EXERCISES ON TENSOR ALGEBRA

In the following exercises $\mathcal{B} := (\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_3)$ is a positively oriented orthonormal basis of $\mathcal{V}$.
$\mathbf{W}(\mathbf{w})$ is the skew symmetric tensor with axial vector $\mathbf{w}$.
$\mathbf{I}$ is the identity.

Problem 1. Consider the three tensors

$$\mathbf{A} := 3\mathbf{e}_1 \otimes \mathbf{e}_1 - 2\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_2 \otimes \mathbf{e}_1 - 4\mathbf{e}_3 \otimes \mathbf{e}_3,$$
$$\mathbf{B} := -\mathbf{e}_1 \otimes \mathbf{e}_1 + 3\mathbf{e}_2 \otimes \mathbf{e}_2 - 2\mathbf{e}_2 \otimes \mathbf{e}_3 - 2\mathbf{e}_3 \otimes \mathbf{e}_1,$$
$$\mathbf{C} := -3\mathbf{e}_1 \otimes \mathbf{e}_2 + 3\mathbf{e}_2 \otimes \mathbf{e}_1 - 2\mathbf{e}_2 \otimes \mathbf{e}_3 + 2\mathbf{e}_3 \otimes \mathbf{e}_2$$

and the vectors $\mathbf{v} := 2\mathbf{e}_1 - \mathbf{e}_2 + 3\mathbf{e}_3$ and $\mathbf{u} := -\mathbf{e}_1 + 3\mathbf{e}_2 + 2\mathbf{e}_3$.

(1) Compute $\mathrm{tr}\mathbf{A}$, $\mathrm{tr}\mathbf{B}$, $\det \mathbf{A}$ and $\det \mathbf{B}$.
(2) Write $\mathbf{A}^\top$, $\mathbf{C}^\top$ and compute $|\mathbf{A}|$ and $|\mathbf{C}|$.
(3) Write the tensor $\mathbf{AB}$.
(4) Compute $\mathbf{A} \cdot \mathbf{B}$.
(5) Compute $\mathbf{Av} \cdot \mathbf{Bu}$ and $\mathbf{Au} \times \mathbf{Bv}$.
(6) Write $\mathbf{W}(\mathbf{v})$ and $\mathbf{W}(\mathbf{u})$.
(7) Are $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ symmetric?
(8) Are $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ skew symmetric?
(9) Compute $\mathrm{tr}\mathbf{C}$ and $\det \mathbf{C}$.
(10) Write the axial vector of $\mathbf{C}$.

Problem 2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{V}$. Prove that:

(1) $\mathbf{W}(\mathbf{a})(\mathbf{b} \otimes \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \otimes \mathbf{c}$
(2) $(\mathbf{a} \otimes \mathbf{b})\mathbf{W}(\mathbf{c}) = \mathbf{a} \otimes (\mathbf{b} \times \mathbf{c})$
(3) $\mathbf{W}(\mathbf{a})\mathbf{W}(\mathbf{b}) = \mathbf{b} \otimes \mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{I}$

Problem 3. Prove that for any versor $\mathbf{e}$

$$\mathbf{W}(\mathbf{e})^3 = -\mathbf{W}(\mathbf{e})$$

Problem 4. Consider the tensor $\mathbf{F} = \mathbf{I} + 3\mathbf{e}_1 \otimes \mathbf{e}_3 + 2\mathbf{e}_2 \otimes \mathbf{e}_3$.

(1) Compute $\det \mathbf{F}$, $\mathrm{tr}\mathbf{F}$ and $|\mathbf{F}|$.
(2) Decompose $\mathbf{F}$ in its symmetric part $\mathrm{sym}\mathbf{F}$ and its skew symmetric part $\mathrm{skw}\mathbf{F}$.
(3) What is the axial vector of $\mathrm{skw}\mathbf{F}$?
(4) Write the tensors $\mathbf{F}^{-1}$ and $\mathbf{F}^*$.
(5) What is the area dilation factor of $\mathbf{F}$ for surfaces parallel to $\mathrm{span}\{\mathbf{e}_1, \mathbf{e}_3\}$?
(6) What is the area dilation factor of $\mathbf{F}$ for surfaces orthogonal to $\mathbf{e}_1$?
(7) What is the area dilation factor of $\mathbf{F}$ for surfaces orthogonal to $-7\mathbf{e}_3$?