# Semi-Decidability of First Order Logic

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# Decidability and automation of $L_{FO}$

- $L_{FO}$ is <u>not</u> decidable

  No Turing machine can tell whether $\Gamma \models \varphi$

  *Are there any hopes for automating the calculus?*

- $L_{FO}$ is ***semi-decidable*** **(Herbrand, 1930)**

  In general, Turing machine can tell (in *finite* time) that

  $$\Gamma \models \varphi$$

  ... but <u>not</u> that

  $$\Gamma \not\models \varphi$$

  In other words, the above Turing machine, when facing the problem "$\Gamma \models \varphi$ ?" :

  1) it will terminate with success if $\Gamma \models \varphi$
  2) it <u>*might*</u> diverge if $\Gamma \not\models \varphi$

# Herbrand's System

Given a universal sentence of the form:

$\forall x_1 \forall x_2 \ldots \forall x_n \varphi$ (where $\varphi$ does not contain quantifiers)

the ***Herbrand's System*** is the set (possibly *infinite*) of *ground* wffs generated by replacing the variables

$\varphi[x_1/t_1, x_2/t_2 \ldots x_n/t_n]$

with all possible combinations of *ground* terms $<t_1, t_2 \ldots t_n>$ of the *signature* $\Sigma$

A term (or a wff ) is *ground* if it does not contain *variables*

Examples:

$H(\forall x\, P(x) \rightarrow Q(x))) = \{P(f(a)) \rightarrow Q(f(a)), P(g(a, b)) \rightarrow Q(g(a, b)), \ldots \}$

$H(\forall x\, \forall y\, R(x, y)) = \{R(f(a), f(a)), R(g(a, b), f(a)), R(f(a), g(a, b)), \ldots \}$

- ***Herbrand's System*** of a theory

Given a theory $\Phi$ of universal sentences, the Herbrand's system $H(\Phi)$ is the union of all Herbrand's systems for the sentences in $\Phi$

Example:

$\Phi = \{\varphi, \psi, \chi\}$

$H(\Phi) = H(\psi) \cup H(\varphi) \cup H(\chi)$

# Herbrand's Theorem

- **Herbrand's Theorem**

  Given a theory of universal sentences $\Phi$,
  $H(\Phi)$ has a model iff $\Phi$ has a model

  ... but what is the utility of that?

  $H(\Phi)$ *may well be infinite even when* $\Phi$ *is finite,*
  *Furthermore, the theorem applies only to sets of* <u>*universal*</u> *sentences…*

# Prenex normal form (PNF)

Any wff $\varphi$ can be transformed into an equivalent formula of the form

$$Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \, \psi$$

($\psi$ is called the ***matrix***)

where $Q_i$ is either $\forall$ or $\exists$ and $\psi$ does not contain quantifiers

1) Replace $\rightarrow$ and $\leftrightarrow$ :

$$\varphi \leftrightarrow \psi \quad \Leftrightarrow \quad (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$$

$$\varphi \rightarrow \psi \quad \Leftrightarrow \quad (\neg\varphi \vee \psi)$$

2) Push negation $\neg$ inwards, as much as possible:

$$\neg(\varphi \wedge \psi) \quad \Leftrightarrow \quad (\neg\varphi \vee \neg\psi)$$

$$\neg(\varphi \vee \psi) \quad \Leftrightarrow \quad (\neg\varphi \wedge \neg\psi)$$

$$\neg\neg\varphi \quad \Leftrightarrow \quad \varphi$$

$$\neg\forall x\, \varphi \quad \Leftrightarrow \quad \exists x\, \neg\varphi$$

$$\neg\exists x\, \varphi \quad \Leftrightarrow \quad \forall x\, \neg\varphi$$

3) Move all quantifiers outwards, respecting order

CAUTION: *variables MUST be renamed - when required - to avoid name clashes*

# Prenex normal form (PNF)

Any wff $\varphi$ can be transformed into an equivalent formula of the form

$Q_1x_1Q_2x_2 \dots Q_nx_n \, \psi$ ($\psi$ is called the ***matrix***)

where $Q_i$ is either $\forall$ or $\exists$ and $\psi$ does not contain quantifiers

Examples:

$\exists y \, (P(y) \rightarrow \forall x \, P(x))$
$\exists y \forall x \, (\neg P(y) \vee P(x))$ PNF

$\exists y \, (\forall x \, P(x) \rightarrow P(y))$
$\exists y \, (\neg \forall x \, P(x) \vee P(y))$
$\exists y \exists x \, (\neg P(x) \vee P(y))$ PNF

$\forall x \exists y \, (Q(x,y) \rightarrow P(y)) \wedge \neg \forall x \, P(x)$
$\forall x \exists y \, (\neg Q(x,y) \vee P(y)) \wedge \exists x \, \neg P(x)$
$\forall x \exists y \, (\neg Q(x,y) \vee P(y)) \wedge \exists z \, \neg P(z)$ (renaming variable)
$\forall x \exists y \exists z \, ((\neg Q(x,y) \vee P(y)) \wedge \neg P(z))$ PNF

# Skolemization

In a sentence in PNF, existential quantifiers can be eliminated by extending the *signature* $\Sigma$ of the *language*

Consider a sentence in PNF $Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \psi$

From left to right, for each $Q_i x_i$ of type $\exists x_i$ :

- Apply to $\psi$ the *substitution* $[x_i / k(x_1, \ldots, x_j)]$
  where $k$ is a <u>new</u> function and $x_1, \ldots, x_j$ are the $j$ variables of the universal quantifiers that come *before* $\exists x_i$ ($k$ is an individual constant if $j = 0$)
- $\exists x_i$ is simply removed

Examples:

$\exists y \, \forall x \, (\neg P(y) \lor P(x))$

$\quad \forall x \, (\neg P(k) \lor P(x))$ ($k$ Skolem's constant)

$\forall x \, \exists y \, \exists z \, ((\neg Q(x,y) \lor P(y)) \land \neg P(z))$

$\quad \forall x \, ((\neg Q(x, k(x)) \lor P(k(x))) \land \neg P(m(x)))$ ($k/1$ and $m/1$ Skolem's functions)

- Theorem

For any sentence $\varphi$

$\varphi$ has a model iff $sko(\varphi)$ (i.e. Skolemization of $\varphi$) has a model

# Semi-decidability of $L_{FO}$

- Corollary of Herbrand's theorem

  For any set of sentences $\Gamma$ and sentence $\varphi$
  these three statements are equivalent:

  - $\Gamma \models \varphi$
  - $\Gamma \cup \{\neg\varphi\}$ is not satisfiable (= it has no model)
  - There exists a ***finite*** subset of $\mathrm{H}(sko(\Gamma \cup \{\neg\varphi\}))$ (= Herbrand's system of the Skolemization of $\Gamma \cup \{\neg\varphi\}$) that is ***inconsistent***

Therefore:

When $\Gamma \models \varphi$, a procedure that generates the finite *subsets* of $\mathrm{H}(sko(\Gamma \cup \{\neg\varphi\}))$ will certainly discover a contradiction (*in finite time*)