FINAL TEST - ANALYTICAL MECHANICS - OCTOBER, 19^{th} 2016

SOLUTIONS

Problem 1.

(1)

$$\det \mathbf{F} = (\mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2) \cdot \mathbf{F}\mathbf{e}_3 = (\mathbf{e}_1 \times (-2\mathbf{e}_1 + \mathbf{e}_2)) \cdot (3\mathbf{e}_2 + \mathbf{e}_3) = 1$$
$$\mathrm{tr}\mathbf{F} = 3$$

(2)

$$\mathbf{F}\mathbf{u} = -2\mathbf{e}_1 + 3\mathbf{e}_2 - 6\mathbf{e}_1 + \mathbf{e}_3 + 3\mathbf{e}_2 = -8\mathbf{e}_1 + 6\mathbf{e}_2 + \mathbf{e}_3$$

(3)

$$\mathrm{sym}\mathbf{F} = \mathbf{I} - \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1 + \frac{3}{2}\mathbf{e}_2 \otimes \mathbf{e}_3 + \frac{3}{2}\mathbf{e}_3 \otimes \mathbf{e}_2$$
$$\mathrm{skw}\mathbf{F} = -\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1 + \frac{3}{2}\mathbf{e}_2 \otimes \mathbf{e}_3 - \frac{3}{2}\mathbf{e}_3 \otimes \mathbf{e}_2$$

(4)

$$\mathbf{w}(\mathbf{F}) = -\frac{3}{2}\mathbf{e}_1 + \mathbf{e}_3$$

(5)

$$\mathbf{F}^{-1} = \mathbf{I} + 2\mathbf{e}_1 \otimes \mathbf{e}_2 - 6\mathbf{e}_1 \otimes \mathbf{e}_3 - 3\mathbf{e}_2 \otimes \mathbf{e}_3$$
$$\mathbf{F}^* = (\det \mathbf{F})\mathbf{F}^{-\top} = \mathbf{I} + 2\mathbf{e}_2 \otimes \mathbf{e}_1 - 6\mathbf{e}_3 \otimes \mathbf{e}_1 - 3\mathbf{e}_3 \otimes \mathbf{e}_2$$

(6)

$$|\mathbf{F}^*\mathbf{e}_1| = |\mathbf{e}_1 + 2\mathbf{e}_2 - 6\mathbf{e}_3| = \sqrt{41}$$

Problem 2.

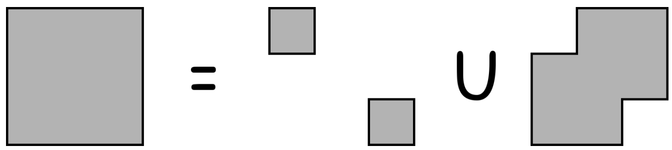
(1) By symmetry,

$$C - O = \frac{3}{2}L(\mathbf{e_x} + \mathbf{e}_y)$$

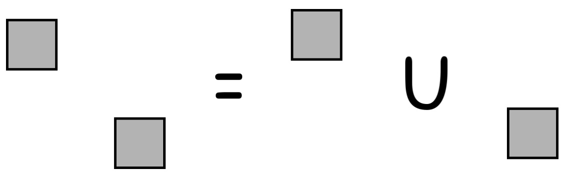
(2) and a principal basis of $\mathbf{I}_C$ is

$$\left(\mathbf{e}_1 := \frac{\sqrt{2}}{2}(\mathbf{e}_x + \mathbf{e}_y), \frac{\sqrt{2}}{2}(-\mathbf{e}_x + \mathbf{e}_y), \mathbf{e}_z := \mathbf{e}_x \times \mathbf{e}_y\right).$$

(3) To compute $\mathbf{I}_C$ we can use the Composition Theorem, indeed:



Therefore $\mathbf{I}_C = \mathbf{I}_{\text{BigSquare}C} - \mathbf{I}_{\text{TwoSquares}C}$, since the center of mass of both BigSquare and TwoSquares is C. Moreover



and hence, by the Composition Theorem again,

$$\mathbf{I}_{\text{TwoSquares}C} = \mathbf{I}_{\text{Square1}C_1} + \mathbf{I}_{\text{Square2}C_2} + \frac{m_1 m_2}{m_1 + m_2}|C_1 - C_2|^2(\mathbf{I} - \mathbf{e}_2 \otimes \mathbf{e}_2)$$

where $|C_1 - C_2|^2 = 8L^2$, $m_1 = m_2 = \rho L^2$ and $m = 7\rho L^2$.

We now need to compute the central tensor of inertia for a generic square with mass density ρ and edge a. We call A the center of mass of the generic square, then by symmetry

$$I_{Axx} = I_{Ayy} = 4\rho \int_0^{\frac{a}{2}} \int_0^{\frac{a}{2}} x^2 dx dy = 2a\rho \int_0^{\frac{a}{2}} x^2 dx = \frac{1}{12}\rho a^4 = \frac{1}{2} I_{Azz}.$$

Since $I_{Axx} = I_{Ayy}$, all the directions on the plane $\text{span}(\mathbf{e}_x, \mathbf{e}_y)$ are principal, and then

$$I_{A11} = I_{A22} = 4\rho \int_0^{\frac{a}{2}} \int_0^{\frac{a}{2}} x^2 dx dy = 2a\rho \int_0^{\frac{a}{2}} x^2 dx = \frac{1}{12}\rho a^4 = \frac{1}{2} I_{Azz}.$$

Therefore

$$\mathbf{I}_{\text{Square1}C_1} = \frac{1}{12}\rho L^4(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + 2\mathbf{e}_z \otimes \mathbf{e}_z)$$

$$\mathbf{I}_{\text{Square2}C_2} = \frac{1}{12}\rho L^4(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + 2\mathbf{e}_z \otimes \mathbf{e}_z)$$

whence

$$\mathbf{I}_{\text{TwoSquares}C} = \frac{1}{6}\rho L^4(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + 2\mathbf{e}_z \otimes \mathbf{e}_z) + 4\rho L^4(\mathbf{I} - \mathbf{e}_2 \otimes \mathbf{e}_2) =$$
$$= \frac{25}{6}\rho L^4 \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{1}{6}\rho L^4 \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{13}{3}\rho L^4 \mathbf{e}_z \otimes \mathbf{e}_z$$

and

$$\mathbf{I}_C = \frac{81}{12}\rho L^4(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + 2\mathbf{e}_z \otimes \mathbf{e}_z) - \frac{25}{6}\rho L^4 \mathbf{e}_1 \otimes \mathbf{e}_1 - \frac{1}{6}\rho L^4 \mathbf{e}_2 \otimes \mathbf{e}_2 - \frac{13}{3}\rho L^4 \mathbf{e}_z \otimes \mathbf{e}_z =$$
$$= \frac{31}{12}\rho L^4 \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{79}{12}\rho L^4 \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{55}{6}\rho L^4 \mathbf{e}_z \otimes \mathbf{e}_z =$$

(4) By Huygens-Steiner's Theorem

$$\mathbf{I}_O = \mathbf{I}_C + m|C - O|^2(\mathbf{I} - \mathbf{e}_1 \otimes \mathbf{e}_1) =$$
$$= \frac{31}{12}\rho L^4 \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{79}{12}\rho L^4 \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{55}{6}\rho L^4 \mathbf{e}_z \otimes \mathbf{e}_z + \frac{63}{2}\rho L^4(\mathbf{I} - \mathbf{e}_1 \otimes \mathbf{e}_1) =$$
$$= \frac{31}{12}\rho L^4 \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{457}{12}\rho L^4 \mathbf{e}_2 \otimes \mathbf{e}_2 + \frac{244}{6}\rho L^4 \mathbf{e}_z \otimes \mathbf{e}_z$$

Problem 3.

(1) The arc-length parameter is

$$
\begin{aligned}
s(t) &= \int |\dot{P}(u)| du = \\
&= \int |(-\sinh(t)\sin(\cosh(t)), \sinh(t)\cos(\cosh(t)), 1)| du = \\
&= \int \sqrt{\sinh^2(t) + 1} du = \\
&= \int \sqrt{\cosh^2(t)} du = \\
&= \sinh(t)
\end{aligned}
$$

(2) Therefore $t(s) = \operatorname{asinh}(s)$ and the arc-length parametrization of the curve is

$$Q(s) = (\cos(\cosh(\operatorname{asinh}(s))), \sin(\cosh(\operatorname{asinh}(s))), \operatorname{asinh}(s))$$

(3) Then

$$\mathbf{t}(s) = Q'(s) = \left(\frac{-s\sin(\cosh(\operatorname{asinh}(s)))}{\sqrt{s^2+1}}, \frac{s\cos(\cosh(\operatorname{asinh}(s)))}{\sqrt{s^2+1}}, \frac{1}{\sqrt{s^2+1}} \right).$$

Since $\frac{d}{ds}\frac{1}{\sqrt{s^2+1}} = -\frac{s}{(\sqrt{s^2+1})^3}$ and $\frac{d}{ds}\frac{s}{\sqrt{s^2+1}} = \frac{1}{\sqrt{s^2+1}} - \frac{s^2}{(\sqrt{s^2+1})^3} = \frac{1}{(\sqrt{s^2+1})^3}$ we get

$$
\begin{aligned}
\frac{d}{ds}\frac{-s\sin(\cosh(\operatorname{asinh}(s)))}{\sqrt{s^2+1}} &= -\frac{\sin(\cosh(\operatorname{asinh}(s)))}{(\sqrt{s^2+1})^3} - \frac{s^2\cos(\cosh(\operatorname{asinh}(s)))}{(\sqrt{s^2+1})^2} \\
\frac{d}{ds}\frac{s\cos(\cosh(\operatorname{asinh}(s)))}{\sqrt{s^2+1}} &= \frac{\cos(\cosh(\operatorname{asinh}(s)))}{(\sqrt{s^2+1})^3} - \frac{s^2\sin(\cosh(\operatorname{asinh}(s)))}{(\sqrt{s^2+1})^2} \\
\frac{d}{ds}\frac{1}{\sqrt{s^2+1}} &= -\frac{s}{(\sqrt{s^2+1})^3}
\end{aligned}
$$

whence

$$c(s) = |\mathbf{t}'(s)| = |Q''(s)| = \sqrt{\frac{1}{(s^2+1)^3} + \frac{s^4}{(s^2+1)^2} + \frac{s^2}{(s^2+1)^3}} = \frac{\sqrt{s^4+1}}{s^2+1}.$$

Problem 4.

(1) Let $P(x) = (x, y(x))$ be the curve that describes the shape of the cable at equilibrium. Then the tangent versor to the curve is $\mathbf{t}(x) = \dfrac{1}{\sqrt{1+\dot{y}^2(x)}}(1, \dot{y}(x))$.
Since the force $\mathbf{F}_O$ is parallel to OB and the stress $\boldsymbol{\varphi}(0) = -\mathbf{F}_O$ at O is equal to $T(0)\mathbf{t}(0)$ (where $T > 0$ is the tension), the tangent $\mathbf{t}(0)$ is parallel to $\mathbf{F}_O$ too (and with opposite direction).
Therefore, $\mathbf{t}(0) = (\cos\alpha, -\sin\alpha)$
and $\mathbf{t}(0) = \dfrac{1}{\sqrt{1+\dot{y}^2(0)}}(1, \dot{y}(0))$
by equating the two components and dividing, we obtain

$$\dot{y}(0) = \frac{-\sin\alpha}{\cos\alpha} = -\tan\alpha.$$

(2) Note that $y(0) = y(x_0) = 0$. Since the only distributed force is the gravitational force due to the weight of the cable, the shape at equilibrium is

$$y(x) = \frac{K}{\lambda g}\cosh\left(\frac{\lambda g}{K}x - C_1\right) - C_2.$$

From the condition $y(0) = 0$ we get $C_2 = \dfrac{K}{\lambda g}\cosh(C_1)$. Given that $\dot{y}(x) = \sinh\left(\dfrac{\lambda g}{K}x - C_1\right)$, from $\dot{y}(0) = -\tan\alpha$ we get $-\tan\alpha = \sinh(-C_1)$, whence $C_1 = \operatorname{asinh}(\tan\alpha)$ and

$$y(x) = \frac{K}{\lambda g}\cosh\left(\frac{\lambda g}{K}x - \operatorname{asinh}(\tan\alpha)\right) - \frac{K}{\lambda g}\cosh\left(\operatorname{asinh}(\tan\alpha)\right).$$

From the condition $y(x_0) = 0$ we get

$$\frac{K}{\lambda g}\cosh\left(\frac{\lambda g}{K}x_0 - \operatorname{asinh}(\tan\alpha)\right) = \frac{K}{\lambda g}\cosh\left(\operatorname{asinh}(\tan\alpha)\right)$$

which means that $\dfrac{\lambda g}{K}x_0 - \operatorname{asinh}(\tan\alpha)$ and $\operatorname{asinh}(\tan\alpha)$ have the same hyperbolic cosine.
Therefore: either $\dfrac{\lambda g}{K}x_0 - \operatorname{asinh}(\tan\alpha) = \operatorname{asinh}(\tan\alpha)$ and $x_0 = \dfrac{2K}{\lambda g}\operatorname{asinh}(\tan\alpha)$, or $\dfrac{\lambda g}{K}x_0 - \operatorname{asinh}(\tan\alpha) = -\operatorname{asinh}(\tan\alpha)$ that yields to $x_0 = 0$ (which does not make any sense in this problem). In conclusion,

$$x_0 = \frac{2K}{\lambda g}\operatorname{asinh}(\tan\alpha)$$

(3) The known length of the cable entails

$$\begin{aligned} L &= \int_0^{x_0}\sqrt{1+\dot{y}^2(x)}dx = \int_0^{x_0}\cosh\left(\frac{\lambda g}{K}x - \frac{\lambda g}{2K}x_0\right)dx = \\ &= \frac{K}{\lambda g}\sinh\left(\frac{\lambda g}{K}x_0 - \frac{\lambda g}{2K}x_0\right) - \frac{K}{\lambda g}\sinh\left(-\frac{\lambda g}{2K}x_0\right) = \\ &= 2\frac{K}{\lambda g}\sinh\left(\frac{\lambda g}{2K}x_0\right) = 2\frac{K}{\lambda g}\tan\alpha \end{aligned}$$

whence

$$K = \frac{\lambda g L}{2\tan\alpha}$$

(4) and

$$y(x) = \frac{L}{2\tan\alpha}\cosh\left(\frac{2\tan\alpha}{L}x - \operatorname{asinh}(\tan\alpha)\right) - \frac{L}{2\tan\alpha}\cosh\left(\operatorname{asinh}(\tan\alpha)\right).$$

(5) Since $x_0 = \dfrac{L}{\tan\alpha}\operatorname{asinh}(\tan\alpha) = 2l\cos\alpha$ then

$$l = \frac{L\operatorname{asinh}(\tan\alpha)}{2\sin\alpha}.$$

(6) Since $T(0) = K\sqrt{1+\dot{y}^2(0)}$, we obtain

$$|\mathbf{F}_O| = T(0) = \frac{\lambda g L}{2\tan\alpha}\sqrt{1+\tan^2\alpha} = \frac{\lambda g L}{2\sin\alpha}$$

Problem 5.
For any $s \in [0, 2L]$, let $\vartheta(s)$ be the angle between $\mathbf{e}_x$ an the tangent versor $t(s)$.

SETUP:

$$\begin{cases} \mathbf{d}_2(s) = \mathbf{e}_z \\ \mathbf{d}_3(s) = \mathbf{t}(s) = \cos\vartheta(s)\mathbf{e}_x + \sin\vartheta(s)\mathbf{e}_y \\ \mathbf{d}_1(s) = \mathbf{d}_2(s) \times \mathbf{d}_3(s) = -\sin\vartheta(s)\mathbf{e}_x + \cos\vartheta(s)\mathbf{e}_y \end{cases} \quad \text{then} \quad \begin{cases} \mathbf{d}_2'(s) = \mathbf{0} \\ \mathbf{d}_3'(s) = \vartheta'(s)\mathbf{d}_1(s) \end{cases}$$

From $\mathbf{d}_i'(s) = \mathbf{u}(s) \times \mathbf{d}_i$, we obtain $\mathbf{u} = \vartheta'(s)\mathbf{d}_2(s)$ and hence $\boldsymbol{\gamma}(s) = B\vartheta'(s)\mathbf{e}_z$.

(1) BOUNDARY CONDITIONS:

$$O: \begin{cases} y(0) = 0 \\ \boldsymbol{\varphi}(0) = -\mathbf{F}_O \text{ [to be determined]} \\ \gamma(0) = -\mathbf{G} = -2FL\mathbf{e}_z \;\Rightarrow\; \vartheta'(0) = -\dfrac{2FL}{B} \end{cases} \qquad 2L: \begin{cases} y(2L) = 0 \\ \boldsymbol{\varphi}(2L) = \mathbf{F}_{2L} = \phi\mathbf{e}_y \text{ [}\phi\text{ to be determined]} \\ \gamma(2L) = \mathbf{0} \;\Rightarrow\; \vartheta'(2L) = 0 \end{cases}$$

(2) CONDITIONS AT $s = L$:

$$s = L: \begin{cases} [\![\boldsymbol{\varphi}(L)]\!] = -\mathbf{F} = 4F\mathbf{e}_y \\ [\![\gamma(L)]\!] = \mathbf{0} \quad \Rightarrow \quad [\![\vartheta'(L)]\!] = 0 \quad \Rightarrow \quad \vartheta' \text{ continuous at } s = L \text{ (hence everywhere)} \end{cases}$$

(3) EQUILIBRIUM EQUATIONS FOR FORCES ACTING ON THE BEAM:

$$\mathbf{F} + \mathbf{F}_O + \mathbf{F}_{2L} = \mathbf{0} \quad \Rightarrow \quad -4F\mathbf{e}_y + \mathbf{F}_O + \phi\mathbf{e}_y = \mathbf{0}$$

hence

$$\mathbf{F}_O = (4F - \phi)\mathbf{e}_y$$

EQUILIBRIUM EQUATIONS FOR TORQUES ACTING ON THE BEAM W.R.T. O:

$$\mathbf{G} + L\mathbf{e}_x \times \mathbf{F} + 2L\mathbf{e}_x \times \mathbf{F}_{2L} = \mathbf{0} \quad \Rightarrow \quad 2FL\mathbf{e}_z - 4FL\mathbf{e}_z + 2\phi L\mathbf{e}_y = \mathbf{0}$$

hence $\phi = F$ and the REACTIVE FORCES are

$$\begin{cases} \mathbf{F}_{2L} = F\mathbf{e}_y \\ \mathbf{F}_O = 3F\mathbf{e}_y \end{cases}$$

(4) EQUILIBRIUM EQUATIONS FOR CONTACT FORCES:

$$\begin{cases} \boldsymbol{\varphi}'(s) = \mathbf{0} \quad s \in [0, L) \\ \boldsymbol{\varphi}'(s) = \mathbf{0} \quad s \in (L, 2L] \end{cases} \quad \text{then} \quad \begin{cases} \boldsymbol{\varphi}(s) = \text{constant} = \boldsymbol{\varphi}(0) = -\mathbf{F}_O = -3F\mathbf{e}_y \quad s \in [0, L) \\ \boldsymbol{\varphi}(s) = \text{constant} = \boldsymbol{\varphi}(2L) = \mathbf{F}_{2L} = F\mathbf{e}_y \quad s \in (L, 2L] \end{cases}$$

EQUILIBRIUM EQUATIONS FOR CONTACT TORQUES:

$$\begin{cases} \boldsymbol{\gamma}'(s) - 3F\cos\vartheta(s)\mathbf{e}_z = \mathbf{0} \quad s \in [0, L) \\ [\![\gamma(L)]\!] = \mathbf{0} \\ \boldsymbol{\gamma}'(s) + F\cos\vartheta(s)\mathbf{e}_z = \mathbf{0} \quad s \in (L, 2L] \end{cases} \quad \text{whence} \quad \begin{cases} B\vartheta''(s) - 3F\cos\vartheta(s) = 0 \quad s \in [0, L) \\ [\![\vartheta'(L)]\!] = 0 \\ B\vartheta''(s) + F\cos\vartheta(s) = 0 \quad s \in (L, 2L] \end{cases}$$

SMALL DEFLECTIONS APPROXIMATION: We suppose $|\vartheta(s)| \ll 1$ for all $s \in [0, 2L]$. Then we can approximate $\sin\vartheta(s)$ with $\vartheta(s)$ and $\cos\vartheta(s)$ with 1.
Then $s \approx x$ and $y'(x) \approx \vartheta(x)$.
Hence

$$\begin{cases} B\vartheta''(x) - 3F = 0 \quad x \in [0, L) \\ \vartheta' \text{ continuous at } x = L \\ B\vartheta''(x) + F = 0 \quad x \in (L, 2L] \end{cases}$$

We start integrating and using the boundary conditions:

$$\begin{cases} B\vartheta'(x) = 3Fx + C_1 \quad x \in [0, L) \\ B\vartheta'(x) = -Fx + C_2 \quad x \in (L, 2L] \end{cases} \quad \text{and} \quad \begin{cases} \vartheta'(0) = -\dfrac{2FL}{B} \\ \vartheta'(2L) = 0 \end{cases}$$

whence

$$\begin{cases} B\vartheta'(x) = 3Fx - 2FL \quad x \in [0, L] \\ B\vartheta'(x) = -Fx + 2FL \quad x \in [L, 2L] \end{cases} \quad \text{and} \quad \begin{cases} B\vartheta(x) = \dfrac{3F}{2}x^2 - 2FLx + C_3 \quad x \in [0, L] \\ B\vartheta(x) = -\dfrac{F}{2}x^2 + 2FLx + C_4 \quad x \in [L, 2L] \end{cases}$$

By the continuity of $\vartheta(x)$ (at $x = L$), we get $C_4 = -2FL^2 + C_3$, then

$$\begin{cases} B\vartheta(x) = \dfrac{3F}{2}x^2 - 2FLx + C_3 \quad x \in [0, L] \\ B\vartheta(x) = -\dfrac{F}{2}x^2 + 2FLx - 2FL^2 + C_3 \quad x \in [L, 2L] \end{cases}$$

and

$$\begin{cases} By(x) = \dfrac{F}{2}x^3 - FLx^2 + C_3x + C_5 \quad x \in [0, L] \\ By(x) = -\dfrac{F}{6}x^3 + FLx^2 - 2FL^2x + C_3x + C_6 \quad x \in [L, 2L] \end{cases}$$

From the boundary conditions $y(0) = 0$ and $y(2L) = 0$ we get

and

$$\begin{cases} By(x) = \dfrac{F}{2}x^3 - FLx^2 + C_3x \quad x \in [0, L] \\ By(x) = -\dfrac{F}{6}x^3 + FLx^2 - 2FL^2x + C_3x - 2C_3L + \dfrac{4F}{3}L^3 \quad x \in [L, 2L] \end{cases}$$

Equating the right sides of the last two equations for $x = L$ (because $y(x)$ is continuous), we obtain $C_3 = \dfrac{1}{3}FL^2$ and hence the function $y(x)$ which represents the shape of the center line of the beam at equilibrium in case of small deflections is

$$\begin{cases} y(x) = \dfrac{F}{2B}x\left(x^2 - 2Lx + \dfrac{2}{3}L^2\right) \quad x \in [0, L] \\ y(x) = -\dfrac{F}{6B}\left(x^3 - 6Lx^2 + 10L^2x - 4L^3\right) = -\dfrac{F}{6B}(x - 2L)\left(x^2 - 4Lx + 2L^2\right) \quad x \in [L, 2L] \end{cases}$$

(5) The function $\vartheta(x) \approx y'(x)$ describes the slope of the center line of the curve at equilibrium

$$\begin{cases} \vartheta(x) = \dfrac{3F}{2B}\left(x^2 - \dfrac{4}{3}Lx + \dfrac{2}{9}L^2\right) = \dfrac{3F}{2B}\left(x - \dfrac{2+\sqrt{2}}{3}L\right)\left(x - \dfrac{2-\sqrt{2}}{3}L\right) \quad x \in [0, L] \\ \vartheta(x) = -\dfrac{F}{2B}\left(x^2 - 4Lx + \dfrac{10}{3}L^2\right) = -\dfrac{F}{2B}\left(x - 2L - \sqrt{\dfrac{2}{3}}L\right)\left(x - 2L + \sqrt{\dfrac{2}{3}}L\right) \quad x \in [L, 2L] \end{cases}$$

and

$$\begin{cases} \vartheta'(x) = \dfrac{F}{B}(3x - 2L) & x \in [0, L] \\ \vartheta'(x) = -\dfrac{F}{B}(x - 2L) & x \in [L, 2L] \end{cases}$$

Therefore $\vartheta(x)$ is decreasing from $\vartheta(0) = \dfrac{FL^2}{3B}$ to $\vartheta\left(\dfrac{2}{3}L\right) = -\dfrac{FL^2}{3B}$ and then increasing to $\vartheta(2L) = \dfrac{FL^2}{3B}$.
Then

$$\vartheta_{\max} = \frac{FL^2}{3B}.$$

(6) The approximation with small deflections is reliable for $\vartheta_{\max} \ll 1$, whence for

$$F \ll 3\frac{B}{L^2}.$$

NOTE: $y(x)$ is increasing from $y(0) = 0$ to $y\left(\dfrac{2-\sqrt{2}}{3}L\right) = 2\dfrac{\sqrt{2}-1}{27}\dfrac{FL^3}{B}$, then decreasing to $y\left(2L - \sqrt{\dfrac{2}{3}}L\right) = -\dfrac{2}{9}\sqrt{\dfrac{2}{3}}\dfrac{FL^3}{B}$, and then increasing to $y(2L) = 0$. Then the maximum deflection is

$$\delta_{\max} = \left| y\left(2L - \sqrt{\frac{2}{3}}L\right)\right| = \frac{2}{9}\sqrt{\frac{2}{3}}\frac{FL^3}{B}.$$