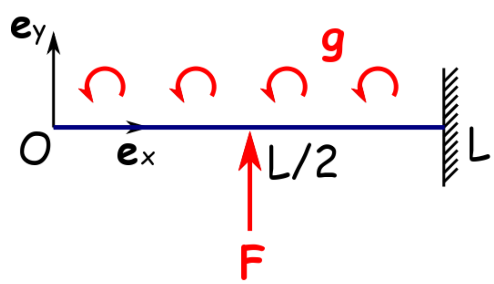
# EXERCISES ON BEAMS

**Problem 1.** A uniformly elastic beam with $EI_2 = B$ and with length $L$ is clamped at one end and free at the other end. The middle point of the beam is subjected to a concentrated force $\mathbf{F} = F\mathbf{e}_y$ (with $F > 0$), and on the entire beam is exerted a distributed couple force $\mathbf{g}(s) = 2F\mathbf{e}_z$, as shown in the following picture:




(1) What are the boundary conditions the beam is subjected to? What are the conditions the beam at subjected to at its middle point?

(2) Write the balance equations for forces and for couples at the equilibrium configuration of the beam.

(3) Write explicitly the function $y(x)$ which represents the shape of the director of the beam at equilibrium, in case of small deflections.

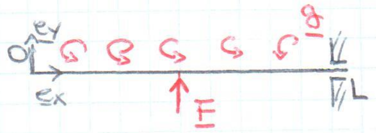
- What is the maximum deflection $\delta_{\max} := \max\{|y(x)|\}$ of the beam, and for which values of $x$ is it attained?

- What is the maximum slope $\theta_{\max} := \max\{|\theta(x)|\}$ of the beam, and for which values of $x$ is it attained?

- What is the condition on the value $F$ that makes the small deflections approximation reliable?

(4) What are the reactions $\mathbf{F_L}$ and $\mathbf{G_L}$ exerted by the clamp on the beam at equilibrium, in case of small deflections?

# EXERCISE:




Beam clamped at $L$ and free at $0$.

Distributed couple $\underline{g}(s) = 2F\underline{e}_z$ $\Big\}$ $F > 0$

At $\frac{L}{2}$ $\quad \underline{F} := F\,\underline{e}_y$

## GOALS:

$y(x)$ with small deflections approximation

$\underline{F}_L$ and $\underline{G}_L$ the clamp exerts on the beam

What is the maximal deflection $\delta_{max} = \max(|y(x)|)$?

Which condition on $F$ makes the small deflections approximation reliable?

## SETUP:

$\underline{d}_2 := \underline{e}_z$

$\underline{d}_3(s) = \underline{t}(s) := \cos\theta(s)\,\underline{e}_x + \sin\theta(s)\,\underline{e}_y$

$\underline{d}_1(s) = \underline{d}_2(s) \times \underline{d}_3(s) = -\sin\theta(s)\,\underline{e}_x + \cos\theta(s)\,\underline{e}_y$

$\partial_s \underline{d}_2(s) = \underline{0} = \underline{u} \times \underline{d}_2 = u_1(s)\,\underline{d}_3(s) - u_3\,\underline{d}_1(s) \Rightarrow u_1(s) = u_3(s) = 0$

$\partial_s \underline{d}_3(s) = \theta'(s)\left(-\sin\theta(s)\,\underline{e}_x + \cos\theta(s)\,\underline{e}_y\right) = \theta'(s)\,\underline{d}_1(s)$

$= \underline{u} \times \underline{d}_3 = u_2(s)\,\underline{d}_1(s) \Rightarrow u_2(s) = \theta'(s)$

$\gamma_2(s) = B u_2(s) = B\theta'(s)$

$\underline{\gamma}(s) = B\theta'(s)\,\underline{e}_z$

## BOUNDARY CONDITIONS: (ANSWER TO 1)

| | | |
|---|---|---|
| $y(L) = 0$ | $\underline{G}_L = \underline{\gamma}(L)$ To be determined | $\underline{\varphi}(0) = \underline{0}$ |
| $\theta(L) = 0$ | $\underline{F}_L = \underline{\varphi}(L)$ to be determined | $\underline{\gamma}(0) = \underline{0}$ |
| | | $\Rightarrow \theta'(0) = 0$ |

## JUMP CONDITIONS: (ANSWER TO 1)

$[\![\underline{\varphi}(\tfrac{L}{2})]\!] = -\underline{F} = -F\,\underline{e}_y$ $\qquad$ $[\![\underline{\gamma}(\tfrac{L}{2})]\!] = \underline{0}$

## SEGMENTS:

| SEG 1) | SEG 2) | SEG 3) |
|---|---|---|
| $s \in [0, \frac{L}{2})$ | $s = \frac{L}{2}$ | $s \in (\frac{L}{2}, L]$ |

EQUILIBRIUM OF BODY FORCES: (ANSWER TO ②)

SEG 1)

$\varphi'(s) = 0$

$\varphi(s) = \varphi(0) = 0$

SEG 2)

$\varphi(\frac{L^+}{2}) - \underbrace{\varphi(\frac{L^-}{2})}_{0} = -\underline{F}$

$\varphi(\frac{L^+}{2}) = -\underline{F} = -F e_y$

SEG 3)

$\varphi'(s) = 0$

$\varphi(s) = -F e_y$

$\Rightarrow \underline{F}_L = \varphi(L) = -\underline{F} = -F e_y$

EQUILIBRIUM OF BODY COUPLES: (ANSWER TO ②)

SEG 1)

$\gamma'(s) + \underbrace{t(s) \times \varphi(s)}_{0} + g(s) = 0$

$\gamma'(s) = -2F e_z = B\theta''(s) e_z$

$B\theta''(s) = -2F$

$B\theta'(s) = -2Fs + c_1$ and $\theta'(0) = 0$

$\Rightarrow c_1 = 0$

$B\theta'(s) = -2Fs \leq 0 \Rightarrow \theta'$ decreasing

$B\theta(s) = -Fs^2 + c_2$

SEG 2)

$\theta'$ and $\theta$ continuous

SEG 3)

$\gamma'(s) + t(s) \times \varphi(s) + g(s) = 0$

$t(s) \times \varphi(s) = -F\cos\theta(s) e_z$

$\gamma'(s) = \underbrace{(F\cos\theta(s) - 2F)}_{B\theta''(s)} e_z$

$B\theta''(s) = F\cos\theta(s) - 2F$

SMALL DEFLECTIONS APPROXIMATION:

$\theta(s) <<$ FOR ALL $s \in [0, L]$

$$\begin{cases} \cos\theta(s) \sim 1 \\ \sin\theta(s) \sim \theta(s) \end{cases} \Rightarrow t \sim e_x + \theta(s) e_y \Rightarrow \begin{cases} x \sim s \\ y'(x) \sim \theta(x) \end{cases}$$

SEG 3)

$B\theta''(x) \sim F - 2F$

$B\theta'(x) = -F x + c_2$

SEG 2)

$[\![\gamma(\frac{L}{2})]\!] = 0$

$[\![\theta'(\frac{L}{2})]\!] B = 0 \Rightarrow B\theta'$ conti. $\Rightarrow B\theta'(\frac{L^-}{2}) = -FL = B\theta'(\frac{L^+}{2})$

$B\theta'(\frac{L}{2}) = -FL = -F\frac{L}{2} + c_2 \Rightarrow c_2 = -\frac{FL}{2}$

$B\theta'(x) = -F x - \frac{FL}{2} \leq 0 \Rightarrow \theta$ decreasing

$B\theta(x) = \frac{-F}{2} x^2 - \frac{FL}{2} x + c_3$ with $\theta(L) = 0$

$\Rightarrow c_3 = FL^2$

$B\theta(x) = \frac{-F}{2} x^2 - \frac{FL}{2} x + FL^2$

$x = \frac{L \pm \sqrt{L^2 + 8L^2}}{-2} = \frac{L \pm 3L}{-2} \left\langle \begin{matrix} -2L \\ L \end{matrix} \right.$

$y'(x) = \frac{-F}{2B} x^2 - \frac{FL}{2B} x + \frac{FL^2}{B} = \frac{F}{2B}(-x^2 - Lx + 2L^2) = \theta(x) \geq 0$ in $[\frac{L}{2}, L]$

$y(x) = \frac{F}{2B}\left(-\frac{x^3}{3} - \frac{L}{2}x^2 + 2L^2 x\right) + c_4$ with $y(L) = 0 \Rightarrow c_4 = -\frac{FL^3}{2B}\frac{7}{6}$

$$y(x) = \frac{F}{2B}\left(-\frac{x^3}{3} - \frac{L}{2}x^2 + 2L^2x - \frac{7}{6}L^3\right) \qquad x \in \left(\frac{L}{2}, L\right]$$

SEG 2)

$$\theta\left(\frac{L}{2}\right) = \frac{F}{2B}\left(-\frac{L^2}{4} - \frac{L^2}{2} + 2L^2\right) = \frac{5FL^2}{8B}$$

$$y\left(\frac{L}{2}\right) = \frac{F}{2B}L^3\left(-\frac{1}{24} - \frac{1}{8} + 1 - \frac{7}{6}\right) = -\frac{FL^3}{6B}$$

SEG 1)

$$B\theta(x) = -Fx^2 + c_2 \quad \text{with } B\theta\left(\frac{L}{2}\right) = \frac{5FL^2}{8}$$

$$\Rightarrow c_2 = \frac{5}{8}FL^2 + \frac{1}{4}FL^2 = FL^2\frac{7}{8}$$

$$\theta(x) = y'(x) = -\frac{F}{B}x^2 + \frac{F}{B}\frac{7}{8}L^2 \geq 0 \quad \text{for } x \in \left[0, \frac{L}{2}\right) \rightsquigarrow y \text{ increasing}$$

$$y(x) = -\frac{F}{3B}x^3 + \frac{7F}{8B}L^2x + c_5 \quad \text{with } y\left(\frac{L}{2}\right) = -\frac{1}{6}\frac{FL^3}{B}$$

$$\Rightarrow c_5 = \frac{FL^3}{B}\left(-\frac{1}{6} + \frac{1}{24} - \frac{7}{16}\right) = \frac{FL^3}{B}\frac{-8+2-21}{48} = -\frac{27}{48}\frac{FL^3}{B}$$

$$y(x) = \frac{F}{B}\left(-\frac{x^3}{3} + \frac{7}{8}L^2x - \frac{27}{48}\right)$$

ANSWER TO 3

$$y(x) = \begin{cases} \frac{F}{B}\left(-\frac{x^3}{3} + \frac{7}{8}L^2x - \frac{27}{48}\right) & x \in \left[0, \frac{L}{2}\right] \\ \frac{F}{2B}\left(-\frac{x^3}{3} - \frac{L}{2}x^2 + 2L^2x - \frac{7}{6}L^3\right) & x \in \left[\frac{L}{2}, L\right] \end{cases}$$

ANSWER TO 4

$$\underline{F}_L = -F\underline{e}_y$$

$$\underline{G}_L = \underline{\gamma}(L) = B\theta'(L)\,\underline{e}_z = -\frac{3}{2}FL\,\underline{e}_z$$

ANSWER TO 5

$$y(x) \text{ increasing and } y(L) = 0 \Rightarrow \delta_{max} = |y(0)| = \frac{27}{48}\frac{F}{B}$$

ATTAINED at $x=0$

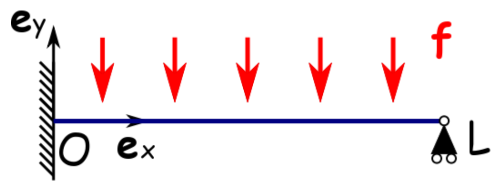
ANSWER TO 3:

$\theta$ decreasing $\Rightarrow |\theta|_{max} = |\theta(0)| = \frac{7}{8}\frac{F}{B}L^2 \ll 1$

ATTAINED AT $x=0$

$\Rightarrow F \ll \frac{8B}{7L^2}$

**Problem 2.** A uniformly elastic beam with $EI_2 = B$ and with length $L$ is clamped at one end and supported by a roller (the beam can rotate and translate horizontally) at the other end. The beam is also subjected to a distributed force $\mathbf{f}(s) = -\dfrac{F}{L}\mathbf{e}_y$ (with $F > 0$), as shown in the following picture:



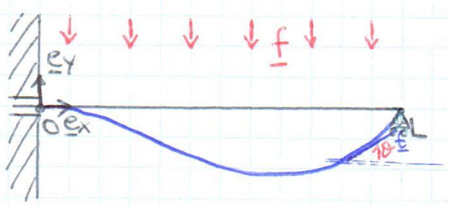

(1) What are the boundary conditions the beam is subjected to?

(2) Write the balance equations for forces and for couples at the equilibrium configuration of the beam.

(3) Write explicitly the function $y(x)$ which represents the shape of the director of the beam at equilibrium, in case of small deflections.

(4) What are the reactions $\mathbf{F_O}$ and $\mathbf{G_O}$ exerted by the clamp on the beam at equilibrium and what is the reaction $\mathbf{F_L}$ exerted by the roller on the beam at equilibrium, in case of small deflections?

# EXERCISE




$\underline{f}(s) = -\frac{F}{L}\,\underline{e}_y$

($y$ and $\theta$ continuous)

CLAMP + ROLLER

## GOALS:

1. Boundary conditions the beam is subjected to.
2. Equilibrium equations for forces and torques.
3. Shape $y(x)$ (with small deflections approximations)
4. Reactions $\underline{F}_0$, $\underline{G}_0$ and $\underline{F}_L$.

## SETUP:

$\underline{d}_2(s) := \underline{e}_z$

$\underline{d}_3(s) = \underline{t}(s) = \cos\theta(s)\,\underline{e}_x + \sin\theta(s)\,\underline{e}_y$

$\underline{d}_1(s) = -\sin\theta(s)\,\underline{e}_x + \cos\theta(s)\,\underline{e}_y$

$\rightsquigarrow u_1 = u_3 = 0$ and $u_2 = \theta' \quad \Rightarrow \underline{\gamma}(s) = B\theta'(s)\,\underline{e}_z$

## BOUNDARY CONDITIONS: (1)

$y(0) = 0$

$\theta(0) = 0$

$\left.\begin{array}{l}\underline{F}_0 = -\underline{\varphi}(0)\\ \underline{G}_0 = -\underline{\gamma}(0)\end{array}\right\}$ UNKNOWN

$y(L) = 0$

$\underline{\gamma}(L) = \underline{0} \Rightarrow \theta'(L) = 0$

$\underline{F}_L = \underline{\varphi}(L)$ } UNKNOWN but $\underline{F}_L \parallel \underline{e}_y$

## EQUILIBRIUM EQUATION FOR FORCES: (2)

$$\underline{\varphi}'(s) = \frac{F}{L}\,\underline{e}_y \rightsquigarrow \begin{cases}\varphi_x'(s) = 0\\ \varphi_y'(s) = \frac{F}{L}\end{cases} \Rightarrow \begin{cases}\varphi_x(s) = C_x\\ \varphi_y(s) = \frac{F}{L}s + C_y\end{cases}$$

with $\underline{\varphi}(L) \parallel \underline{e}_y$
$\Rightarrow C_x = 0$

$$\underline{\varphi}(s) = \left(\frac{F}{L}s + C_y\right)\underline{e}_y$$

## EQUILIBRIUM EQUATION FOR TORQUES: (2)

$$B\theta''(s)\,\underline{e}_z + \left(\frac{F}{L}s + C_y\right)\cos\theta(s)\,\underline{e}_z = \underline{0}$$

## SMALL DEFLECTIONS APPROXIMATION:

$$|\theta(s)| \ll 1 \quad \rightsquigarrow \quad \underline{t}(s) \sim \underline{e}_x + \theta(s)\underline{e}_y \quad \rightsquigarrow \quad \begin{cases} x = s \\ y'(x) = \theta(x) \end{cases}$$

$$B\theta''(x) = -\frac{F}{L}x - c_y$$

$$B\theta'(x) = -\frac{F}{2L}x^2 - c_y x + c_1 \qquad \text{with } \theta'(L) = 0$$

$$\Rightarrow c_1 = \left(\frac{F}{2} + c_y\right)L$$

$$B\theta(x) = -\frac{F}{6L}x^3 - \frac{c_y}{2}x^2 + \left(\frac{F}{2} + c_y\right)Lx + c_2 \qquad \text{with } \theta(0) = 0 \Rightarrow c_2 = 0$$

$$y'(x) = \theta(x) = -\frac{F}{6BL}x^3 - \frac{c_y}{2B}x^2 + \left(\frac{F}{2B} + \frac{c_y}{B}\right)Lx$$

3 $$y(x) = -\frac{F}{24BL}x^4 - \frac{c_y}{6B}x^3 + \left(\frac{F}{4B} + \frac{c_y}{2B}\right)Lx^2 + c_3 \qquad \text{with } y(0) = 0 \Rightarrow c_3 = 0$$

$$= \frac{1}{24BL}x^2\left(-Fx^2 - 4c_yLx + (6F + 12c_y)L^2\right) \qquad \text{with } y(L) = 0$$

$$\Rightarrow -FL^2 - 4c_yL^2 + 6FL^2 + 12c_yL^2 = 0$$

$$\Rightarrow 8c_y = -5F \Rightarrow c_y = -\frac{5}{8}F$$

$$\rightsquigarrow y(x) = \frac{F}{48BL}x^2\left(-2x^2 + 5Lx - 3L^2\right)$$

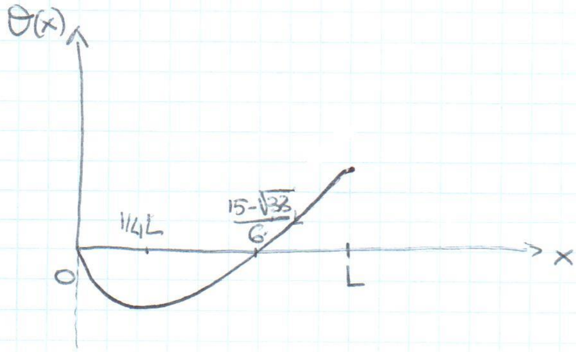
$$\theta(x) = \frac{F}{48BL}x\left(-8x^2 + 15Lx - 6L^2\right)$$

$$\theta'(x) = \frac{F}{48BL}\left(-24x^2 + 30Lx - 6L^2\right) = \frac{F}{24BL}\left(-12x^2 + 15Lx - 3L^2\right)$$

* $y(x) = 0$ for $x = 0$ or $x = L\,\dfrac{-5 \pm \sqrt{25 - 24}}{-4} = \begin{cases} \frac{3}{2}L > L \\ L \end{cases}$

$$\rightsquigarrow y(x) \leq 0, \quad x \in [0, L]$$

* $\theta(x) = 0$ for $x = 0$ or $x = L\,\dfrac{-15 \pm \sqrt{225 - 192}}{-16} = \dfrac{+15 \pm \overset{5,74}{\sqrt{33}}}{16}L \begin{cases} \frac{15+\sqrt{33}}{16}L > L \\ \frac{15-\sqrt{33}}{16}L \in (0, L) \end{cases}$




* $\theta'(x) = 0$ for

$$x = \frac{15 \pm \sqrt{\overset{81}{225 - 144}}}{24} \begin{cases} L \\ \frac{1}{4}L \end{cases}$$

$\Rightarrow$ $y$ decreasing in $\left[0, \frac{15-\sqrt{33}}{16}L\right]$ and then increasing

$$\delta_{max} = \left| y\left(\frac{15-\sqrt{33}}{16}L\right)\right| \simeq \qquad \text{at } x = \frac{15-\sqrt{33}}{16}L$$

$$\Rightarrow \quad \vartheta_{max} = \max\left(|\vartheta(\tfrac{1}{4}L)|, \vartheta(L)\right) =$$

$$= \max\left(\frac{11}{16}\,\frac{1}{48}\,\frac{F}{B}L^2, \frac{1}{48}\,\frac{F}{B}L^2\right) = \frac{FL^2}{48B} \quad \text{at } L$$

$$\vartheta\left(\frac{1}{4}L\right) = \frac{F}{4\cdot 48B}\left(-\frac{8}{16}L^2 + \frac{15}{4}L^2 - 6L^2\right) =$$

$$= \frac{F}{4\cdot 16\cdot 48B}L^2(-8+60-96) = -\frac{11\,FL^2}{768\;B}$$

$$\vartheta(L) = \frac{F}{48B}L^2$$

$$\rightsquigarrow F \ll \frac{48B}{L^2}$$

4

$$\underline{F}_L = \underline{\varphi}(L) = \frac{3}{8}F\underline{e}_y$$

$$\underline{F}_0 = -\underline{\varphi}(0) = \frac{5}{8}F\underline{e}_y$$

$$\underline{G}_0 = -\underline{\gamma}(0) = -B\vartheta'(0)\underline{e}_z = \frac{1}{8}FL\,\underline{e}_z$$