EXERCISES ON LINEAR ALGEBRA

Problem 1. For each of the following matrices $\mathbf{A}$ write the transpose matrix $\mathbf{A}^\top$, the matrix $-2\mathbf{A}$, compute the determinant $\det \mathbf{A}$, the determinant $\det(\mathbf{A}^4)$, the determinant $\det -2\mathbf{A}$, the rank $\mathrm{rank}\mathbf{A}$, the trace $\mathrm{tr}\mathbf{A}$, and, if possible, the inverse matrix $\mathbf{A}^{-1}$:

(1)

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 0 & 2 \\ 0 & 5 & 0 \end{bmatrix}$$

(2)

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & 3 \\ 4 & 1 & 1 \end{bmatrix}$$

(3)

$$\mathbf{A} = \begin{bmatrix} -8 & 3 & -1 \\ 0 & 0 & 1 \\ 5 & 7 & 2 \end{bmatrix}$$

(4)

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & -2 \\ 2 & -1 & 5 \end{bmatrix}$$

(5)

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For each of the previous matrices $\mathbf{A}$, tell which of the following vectors are in the image of $\mathbf{A}$ or in the null space of $\mathbf{A}$:

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Problem 2. For each of the following pairs of matrices $\mathbf{B}$ and $\mathbf{C}$ write the matrices $\mathbf{B}+\mathbf{C}$, $\mathbf{B}-\mathbf{C}$, $\mathbf{BC}$ and $\mathbf{CB}$:

(1)
$$\mathbf{B} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & -2 \\ 3 & -1 & 1 \end{bmatrix}$$

(2)
$$\mathbf{B} = \begin{bmatrix} -1 & 0 & 0 \\ 3 & 0 & -1 \\ 2 & -1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 0 & -2 & 1 \\ 3 & 2 & 3 \\ -1 & 3 & -2 \end{bmatrix}$$

Problem 3. For each of the following pairs of vectors tell if they are linearly independent or orthogonal. Write the norm of each vector of each pair.

(1)
$$\mathbf{u} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}$$

(2)
$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

(3)
$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

(4)
$$\mathbf{u} = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1/3 \\ 0 \end{bmatrix}$$

Problem 4. Which of the following sets of vectors are not linear subspaces of $\mathbb{R}^3$? Explain why.
For each of them that is a linear subspace of $\mathbb{R}^3$, write a basis.

(1)
$$\mathcal{V} = \left\{ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} : 2v_1 = 0 \right\}$$

(2)
$$\mathcal{V} = \left\{ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} : v_2 = v_3 = -3 \right\}$$

(3)
$$\mathcal{V} = \left\{ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} : v_1 v_3 = 2 \right\}$$

(4)
$$\mathcal{V} = \left\{ \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} : v_2^2 - v_3 = 0 \right\}$$

Problem 5. Consider the following matrices

$$\mathbf{A} = \begin{bmatrix} 0 & a & -2a \\ 1 & -1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b & 0 & 2-2b \\ b-1 & b+1 & 0 \\ 0 & 0 & b \end{bmatrix} \qquad \text{with } a \in \mathbb{R} \text{ and } b \in \mathbb{R}$$

(1) For which values of a and b the matrices $\mathbf{A}$ and $\mathbf{B}$ are invertible?

(2) For which values of a and b the matrices $\mathbf{A}$ and $\mathbf{B}$ are symmetric?

(3) For which values of a and b the matrices $\mathbf{A}$ and $\mathbf{B}$ are diagonal?

(4) Depending on a and b, study the rank of $\mathbf{A}$ and $\mathbf{B}$.

Problem 6. Compute the eigenvalues of the following matrices and, for each of them, find a basis of the associated eigenspace:

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$$

Problem 7. Consider the two following bases of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$\mathcal{C} = \left\{ \mathbf{b}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \ \mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(1) Is $\mathcal{C}$ an orthonormal basis of $\mathbb{R}^3$?

(2) Write the change of basis matrices $\mathbf{M}_{\mathcal{B}}^{\mathcal{C}}$ (from the basis $\mathcal{C}$ to the basis $\mathcal{B}$) and $\mathbf{M}_{\mathcal{C}}^{\mathcal{B}}$ (from the basis $\mathcal{B}$ to the basis $\mathcal{C}$).

(3) Write the vector $\mathbf{v} = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ in the basis $\mathcal{B}$.

(4) Consider the linear application $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\begin{aligned} T(\mathbf{e}_1) &= \mathbf{e}_1 \\ T(\mathbf{e}_2) &= -\mathbf{e}_1 + \mathbf{e}_2 \\ T(\mathbf{e}_3) &= \mathbf{e}_2 + \mathbf{e}_3 \end{aligned}$$

and write the matrix $\mathbf{T}_{\mathcal{C}}$ that represents T in the basis $\mathcal{C}$ and the matrix $\mathbf{T}_{\mathcal{B}}$ that represents T in the basis $\mathcal{B}$.