2016 23rd International Conference on Pattern Recognition (ICPR)
Cancún Center, Cancún, México, December 4-8, 2016

A New Approach to Mathematical Morphology on One Dimensional Sampled Signals

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***Abstract*—We present a new approach to approximate continuous-domain mathematical morphology operators. The approach is applicable to irregularly sampled signals. We define a dilation under this new approach, where samples are duplicated and shifted according to the flat, continuous structuring element. We define the erosion by adjunction, and the opening and closing by composition. These new operators will significantly increase precision in image measurements. Experiments show that these operators indeed approximate continuous-domain operators better than the standard operators on sampled one-dimensional signals, and that they may be applied to signals using structuring elements smaller than the distance between samples. We also show that we can apply the operators to scan lines of a two-dimensional image to filter horizontal and vertical linear structures.**

I. Introduction

Mathematical morphology has long been successfully applied to digital images for processing and analysis [11]. One-dimensional operators have applications in, for example, line-detection [13], and path openings [4, 14, 9]. They may also be used to create more complex operators in higher dimensions by combining operators along various directions in a higher dimensional signal [11].

Traditionally, the grayscale morphological operators are defined as operators on functions from $\mathbb{E}$ to $\mathbb{D}$, where $\mathbb{E}$ is the positions of data points (e.g. pixels), and $\mathbb{D}$ is the range of values these points may take. In this paper we will only consider one-dimensional, sampled signals.

The basic morphological operators may be defined, taking a signal $F : \mathbb{Z} \to \mathbb{D}$ and a structuring element (SE) $G : \mathbb{Z} \to \mathbb{D}$ as arguments [11]. If we restrict ourselves to *flat*, contiguous (without holes) structuring elements (i.e. functions G that only take the values $-\infty$ and 0, and for which there are no three points x_0, x_1, and x_2, such that $x_0 < x_1 < x_2$ and $G(x_0) = 0$, $G(x_1) = -\infty$, and $G(x_2) = 0$), the operators may be defined as:

- Dilation:
$$(F \oplus G)(x) = \bigvee_{h \in [a,b]} F(x-h), \tag{1}$$
- Erosion:
$$(F \ominus G)(x) = \bigwedge_{h \in [a,b]} F(x+h), \tag{2}$$
- Opening:
$$(F \circ G)(x) = \left((F \ominus G) \oplus G\right)(x), \tag{3}$$
- Closing:
$$(F \bullet G)(x) = \left((F \oplus G) \ominus G\right)(x), \tag{4}$$

where x is any point in $\mathbb{Z}$, and $[a, b]$ represents a range of integers from a to b, where $a < b$, and $a, b \in \mathbb{Z}$. This range is the extent of the flat structuring element. One may think of these operators as constructing a new signal by sliding a line-segment along the original signal in different ways, thereby tracing out the new signal.

These definitions assume that the signal, F, is defined for all points in $\mathbb{Z}$, i.e. a regular sampling is required. The samples represent a continuous, band-limited signal, i.e. this signal can be reconstructed in the continuous domain from the samples. However, the samples will not generally fall on the maxima and minima of the continuous signal, even if the sampling frequency is high enough to accurately represent the signal [10, 12], which is a problem, since the morphological operators depend on local extrema. Thus the discrete operators will fail to accurately reflect their continuous counterparts. Another issue is the fact that the continuous operators may yield signals with discontinuous first and higher order derivatives, thereby introducing infinite frequencies, meaning the signals cannot be accurately represented using traditional sampling. Finally, the structuring element, too, is dependent on the sampling, since the interval $[a, b]$ must be a subset of $\mathbb{Z}$. This means that one cannot, for example, dilate a signal with a structuring element of non-integer length.

Some attempts to deal with these problems have been made. In a paper from 1992, Brockett and Maragos [2] develop partial differential equations that can be evolved to compute continuous morphology in the discrete domain. Many authors have improved on this work [16, 3, 8, 1], however, these methods still yield worse approximations to the continuous-domain operators than the discrete operators. The main reason for this is that the results of the morphological operators are not band-limited, and therefore cannot be represented on a regular grid. Moreover, these methods are very slow [17].

In his thesis [15], Thurley develops a morphology on irregularly sampled data using continuous structuring elements. These operators only shift sample points vertically, thus the morphologically transformed signal is sampled at the same points as the original signal. This means that these operators have some problems similar to the ones of the traditional morphological operators mentioned above.

Luengo Hendriks and van Vliet [5] define one-dimensional morphological operators that apply continuous morphology to sampled signals by interpolation. Two later papers [6, 7] proposed operations that are not strictly morphological operators, but their results approximate continuous-domain morphology better than the traditional discrete operators

In this paper, a new approach to morphology is suggested. The following sections will give a description of operators that behave similarly to the traditional ones, however, the new operators loosen the restrictions of the traditional approach by allowing for irregular samplings, $V = \{(x_i, y_i)\}_{i \in [1,N]}$, i.e. sets of $N \in \mathbb{Z}^+$ pairs from $\mathbb{E} \times \mathbb{D}$, such that

$$\text{for any two pairs } (x_j, y_j), (x_k, y_k) \in V, x_j = x_k \Rightarrow j = k \tag{5}$$

where x_i denotes position and y_i denotes value. This will also yield operators that may use structuring elements of lengths shorter than the minimal distance between samples. The operators presented in this paper never leave the discrete domain. This leads to fast, simple implementations.

II. Duplicate-and-Shift Operators

In this section we will define four new operators that are the counterparts of the traditional dilation, erosion, opening, and closing by flat structuring elements defined in the previous section. We will call these operators *duplicate-and-shift* (DAS) operators, for reasons that will be explained shortly. We will first define the DAS-dilation (here denoted by $\oplus_{DAS}$), which will then be used to construct the other three operators.

A. DAS-Dilation

The dilation of a continuous signal by a flat structuring element may be thought of as the result of sliding the origin of the structuring element along the signal and tracing out the area shaded by the line segment of the SE. The dilation is the result of taking the maximum of the shaded area. This hints at a way of defining the dilation for sampled signals. For each sample do the following:

- Align the origin of the structuring element with the sample;
- Make two copies (i.e. duplicates) of the sample;
- Shift one of the copies to the left extreme of the SE, and the other to the right extreme.

Hence the name of the operator. A line segment can be represented by two points that fall on the endpoints of that segment, hence the shifted samples define a line segment with its origin at the original sample. These operations are analogous to the process of sliding the SE along the signal and tracing the line segment described above. There is, however, the issue that this does not create a shaded area. In the continuous case, the final signal is found by taking the maximum of the traced out area. In the case of discrete, irregular samples, however, there is no obvious way of defining a corresponding operation, if only the samples are considered.

To solve this problem, one may use the information about the structuring element along with the knowledge of which sample spawned the copies and which traced signal a sample is associated with. The idea is to note that the two nodes spawned by a sample define a line segment. Instead of shifting

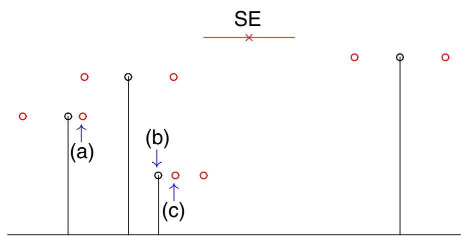

Fig. 1: DAS-dilation: The black nodes are the samples of the original signal, the red nodes are the duplicated nodes. The duplicated nodes are shifted according to the structuring element (SE).

the duplicates to the extremes of the SE, they are only shifted as far as possible without them falling in the shadow of any line segment defined by duplicates above. In fact, the right node may be shifted to the left of the original sample and the left node may be shifted to the right of it, if this is necessary to avoid falling in the shadow of a higher line segment. Figure 1 illustrates this process on an irregularly sampled signal. The duplicated node (a) is not allowed to be moved all the way to the right extreme, since it would then end up under a line segment. Similarly, (c) is moved right until it emerges from the shadow of the line segment defined by the two nodes spawned by the second original sample. The DAS-dilated signal is given by the black and red nodes, except node (b), which is deleted (or equivalently, shifted to the same position as (c)).

B. DAS-Operators

One may use the DAS-dilation defined above to generate counterparts to erosion, opening, and closing:

- DAS-erosion:
$$(F \ominus_{DAS} G) = (F^{\complement} \oplus_{DAS} \hat{G})^{\complement}, \tag{6}$$
- DAS-opening:
$$(F \circ_{DAS} G) = ((F \ominus_{DAS} G) \oplus_{DAS} G), \tag{7}$$
- DAS-closing:
$$(F \bullet_{DAS} G) = ((F \oplus_{DAS} G) \ominus_{DAS} G), \tag{8}$$

where $\hat{G} = \{-x \mid x \in G\}$ is the reflection of G. Thus, the erosion is defined by duality, and the opening and closing may then be defined by composition [11].

III. Implementation

We implemented the four operators in C++ with an interface to MATLAB. Some pseudocode for the DAS-dilation is shown in Alg. 1. The function takes three inputs:

- NODES: a list containing the samples. Each element of NODES has a position and a value.
- SE^-: the left endpoint of the structuring element.
- SE^+: the right endpoint of the structuring element.

The function returns a list DNODES, which contains the samples of the DAS-dilation. Each element of DNODES has a position and a value. The implementation can be thought of as dropping line segments onto each sample point (from highest to lowest), and cutting off the parts of the segment that hit previous segments. The extra amount that is cut off is defined by $\varepsilon > 0$, which is used since we do not want two nodes to have the exact same position. Choosing ε small results in a "staircase" effect (see (a) and (c) in Fig. 1).

The returned list, DNODES, is the nodes that lie on the endpoints of these segments when they have come to rest upon their associated sample, along with the original sample, unless it has been suppressed. An original sample is suppressed unless it falls between the duplicated nodes after shifting.

Using the dilation and the equations (6) – (8) we may then implement the remaining operators.

IV. Morphological Properties

Consider a signal V under the following assumptions:

1) $\varepsilon = q \cdot d_{min}$, where $q \in \mathbb{Q}^+$, $q \leq 1$ and d_{min} is the smallest distance between samples,
2) $\forall (x_i, y_i) \in V$, $\exists n_i \in \mathbb{Z}$ such that $x_i = n_i\varepsilon$, and
3) the length of the SE is $m\varepsilon$, where $m \in \mathbb{Z}^+$.

The proposed operators have clear ties to the traditional morphology [11]. First of all, it is clear that the structuring elements may be decomposed [18]. There are some issues, since the position of nodes created on plateaus, as well as their number, may differ depending on whether the SE is applied directly or decomposed and applied in sequence. However, ignoring all nodes on a plateau except the nodes at the endpoints, the results are the same. This stems from the fact that the DAS-dilation moves nodes to the extremes of the SE. Shifting nodes twice with distance d_1 first, then d_2 (or v.v.) is the same as shifting nodes once with distance $d_1 + d_2$.

Idempotence of opening and closing holds if we mask the filtered samples by removing nodes with a position that falls outside the range of the original samples (since dilation extends the filtered samples beyond the limits of the original signal). We still have the issue of creating additional nodes on plateaus, however we can deal with this, again, by only considering the endpoints of plateaus. If there exists a gap between two neighboring samples that is larger than the SE, the DAS-opening as well as closing will keep creating nodes in this gap, for each iterative application, until it is filled. This breaks idempotency in the sense of the sets of nodes being equal after removing redundant nodes on plateaus.

An interesting special case is $\varepsilon = 1$ and a regular sampling rate where the distance between samples is 1. In this case, choosing a SE of integer length satisfies the assumptions 1, 2, and 3. The DAS-operators will, under these conditions, give results equal to the traditional operators (again considering only endpoints of plateaus) except near the border (see Fig 2).

When breaking the initial assumptions problems arise. If our initial assumptions do not hold, the shifted samples may change the endpoints of plateaus by a small distance, which means the signals will not be equal in the sense discussed above.

Other issues include the properties of increasingness and (anti-)extensivity, which are not straightforward to verify, since there is no immediately obvious way to define a partial order, $\preceq$, on the set of irregularly sampled signals. One possibility might be to define a *top*, T, of the samples by interpolation, i.e. for a sampled signal V

$$T(V)(x) = \begin{cases} f_V(x), & \text{if } x \in [x_{inf}, x_{sup}] \\ -\infty, & \text{otherwise} \end{cases} \tag{9}$$

where

$$x_{inf} = \bigwedge_{(x,y)\in V} x,$$

$$x_{sup} = \bigvee_{(x,y)\in V} x, \quad \text{and}$$

$f_V(x)$ is an interpolating function on the samples of V.

Then we may compare a pair of sampled signals, V and W, by comparing their tops, i.e.

$$V \preceq W \iff T(V) \leq T(W). \tag{10}$$

It is not obvious that the DAS-operators will satisfy the properties of the traditional morphological operators using this definition of a partial order. In fact, for certain choices of interpolating function, it is clear that the proposed operators do *not* satisfy the morphological properties.

One interpolating function that may be interesting to consider gives the top

$$T(V)(x) = \begin{cases} \min(y_i, y_{i+1}), & \text{if } x_i < x < x_{i+1} \\ y_i, & \text{if } x = x_i \\ -\infty, & \text{if } x < x_{inf} \text{ or } x > x_{sup} \end{cases} \tag{11}$$

where, without loss of generality, we assume that the samples, V, are in increasing order with respect to position. This top describes a set of line-segments and isolated points. The plateaus discussed previously will be extended until they fall under a higher plateau. Thus, the slight variability of endpoints defined by the shifted nodes becomes a non-issue.

We shall see that, experimentally, the results are sound also when not adhering to the assumptions (see V-B and V-C).

V. Results

A. Application to Arbitrary Signal

We illustrate the results of the DAS-operators on regularly sampled signals and compare them to the traditional operators. For SEs that have a length that is a multiple of the distance between samples, the results are very similar (as discussed in the previous section) using $\varepsilon = d$ where d is the distance between consecutive samples. The DAS-operators may skip samples at plateaus (if this is undesirable, one may simply interpolate these samples; since it is a plateau the correct value is easily found). The DAS-operators also extend the signal outside the original samples. Figure 2 shows the traditional dilation, erosion, opening, and closing along with the corresponding DAS-operators on a regularly sampled signal using a structuring element of length 41 with the origin in the middle. Note that the result after applying the DAS-operators is not regularly sampled. Also note that the DAS-opening is computed by first applying the DAS-erosion on the regularly sampled signal, yielding an *irregularly* sampled signal upon

function DAS–DILATE(NODES, SE^-, SE^+)
let DNODES be an empty array
sort NODES according to y-value in descending order
for each node $i \in$ NODES **do**
let i^- and i^+ be duplicates of i
let pos(i^-) = pos(i) + SE^-
let pos(i^+) = pos(i) + SE^+
let NODES^- be the list of nodes that precede i
for each node $j \in \text{NODES}^-$ **do**
let j^+ be pos(j) + SE^+
let j^- be pos(j) − SE^-
if pos$(i^-) \leq j^+$ and pos$(i^-) \geq j^-$ **then**
pos$(i^-) = j^+ + \varepsilon$
end if
if pos$(i^+) \geq j^-$ and pos$(i^+) \leq j^+$ **then**
pos$(i^+) = j^- - \varepsilon$
end if
end for
if pos$(i^-) \leq$ pos(i^+) **then**
if $\nexists n \in$ DNODES : pos(i^-) = pos(n) **then**
insert i^- into DNODES
end if
if $\nexists n \in$ DNODES : pos(i^+) = pos(n) **then**
insert i^+ into DNODES
end if
if pos(i^-) < pos(i) < pos(i^+) **then**
insert i into DNODES
end if
else
drop nodes i^-, i, and i^+
end if
end for
return DNODES
end function

Alg. 1: Duplicate-and-shift dilation pseudocode. Note that the number of nodes in NODES^- can be reduced by only considering nodes within a neighborhood the size of the SE. A red-black tree enables quickly querying only the nearest left and right neighbor, yielding a bound on time complexity of $\mathcal{O}(N \log N)$.

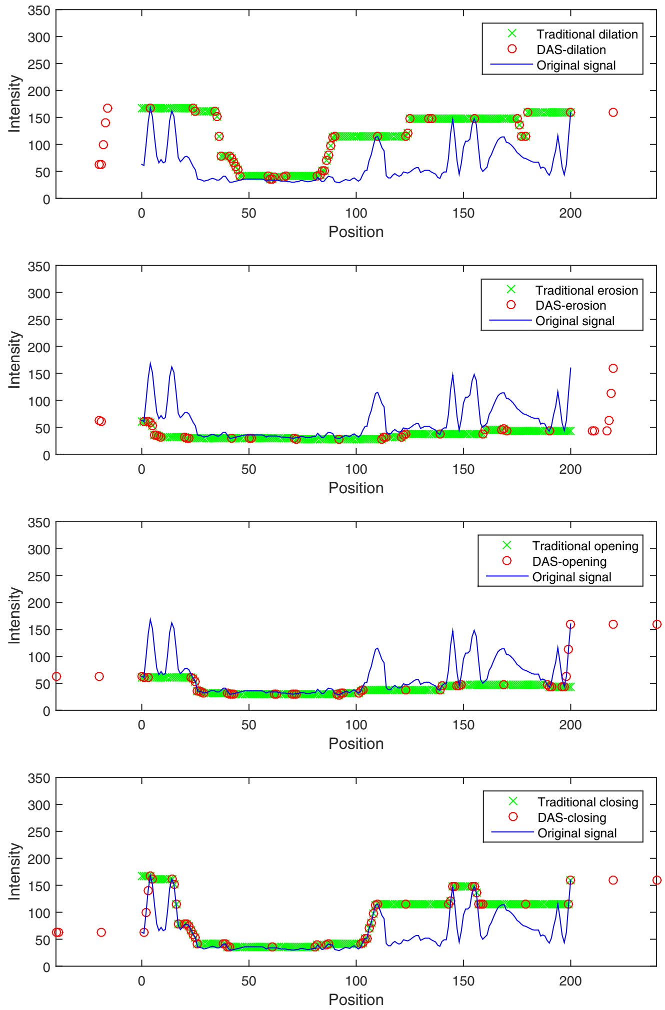

Fig. 2: The four DAS-operators (red) and their corresponding traditional operators (green) applied to the original signal (blue). From top to bottom: dilation, erosion, opening, and closing.

which the DAS-dilation is applied (and v.v. for the DAS-closing). The DAS-opening and closing are very similar to the traditional opening and closing, indicating that the method of duplicating and shifting is sound also on irregularly sampled signals.

The original signal is regularly sampled with 201 sample points. The traditional operators will, of course, produce a signal with 201 samples as well, however this is not the case for the DAS-operators, since they can produce irregularly sampled signals. Thus, the DAS-dilation contains 47 samples, the DAS-erosion contains 38 samples, the DAS-opening contains 45 samples, and the DAS-closing contains 53 samples. In general, the number of samples in the DAS-filtered signals will decrease with increasing size of the SE. This is reasonable, since a larger SE will create larger plateaus, which do not need many samples to be accurately represented. On the other hand, if the SE is small enough, the transformed signal will contain *more* samples than the original signal.

In the remaining experiments the length of the SE is chosen independently of d and ε. We also choose $\varepsilon = 0.99 * d$, since values less than d, but close to it, give good results (using $\varepsilon = d$ gives slightly worse results, and choosing ε very small creates a "staircase" effect).

B. Quantifying Errors

To compare the results when using non-integer length structuring elements, we look at the results of sampling a sine-wave (approximately 7 samples per 2π) and dilating it. The results are compared against the analytical solution when dilating the continuous sine-wave. The samples are quantized into 256 levels. Figure 3 shows the results. The samples in the traditionally dilated signal must have the same position on the horizontal axis as the original sampled signal and can only use the information of samples that fall within the structuring element. These restrictions make it unsuitable to the task. If the SE is too small, the traditional dilation leaves the signal unchanged (see the lower image of Fig. 3). The DAS-dilation on the other hand does not require the samples of the dilation to fall on the same positions as the original signal. Moreover, it also uses information about the structuring element itself, meaning the parts of the SE that fall between samples still

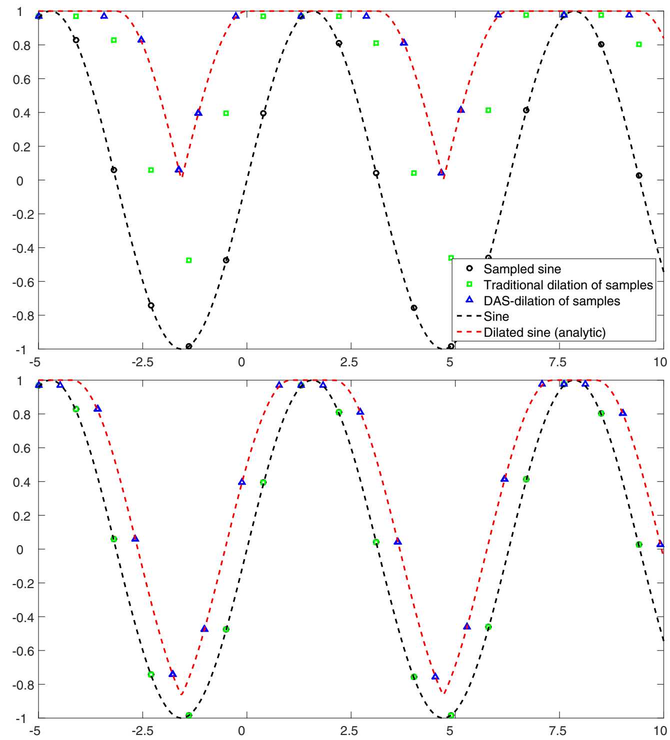

Fig. 3: Two examples of dilations of a sine-wave (black). The samples of the traditional dilation (green) do not manage to capture the continuous dilation (red) very well. The samples of the DAS-dilation (blue) fall very close to the continuous dilation. The upper figure uses a SE with a length corresponding to $\pi/2$. The lower figure uses a length of $\pi/6$.

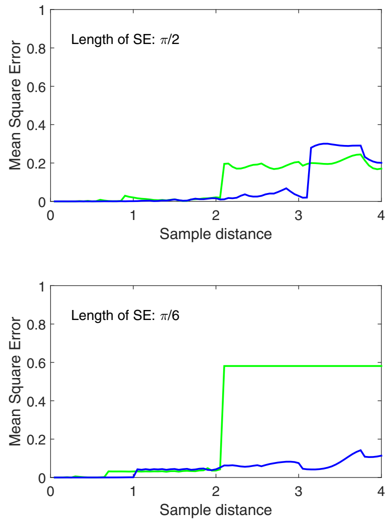

Fig. 4: Green is the mean square error (MSE) of the traditional dilation, and blue is the MSE of the DAS-dilation. Using linear interpolation between samples in the dilations we find the mean square error for structuring elements of different lengths with increasing distance between samples when dilating the sine-wave shown in Fig. 3.

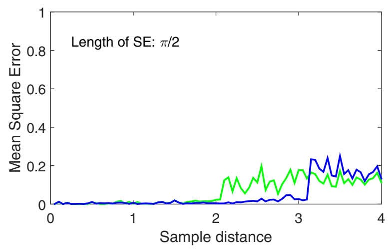

Fig. 5: The mean square error when dilating the sine-wave shown in Fig. 3 with added uniform noise between 0 and 0.5 before quantization. Green is the MSE of the traditional dilation, and blue is the MSE of the DAS-dilation.

contributes to the result. The effect of this is that SEs are decoupled from the sampling frequency of the original sampled signal.

The errors of the dilations are quantified on the sampled sine-wave in Fig 3. Using SEs of varying lengths, we measure the mean square error between the analytic dilation and the traditional dilation as well as the DAS-dilation. The discrete dilations are interpolated in order to be comparable to the continuous dilation. The continuous dilation contains cusps, therefore a linear interpolation is used, since it can construct points where the derivative is discontinuous. The results of these experiments are summarized in Figure 4.

Finally, we ran a similar experiment while adding noise to the sine-wave. The performance of the proposed DAS-dilation seems to be robust with regard to noise. Figure 5 shows the result when adding uniform noise.

C. DAS-Closing vs. Traditional Closing Using Linear SE

We applied the DAS-closing and the traditional closing to a 654×840 pixels, 8-bit 2D-image of a piece of graph paper with a curved line. A linear structuring element of length 201 with the origin in the middle was used. Since this is a one-dimensional structuring element, we may apply the 1D operators defined in this paper by extracting the line of pixels that will be affected and applying the DAS-closing line by line. The squares of the graph paper are roughly aligned such that the sides are horizontal/vertical. Therefore, we combine two closings: one where the SE is horizontally aligned, and one where it is aligned vertically. Since the DAS-closing yields irregularly sampled signals, we have to resample the closings onto the square grid. Linear interpolation is used for this purpose (line by line).

The original image, as well as the resulting images of the traditional closing and the DAS-closing, is shown in Fig. 6. Because of different boundary conditions the DAS-closing preserves some parts of the curve near the edges. A bright border around the image will give behavior similar to the traditional closing near the edges.

VI. Conclusion

In this paper a new approach to mathematical morphology on one-dimensional, sampled signals is presented. Four operators are defined, which correspond to the traditional morphological operators of dilation, erosion, opening, and closing. We show that the proposed operators can yield more accurate results on sampled, one-dimensional signals. The proposed operators are not restricted to regular samplings and structuring elements are not required to be of lengths that are

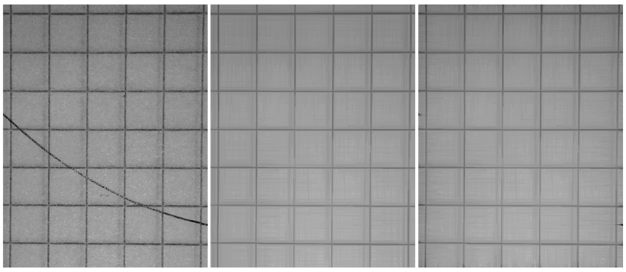

Fig. 6: Left to right: Original image, traditional closing with vertical and horizontal line SE, DAS-closing along vertical and horizontal lines (resampled onto square grid). The DAS-dilation yields similar results, except on the edges, where parts of the curve are preserved. This difference stems from the different boundary conditions used (see Fig. 2). For display purposes, the contrast of these images has been increased in a post-processing step.

multiples of the distance between samples.

The operators use flat structuring elements without holes. Allowing for holes (i.e. using SEs consisting of multiple line segments) should be possible by creating more copies in the duplication step and carefully shifting these in a similar manner as the two copies in Fig. 1. It should also be possible to allow for non-flat structuring elements by shifting duplicated nodes along both axes. We would like to develop the operators in this direction as well.

In the future, a more thorough examination of the proposed operators and their relation to traditional morphology is planned. Section IV briefly discusses these issues, however a more rigorous, theoretical examination is desirable. Especially interesting is what happens when we move away from the assumptions we initially make in Section IV (which, for example, traditional morphology satisfy). The experiments in sections V-B and V-C show promising results for such cases. Defining a suitable complete lattice structure on the set of irregularly sampled signals is also desirable.

We are also interested in generalizing these operators to higher-dimensional signals. There is an abundance of irregularly sampled 3D data and being able to apply morphology directly to such data is of interest.

Finally, we would like to examine the results of adaptively sampling signals, such that plateaus are sparsely sampled while parts of the signal that vary quickly are sampled with a higher frequency. This should lead to discrete operators that better approximate their continuous counterparts while not sacrificing speed by generating a huge number of samples.

Acknowledgment

Teo Asplund was funded through grant 2014-5983 from the Swedish Research Council. The authors would like to thank Gunilla Borgefors for providing comments and suggestions during the writing of this paper.

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