Rotations and orthonormal frames

Problem 1. Consider the vector $v = (1/\sqrt{2}, 1/\sqrt{2}, 0)$.

- Write the rotation matrix $R(v,\theta)$ around the axis directed by v, with positive rotation angle θ.
- Write the skew-symmetric tensor $W(v)$.
- Check that the derivative of $R(v,\theta)$ with respect to θ, at $\theta = 0$, satisfies:

$$\left.\frac{d}{d\theta}\right|_{\theta=0} R(v,\theta) = W(v)\,.$$

Problem 2. Consider the triple

$$d_1(t) = (\cos(e^t), \sin(e^t), 0) \quad d_2(t) = (-\sin(e^t), \cos(e^t), 0) \quad d_3 = (0,0,1)$$

- Check that (d_1, d_2, d_3) is an orthonormal triple;
- Write the matrix representing the twist tensor with respect to the basis (d_1, d_2, d_3).
- Write the matrix representing the twist tensor with respect to the standard basis.
- Write the twist vector.

Problem 3. Let $u(t), v(t)$ be two differentiable vector-valued functions. Prove that:

$$\frac{d}{dt}(u(t) \times v(t)) = \left(\frac{d}{dt}u(t)\right) \times v(t) + u(t) \times \left(\frac{d}{dt}v(t)\right)$$

and

$$\frac{d}{dt}(u(t) \otimes v(t)) = \left(\frac{d}{dt}u(t)\right) \otimes v(t) + u(t) \otimes \left(\frac{d}{dt}v(t)\right)$$

Problem 4. With respect to the standard basis,

- Write the rotation matrix $R_x^{\pi/2}$ with axis the x-line, which rotates of $\pi/2$ and sends $(0,1,0)$ to $(0,0,1)$.
- Check that $(R_x^{\pi/2})^{-1}$ coincides with $R_x^{-\pi/2}$.
- Write the rotation matrix R_y^{θ} with axis the y-line, which rotates of angle θ. Analogously write R_z^{θ}, which has axis the z-line.
- Check that $R_x^{\pi/2} \circ R_y^{\theta} \circ (R_x^{\pi/2})^{-1} = R_z^{\theta}$.
- Deduce that in general, for matrices of rotations A and B, $AB \neq BA$.

Problem 5. Consider the matrix

$$A(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

- Prove that $A(t)$ sends a square to a parallelogram of the same area. Observe that $A(t)$ does not preserve lengths and angles.
- Prove that, given two vectors $v = (v_1, v_2)$ and $w = (w_1, w_2)$ in $\mathbb{R}^2$, the area of the parallelogram with sides v and w satisfies:

$$(\text{Area}(v,w))^2 = \left(\det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}\right)^2.$$

- Deduce that a matrix B preserves the area if and only if $|\det B| = 1$.
- Prove that the matrix
$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$
preserves the area if and only if $\alpha = \pm 1/\beta$.
- Given a 2x2 matrix-valued function
$$C(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}$$
such that $C(0)$ is the identity, prove that
$$\left.\frac{d}{dt}\right|_{t=0} \det C(t) = \operatorname{tr} C'(0)\,.$$
- Deduce that, if $C(t)$ is a 2x2 matrix-valued function such that $C(0) = \mathrm{id}$ and $\det C(t) \equiv 1$, then $C'(0)$ is traceless (i.e. $\operatorname{tr} C'(0) = 0$).
- Find an example of a traceless matrix which is not skew-symmetric.