TEST ON LINEAR ALGEBRA 09/15/2016 - SOLUTIONS

Problem 1

(1)

$$C = \begin{bmatrix} 1 & -2 & -5 \\ -3 & -6 & -5 \\ -2 & -2 & 1 \end{bmatrix}$$

(2)

$$A^T = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 7 & 2 \\ 5 & 2 & 0 \end{bmatrix} \qquad AB = \begin{bmatrix} 7 & 0 & 2 \\ 8 & 4 & -19 \\ 6 & -1 & -6 \end{bmatrix}$$

(3) No:

$$BA = \begin{bmatrix} -1 & \cdots \\ \cdots & \\ \cdots & \end{bmatrix} \neq AB$$

(4)

$$-3B = \begin{bmatrix} -6 & 3 & 0 \\ 0 & -3 & 9 \\ -3 & 0 & -3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] + 2^{nd} + 3 \text{ times } 3^{rd}$$

$$\left[\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & 1 & 3 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \times \frac{1}{5} \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 1/5 & 3/5 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] - 1^{ST}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 1/5 & 3/5 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/5 & -1/5 & 2/5 \end{array}\right] + 3 \text{ times } 3^{rd}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 1/5 & 3/5 \\ 0 & 1 & 0 & -3/5 & 2/5 & 6/5 \\ 0 & 0 & 1 & -1/5 & -1/5 & 2/5 \end{array}\right] \Rightarrow B^{-1} = \begin{bmatrix} 1/5 & 1/5 & 3/5 \\ -3/5 & 2/5 & 6/5 \\ -1/5 & -1/5 & 2/5 \end{bmatrix}$$

$$\left(\text{Check } BB^{-1} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

Problem 2

* $\underline{v} = -2\underline{u}$ ⇒ NOT LIN. IND. (NO)
  NOT ORTH. (NO)

* $\underline{a}$ and $\underline{b}$ LIN. IND. (YES)
  $\underline{a} \cdot \underline{b} = 1$ ⇒ NOT ORTH. (NO)

* $\underline{x} \cdot \underline{y} = 0$ ⇒ LIN. IND. (YES)
  ORTHOG. (YES)

Problem 3

① $|\underline{u}| = 1$

$|\underline{v}| = \sqrt{4+4} = 2\sqrt{2}$

$|\underline{w}| = 1$

② $\underline{u}$ and $\underline{w}$ are NORMAL vectors

③ $\underline{u}$, $\underline{v}$ and $\underline{w}$ are linearly independent:

$$a_1 \underline{u} + a_2 \underline{v} + a_3 \underline{w} = \underline{0}$$

$$\Rightarrow \begin{cases} a_1 + \frac{1}{\sqrt{3}} a_3 = 0 \\ 2a_2 - \frac{1}{\sqrt{3}} a_3 = 0 \\ 2a_2 + \frac{1}{\sqrt{3}} a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases}$$

④ $\underline{a} = a_1 \underline{u} + a_2 \underline{v} + a_3 \underline{w}$

$$\Rightarrow \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2a_2 \\ 2a_2 \end{bmatrix} + \begin{bmatrix} a_3/\sqrt{3} \\ -a_3/\sqrt{3} \\ a_3/\sqrt{3} \end{bmatrix}$$

$$\Rightarrow \begin{cases} -\sqrt{3} = \sqrt{3} a_1 + a_3 \\ 4\sqrt{3} = 2\sqrt{3} a_2 - a_3 \\ 2\sqrt{3} = 2\sqrt{3} a_2 + a_3 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = \frac{3}{2} \\ a_3 = -\sqrt{3} \end{cases}$$

$$\Rightarrow \underline{a} = \frac{3}{2} \underline{v} - \sqrt{3}\, \underline{w}$$

Problem 4

$\mathcal{U}$: * $\underline{u} \in \mathcal{U} \overset{?}{\Rightarrow} a\underline{u} \in \mathcal{U} \ \forall a \in \mathbb{R}$

$\underline{u} \in \mathcal{U} \Rightarrow u_1 + u_2 + u_3 = 0 \Rightarrow au_1 + au_2 + au_3 = a(u_1+u_2+u_3) = 0$

$\Rightarrow a\underline{u} \in \mathcal{U}$ ok!

* $\underline{u} \in \mathcal{U}$ and $\underline{w} \in \mathcal{U} \overset{?}{\Rightarrow} \underline{u} + \underline{w} \in \mathcal{U}$

$\underline{u}, \underline{w} \in \mathcal{U} \Rightarrow \begin{cases} u_1 + u_2 + u_3 = 0 \\ w_1 + w_2 + w_3 = 0 \end{cases}$

$\Rightarrow (u_1 + w_1) + (u_2 + w_2) + (u_3 + w_3) =$

$= u_1 + u_2 + u_3 + w_1 + w_2 + w_3 = 0$

$\Rightarrow \underline{u} + \underline{w} \in \mathcal{U}$ ok!

YES: $\mathcal{U}$ is a linear subspace of $\mathbb{R}^3$

BASIS: $u_1 + u_2 + u_3 = 0 \rightarrow u_3 = -u_1 - u_2$

$\rightsquigarrow$ 2 degrees of freedom: u_1 and u_2

$$\Rightarrow \mathcal{U} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

$\mathcal{V}$: * $\underline{v} \in \mathcal{V} \overset{?}{\Rightarrow} a\underline{v} \in \mathcal{V} \ \forall a \in \mathbb{R}$

$\underline{v} \in \mathcal{V} \Rightarrow v_1 v_2 v_3 = 0 \Rightarrow av_1 \, av_2 \, av_3 = a^3 v_1 v_2 v_3 = 0$

$\Rightarrow a\underline{v} \in \mathcal{V}$ ok!

* $\underline{v} \in \mathcal{V}$ and $\underline{w} \in \mathcal{V} \overset{?}{\Rightarrow} \underline{v} + \underline{w} \in \mathcal{V}$

$\underline{v}, \underline{w} \in \mathcal{V} \Rightarrow \begin{cases} v_1 v_2 v_3 = 0 \\ w_1 w_2 w_3 = 0 \end{cases}$

$(v_1 + w_1)(v_2 + w_2)(v_3 + w_3) =$

$= (v_1 + w_1)(v_2 v_3 + v_2 w_3 + v_3 w_2 + w_2 w_3) =$

Nothing tells me this is 0! ←

$= \underbrace{v_1 v_2 v_3}_{0} + v_1 v_2 w_3 + v_1 v_3 w_2 + v_1 w_2 w_3 +$

$+ w_1 v_2 v_3 + w_1 v_2 w_3 + w_1 v_3 w_2 + \underbrace{w_1 w_2 w_3}_{0}$

Can I find a counterexample such that

$\underline{v}, \underline{w} \in V$ but $\underline{v}+\underline{w} \notin V$?

YES: $\underline{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\underline{v}+\underline{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow V$ IS NOT a linear subspace of $\mathbb{R}^3$ because (for example)

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are in V but their sum

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is NOT.

Problem 5

(1) $\det(A) = 3(2-2) = 0$
$\operatorname{tr}(A) = 1+3+2 = 6$

(2) A is NOT invertible (because $\det(A)=0$).

(3) $\operatorname{rank}(A) = 2$ (1^{st} and 2^{nd} columns are linearly independent)

(4) $\underline{u} = 2$ times 3^{rd} column of A

$\Rightarrow$ YES, $\underline{u} \in \operatorname{Im}(A)$.

(5) $A\underline{v} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} \neq \underline{0}$ $\Rightarrow$ NO, $\underline{v} \notin \operatorname{Null}(A)$

(6) $\operatorname{rank}(A) + \dim(\operatorname{Null}(A)) = 3$

$\Rightarrow \dim(\operatorname{Null}(A)) = 1$

(7) $A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} v_1 + v_3 = 0 \\ -v_1 + 3v_2 + 2v_3 = 0 \\ 2v_1 + 2v_3 = 0 \end{cases} \Rightarrow \begin{cases} v_3 = -v_1 \\ v_2 = v_1 \end{cases}$

$\Rightarrow$ BASIS: $\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right)$

Problem 6

(1) $2k = 3k \Rightarrow k = 0$

(2) $k = 1 \Rightarrow$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\det(H - \lambda I) = (1-\lambda)\left((2-\lambda)(1-\lambda) - 6\right) =$$

$$= (1-\lambda)(\lambda^2 - 3\lambda - 4) = (1-\lambda)(\lambda - 4)(\lambda + 1)$$

* $\boxed{\lambda = 1}$ $\Rightarrow H\underline{v} = \underline{v}$

$$\begin{cases} v_1 = v_1 \\ 2v_2 + 3v_3 = v_2 \\ 2v_2 + v_3 = v_3 \end{cases} \Rightarrow \begin{cases} v_1 \text{ ANYTHING} \\ v_2 = 0 \\ v_3 = 0 \end{cases}$$

BASIS: $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$

* $\boxed{\lambda = 4}$ $\Rightarrow H\underline{v} = 4\underline{v}$

$$\begin{cases} v_1 = 4v_1 \\ 2v_2 + 3v_3 = 4v_2 \\ 2v_2 + v_3 = 4v_3 \end{cases} \Rightarrow \begin{cases} v_1 = 0 \\ v_2 = \frac{3}{2}v_3 \\ v_3 = \text{ ANYTHING} \end{cases}$$

BASIS: $\left(\begin{bmatrix} 0 \\ 3/2 \\ 1 \end{bmatrix} \right)$

* $\boxed{\lambda = -1}$ $\Rightarrow H\underline{v} = -\underline{v}$

$$\begin{cases} v_1 = -v_1 \\ 2v_2 + 3v_3 = -v_2 \\ 2v_2 + v_3 = -v_3 \end{cases} \Rightarrow \begin{cases} v_1 = 0 \\ v_2 = -v_3 \\ v_3 = \text{ ANYTHING} \end{cases}$$

BASIS: $\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right)$

③ $h=0 \Rightarrow H=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\det(H-\lambda I) = \lambda(1-\lambda)(2-\lambda)$

* $\boxed{\lambda=0}$ $H\underline{v}=\underline{0}$ (it is the Null Space!)

$$\begin{cases} v_1=0 \\ 2v_2=0 \\ 0=0 \end{cases} \Rightarrow \begin{cases} v_1=0 \\ v_2=0 \\ v_3=\text{ANYTHING} \end{cases}$$

BASIS: $\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

* $\boxed{\lambda=1}$ $H\underline{v}=\underline{v}$

$$\begin{cases} v_1=\text{ANYTHING} \\ v_2=0 \\ v_3=0 \end{cases}$$

BASIS: $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

* $\boxed{\lambda=2}$ $H\underline{v}=2\underline{v}$

$$\begin{cases} v_1=0 \\ v_2=\text{ANYTHING} \\ v_3=0 \end{cases}$$

BASIS: $\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

Problem 7

(1) linearly independent rows $\iff$ invertible matrix $\iff$ $\det M \neq 0$

$\det(M) = -m^2(1-m)$

$\Rightarrow$ FOR $m \neq 0$ AND $m \neq 1$

(2) for $m \neq 0$ AND $m \neq 1$

(3) • $m \neq 0$ and $m \neq 1 \Rightarrow \text{rank}(M) = 3$

• $m = 0 \Rightarrow$ $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(M) = 1$

$\dim(\text{Im}(M)) = 1$ because $\text{Im}(M) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$

• $m = 1 \Rightarrow$ $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(M) = 2$

lin. ind. $\Rightarrow \dim(\text{Im}(M)) = 2$

(4) $m = -1 \Rightarrow \det(M) = -2$

$\det(M^8) = (-2)^8 = 256$

$\det(-5M) = (-5)^3(-2) = +150$

$\det(M^{-1}) = \dfrac{1}{\det(M)} = -\dfrac{1}{2}$ because $\det(MM^{-1}) = \det(I) = 1$

$\det(M)\det(M^{-1})$

Problem 8

(1) $\mathcal{C}$ is orthonormal (we saw it during the 1st lesson, and it is very easy to verify)

$|\underline{b}_1| = |\underline{b}_2| = |\underline{b}_3| = 1$

$\underline{b}_1 \cdot \underline{b}_2 = 0$

$\underline{b}_1 \cdot \underline{b}_3 = 0$

$\underline{b}_2 \cdot \underline{b}_3 = 0$

$\Rightarrow$ $\mathcal{B}$ is orthonormal, too.

(2) FROM $\mathcal{B}$ TO $\mathcal{C}$: $M_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}$

FROM $\mathcal{C}$ TO $\mathcal{B}$: $M_{\mathcal{C}}^{\mathcal{B}} = (M_{\mathcal{B}}^{\mathcal{C}})^{-1} \underset{\uparrow}{=} (M_{\mathcal{B}}^{\mathcal{C}})^{T} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}$

because BOTH $\mathcal{B}$ and $\mathcal{C}$ are ORTHONORMAL

(3) $[T]_{\mathcal{C}} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ (it is $\begin{bmatrix} T(\underline{e}_1) & T(\underline{e}_2) & T(\underline{e}_3) \end{bmatrix}$)

written in the basis $\mathcal{C}$

$$[T]_{\mathcal{B}} = M_{\mathcal{C}}^{\mathcal{B}} [T]_{\mathcal{C}} M_{\mathcal{B}}^{\mathcal{C}}$$

$$= \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix} =$$

collect $\frac{\sqrt{2}}{2}$ in 1st and 3rd matrices

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} 5 & -3 & 1 \\ 3\sqrt{2} & -5\sqrt{2} & 3\sqrt{2} \\ -7 & 9 & -7 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{bmatrix} =$$

$$= \frac{1}{2} \begin{bmatrix} -4 & -3\sqrt{2} & -6 \\ 0 & -10 & -6\sqrt{2} \\ 0 & 9\sqrt{2} & 14 \end{bmatrix} = \begin{bmatrix} -2 & -3/\sqrt{2} & -3 \\ 0 & -5 & -3\sqrt{2} \\ 0 & 9/\sqrt{2} & 7 \end{bmatrix}$$