Probabilistic reasoning:
representation & inference

Marco Piastra

Probability Space

Probability: events as *subsets of possible worlds*

- ***Probability space***

A triple $\langle W, \Sigma, P \rangle$

Possible worlds (a.k.a. Sample Space)

Event Space (a collection of subsets over W)

Probability Measure $P : \Sigma \rightarrow [0, 1]$

The intuitive definition is simple enough, its mathematical translation ... not so much

Probability: events as *subsets of possible worlds*

- Boolean algebra

  A non-empty collection of subsets Σ of a set W such that:

  1) $A, B \in \Sigma \implies A \cup B \in \Sigma$
  2) $A \in \Sigma \implies A^c \in \Sigma$
  3) $\varnothing \in \Sigma$

  Corollary:
  The sets $\varnothing$ e W belong to any Boolean algebra generated on W
  Σ is also closed under *binary intersection*

- σ-algebra

  A non-empty collection of subsets Σ of a set W such that:

  1) $A_k \in \Sigma, \; \forall k \in \mathbb{N}^+ \implies (\bigcup_{k=1}^{\infty} A_k) \in \Sigma$
  2) $A \in \Sigma \implies A^c \in \Sigma$
  3) $\varnothing \in \Sigma$

  This is a stronger requirement: closeness under countable union Hence a σ-algebra is a boolean algebra but not vice-versa

  Corollary:
  The sets $\varnothing$ and W belong to any σ- algebra generated on W
  Σ is also closed under *countable intersection*

Probability: events as *subsets of possible worlds*

- σ-algebra (*Event Space*)

  A non-empty collection of subsets Σ of a set W such that:

  1) $A_k \in \Sigma,\ \forall k \in \mathbb{N}^+ \implies (\bigcup_{k=1}^{\infty} A_k) \in \Sigma$
  2) $A \in \Sigma \implies A^c \in \Sigma$
  3) $\varnothing \in \Sigma$

- Probability <u>*measure*</u> over a σ-algebra (i.e., over the events)

  A function $P : \Sigma \to [0, 1]$

  i.e. P assigns a measure (i.e. a real number)
  to each elements of a *σ-algebra* Σ of subsets of W

  1) $\forall A \in \Sigma, P(A) \geq 0$
  2) $A_k \in \Sigma,\ \forall k \in \mathbb{N}^+$ are <u>*disjoint*</u> $\implies P(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} P(A_k)$
  3) $P(\varnothing) = 0$
  4) $P(A^c) = 1 - P(A)$ (which implies $P(W) = 1$)

Probability: events as *subsets of possible worlds*

- ***Probability space***

A triple $\langle W, \Sigma, P \rangle$

Possible worlds — W

Event Space (*a* σ*-algebra over* W) — Σ

Probability Measure $P : \Sigma \to [0, 1]$ — P

Why bothering so much with these (very) technical definitions?

- **Rationale** *(just a few hints)*

Closure w.r.t. *countable unions* of a σ-algebras (as well as *countable additivity* of P) is required for dealing with *infinite sequences of events*

In such case, assuming *countable* union and additivity is a *restriction*, to ensure *measurability*

(see the so-called Banach-Tarski paradox for counterexamples)

Probability: events as *subsets of possible worlds*

- *Probability measure* over a σ-algebra
- *Disjoint* events

In general

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If $A \cap B = \varnothing$ then events *A* and *B* are *disjoint*

$$P(A \cup B) = P(A) + P(B)$$

(*) Note that $A \cap B = \varnothing \implies P(A \cap B) = 0$
but not vice-versa: as an event can have zero probability without being empty

(**) Unlike in propositional logic, knowing $P(A)$ and $P(B)$ is not sufficient for determining $P(A \cup B)$

Namely, probability is not *compositional* …

Discrete Probability

Studying basic properties: *a finitary setting*

A simpler setting that allows a more intuitive definition of fundamental properties

- Finite *event space*

  Σ is a finite collection of subsets

  In this setting

  boolean algebra ≡ σ-algebra

  Events could also be defined via propositional logic

  (à la de Finetti, 1937)

- Finitely additive *probability measure*

  Just summations, no integrals

  Computability will be always guaranteed

Partitions, random variables*

(*) In a finitary setting

- Partition

  A *finite* collection A_i of *disjoint* subsets (i.e. events) such that

$$\bigcup_i A_i = W$$

A σ-algebra can be generated from a *partition*
by taking its closure under union and complement

Random Variables

Partitions, random variables*

- **Random Variable** (i.e. a convenient way to define a *σ-algebra*)

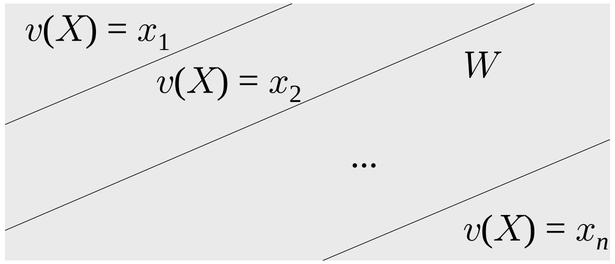
  Let X be a variable having a *finite* set of possible values $\{x_1, x_2, \ldots, x_n\}$
  In each possible world, the variable X is assigned a specific value x_i

  - The set of possible assignments $\{X = x_1, X = x_2, \ldots X = x_n\}$ defines a *partition* of W
  - A *σ-algebra* can be obtained by taking the closure of the partition under union and complement

  - $X = x_i$ defines an *event* (i.e. a subset of W)
  - $X = x_i$ and $X = x_j$ are *disjoint events*, whenever $i \neq j$

$$P(X = x_i \cup X = x_j) = P(X = x_i) + P(X = x_j)$$

Random variables having binary values are also said to be binomial (also Bernoullian)
Random variables with multiple values are also said to be multinomial



Random variables, joint distribution*

(*) In a finitary setting

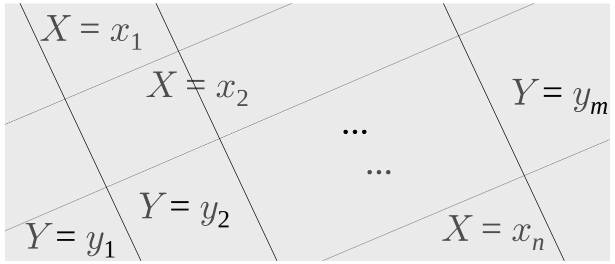
Multiple random variables

In practice, in a probabilistic representation, there will be *multiple* *random variables*

Example:
X_i occurrence of a *word* i in the body of an email (binomial)
Y classification of that email as *spam* (binomial)

The intersection of two or more σ-algebras is a σ-algebra

Together, a collection of random variables defines a partition of W



- Joint Probability Distribution

for a given set of random variables, e.g. X, Y, Z

It is a *function* that associates a value in $[0, 1]$ to each individual combination of values

$$P(X = x, Y = y, Z = z)$$

Given that X, Y e Z define each a *partition* of W:

$$\sum_x \sum_y \sum_z P(X = x, Y = y, Z = z) = 1$$

Random variables: notation

(*) In a finitary setting

- Random variables, events and σ-algebras

  Sometimes the notation can be ambiguous

  Examples:

$$P(X)$$

This is the probability measure over the σ-algebra *generated by the* random variable X

$$P(X = x)$$

This the probability (i.e. a value in $[0,1]$ *) associated to the* event $X = x$

$$P(X, Y = y)$$

This is the probability measure over the σ-algebra *generated by the* random variable X *in* the subspace of W corresponding to the event $Y = y$

Fundamental Operations

Marginalization

(*) In a finitary setting

Removing a random variable from a joint distribution

Given a joint probability distribution

$$P(X = x, Y = y)$$

The *marginal probability* $P(X = x)$ is obtained via summation:

$$P(X = x) \;=\; \sum_{y} P(X = x, Y = y)$$

A marginal probability can be a joint probability too …

Marginal probability of an event (shorthand notation, values of Y omitted):

$$P(X = x) \;=\; \sum_{Y} P(X = x, Y)$$

Marginal probability of a σ-algebra (shorthand notation, values of Y omitted):

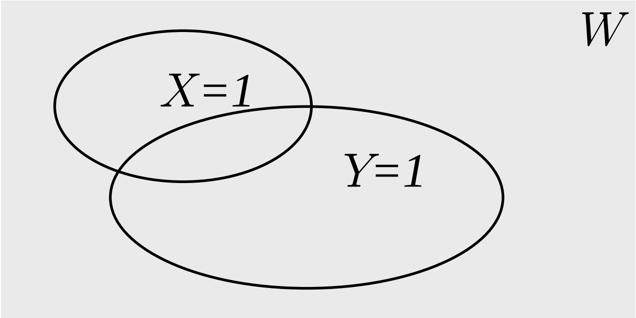
$$P(X) \;=\; \sum_{Y} P(X, Y)$$

Conditional probability

(*) In a finitary setting

- Definition

$$P(X|Y=y) \ := \ \frac{P(X,Y=y)}{P(Y=y)}$$



It is a form of *inference*: from a set W to a set W'

i.e., from a probability space to another probability space

Example: W is the set of possible worlds, X, Y are binary random variables and $P(X, Y)$ is the joint probability distribution

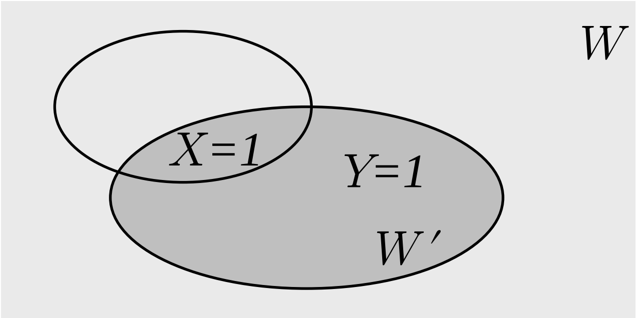
Suppose the agent learns that event $Y = 1$ has occurred:
the event $Y = 0$ is then *impossible* (to him/her)

$W' \ := \ \{w \in W | Y = 1\}$ is the new set of possible worlds

$P(X|Y=1)$ is the new probability of X

More in general

$$P(X|Y) \ := \ \frac{P(X,Y)}{P(Y)}$$



Denotes the conditional probabilities for the whole σ-algebra of events generated by Y *(i.e. a family of probability measures)*

Bayes' Theorem (T. Bayes, 1764)

- Definition

A relation between conditional and marginal probabilities

$$P(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$$

$P(Y|X)$ is also called *likelihood* $L(X|Y)$

The theorem follows from the definition of conditional probability *(chain rule)*

$$P(X,Y) = P(X|Y)P(Y) = P(Y|X)P(X)$$

Furthermore, given the definition of marginalization:

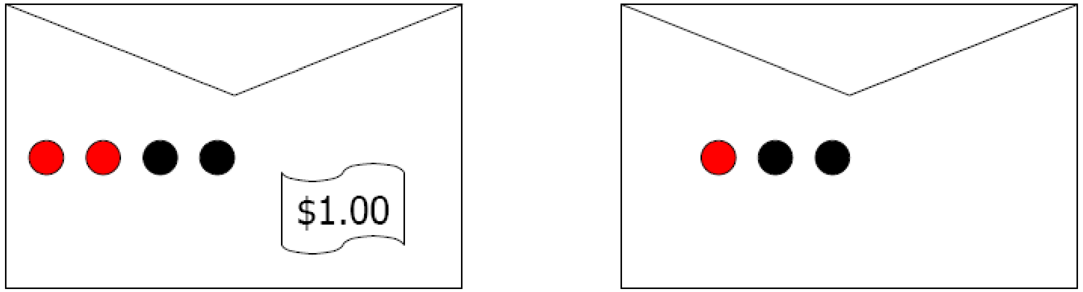
$$P(Y) = \sum_X P(X,Y) = \sum_X P(Y|X)P(X)$$

Also called 'law of total probability'

it follows an alternative formulation of the Bayes' theorem:

$$P(X|Y) = \frac{P(Y|X)P(X)}{\sum_X P(Y|X)P(X)}$$

Example: information and bets



- Two envelopes, only one is extracted

    One envelope contains two red tokens and two black tokens, it is worth $1.00
    One envelope contains one red token and two black tokens, it is valueless

  The envelope has been extracted.

  Before posing you bet, you are allowed to extract on token from it

  a) The token is black. How much do you bet ?

  b) The token is red. How much do you bet ?

Purpose: showing that Bayes' Theorem makes the representation easier

Independence

Independence, conditional independence

- **Independence** (also *marginal independence*)

  Two events are independent
  iff their joint probability is equal to the product of the marginals

$$< X \perp Y > \;\Rightarrow\quad P(X,Y) = P(X)P(Y)$$

$$\Rightarrow\quad P(X|Y) = \frac{P(X,Y)}{P(Y)} = \frac{P(X)P(Y)}{P(Y)} = P(X)$$

- **Conditional independence**

  Two events are conditional independent, given a third event,
  iff their joint conditional probability is equal to the product of the *conditional marginals*

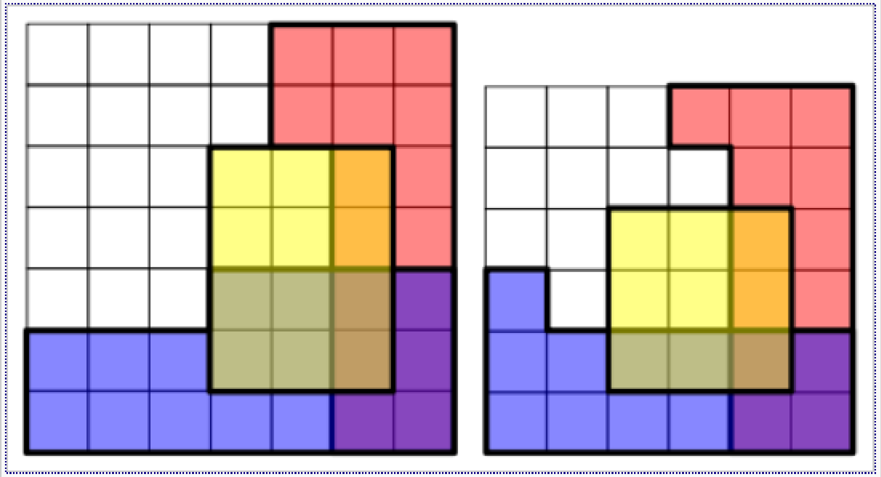
$$< X \perp Y | Z > \;\Rightarrow\quad P(X,Y|Z) = P(X|Z)P(Y|Z)$$

$$\Rightarrow\quad P(X|Y,Z) = \frac{P(X,Y|Z)}{P(Y|Z)} = \frac{P(X|Z)P(Y|Z)}{P(Y|Z)} = P(X|Z)$$

CAUTION: *the two forms of independence are distinct!*

$$< X \perp Y > \;\not\Rightarrow\; < X \perp Y | Z >, \quad < X \perp Y | Z > \;\not\Rightarrow\; < X \perp Y >$$

Independence, conditional independence



These are two examples illustrating **conditional independence**. Each cell represents a possible outcome. The events *R*, *B* and *Y* are represented by the areas shaded red, blue and yellow respectively. And the probabilities of these events are shaded areas with respect to the total area. In both examples *R* and *B* are conditionally independent given *Y* because:

$$\Pr(R \cap B \mid Y) = \Pr(R \mid Y)\Pr(B \mid Y)$$ [1]

but not conditionally independent given not *Y* because:

$$\Pr(R \cap B \mid \text{not } Y) \neq \Pr(R \mid \text{not } Y)\Pr(B \mid \text{not } Y).$$

[from Wikipedia, "Conditional Independence"]

R, B and Y here are subsets, i.e. <u>events</u>, not random variables

The example above shows that (marginal or conditional) independence of two specific <u>events</u> does NOT imply (marginal or conditional) independence of the whole σ-algebras

Inference
(without *learning*)

Probabilistic Inference (no *learning*)

- General structure of probabilistic inference problems

  The starting point is a fully-specified joint probability distribution

  $$P(X_1, X_2, \ldots, X_n)$$

  In an *inference* problem, the set of random variables $\{X_1, X_2, \ldots, X_n\}$
  is divided into three categories:

  1) *Observed variables* $\{X_o\}$, i.e. having a definite (and certain) value
  2) *Irrelevant variables* $\{X_i\}$, i.e. which are not directly part of the answer
  3) *Relevant variables* $\{X_r\}$, i.e. which are part of the answer we seek

  In general, the problem is finding:

  $$P(\{X_r\}|\{X_o\}) = \sum_{\{X_i\}} P(\{X_r\}, \{X_i\}|\{X_o\})$$

  - "Decidability" (actually "computability") is not an issue (*in a finitary setting)

    Given that the joint probability distribution is completely specified

  - Computational efficiency can be a problem

    The number of value combinations grows exponentially
    with the number of random variables

A few more concepts

Continuous random variables (hints)

Although intuitively similar, dealing with continuous random variables is technically difficult

Consider a **continuous** random variable $X \in \mathcal{X}$

A continuous domain e.g. the real interval $[0, 1]$

$X = x$ does not describe a proper *event*

For technical reasons (i.e. *measurability*), a point must have probability zero

Events need to be *subsets*, or better, intervals:

$X \leq a \, , X \leq b \, , \quad a < X \leq b$

Assuming $a < b$

Probability measures these *subsets*

$$P(X \leq b) = P(X \leq a) + P(a < X \leq b)$$

These two events are disjoint

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a)$$

Sometimes written also as (see next slide)

Density and Cumulative Distribution

- **Probability Density Function (pdf)**

Assume that the derivative $p(X) := \frac{dP(X)}{dX}$ *exists everywhere*

It is due to be non-negative

$$p(X = x) \geq 0$$

usually written as $p(x) \geq 0$

- **Probability Measure as Cumulative Distribution Function (CDF)**

cumulative distribution function (cdf)

$$P(a < X \leq b) := \int_a^b p(x)\, dx$$

probability density function (pdf)

As a probability measure, it must integrate to unity

$$P(W) = \int_{x \in \mathcal{X}} p(x)\, dx = 1$$

Note that $p(x)$ *may well be above* 1 *(it is its integral that equals unity)*

Expected value of a random variable

(also *expectation*)

Basic definition

$$\mathbb{E}_X[X] := \sum_{x \in \mathcal{X}} x \; P(X = x)$$

More concise notation

$$\mathbb{E}[X] := \sum_{x \in \mathcal{X}} x \; P(x)$$

Continuous case

$$\mathbb{E}[X] := \int_{x \in \mathcal{X}} x \; p(x) dx$$

Expectation is a linear operator

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\mathbb{E}[cX] = c\mathbb{E}[X]$$

Conditional expectation

$$\mathbb{E}_X[X|Y = y] = \mathbb{E}[X|Y = y] := \sum_{x \in \mathcal{X}} x \; P(X = x|Y = y)$$

Variance of a random variable

Basic definition

$$\mathrm{Var}(X) := \mathbb{E}_X[(X - \mathbb{E}_X[X])^2] = \mathbb{E}_X[(X - \mu_X)^2]$$

where $\mu_X := \mathbb{E}_X[X]$

$$\mathrm{Var}(X) := \sum_{x \in \mathcal{X}} P(X = x)\ (x - \mu)^2$$

variance is *not* a linear operator

Conditional variance

$$\mathrm{Var}(X|Y = y) := \mathbb{E}_X[(X - \mathbb{E}_X[X|Y = y])^2\ |Y = y]$$

Variance lemma

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}[(X - \mu_X)^2] = \mathbb{E}[X^2] - 2\mu_X \mathbb{E}[X] + \mu_X^2 \\ &= \mathbb{E}[X^2] - 2\mu_X^2 + \mu_X^2 = \mathbb{E}[X^2] - \mu_X^2\end{aligned}$$

$$\mathbb{E}[X^2] = \mu_X^2 + \sigma_X^2$$

where $\sigma_X := \sqrt{\mathrm{Var}(X)}$ *standard deviation*