM adaptation process**

  Either a curve or a surface from the same input

  The dimension of the manifold to be reconstructed (i.e. either 1 or 2) is the *main parameter* of the algorithm



This is the input dataset

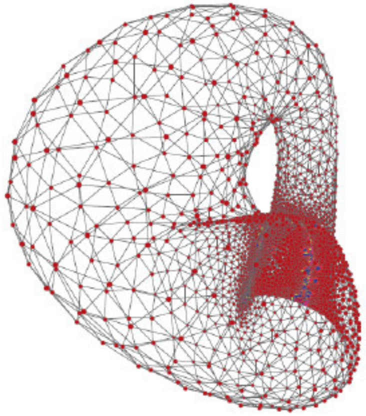
The SOAM reconstructs a curve in 3D

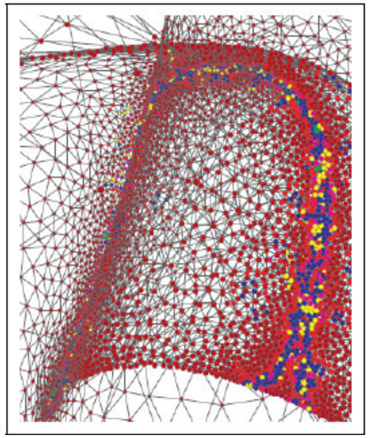
The SOAM reconstructs a surface in 3D

Self-Organizing Adaptive Map (SOAM)

- **SOAM adaptation process**

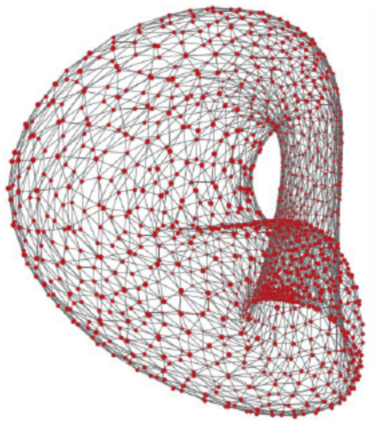
  Higher dimensions (i.e. beyond 3D)

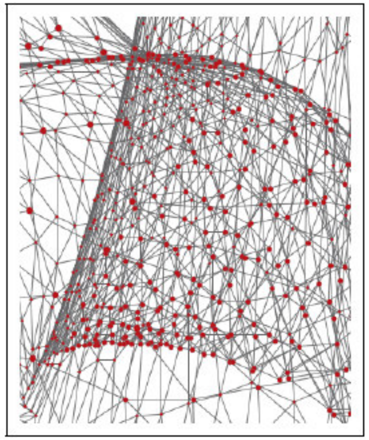




In 3D the Klein bottle
is not a manifold, as it must self-intersect:
the SOAM cannot converge

In 4D (and beyond)
the Klein bottle is a manifold
and the SOAM converges





Self-Organizing Adaptive Map (SOAM)

- **Pre-print**

  See http://arxiv.org/abs/0812.2969