First-Order Resolution

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Propositional Resolution

a) Refutation $\Gamma \cup \{\neg\varphi\}$ and translation into *conjunctive normal form* (CNF)

   $\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_n$ where each β_i is a disjunction of literals (i.e. A or $\neg A$)

b) Translation of $\Gamma \cup \{\neg\varphi\}$ in *clausal form* (CF)

   $\{\beta_1, \beta_2, \ldots, \beta_n\}$ where each β_i is a *clause* (i.e. a set of literals, representing a disjunction)

c) Exhaustive application of the resolution rule

   1) Selection of two clauses $\{\beta_1, \beta_2, \ldots, \beta_n, \alpha\}$, $\{\neg\alpha, \gamma_1, \gamma_2, \ldots, \gamma_m\}$
   2) Generation of the *resolvent*
      $\{\beta_1, \beta_2, \ldots, \beta_n, \alpha\}, \{\neg\alpha, \gamma_1, \gamma_2, \ldots, \gamma_m\} \vdash \{\beta_1, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2, \ldots, \gamma_m\}$

   Termination conditions:
   1) The empty clause has been derived (*success*)
   2) No further resolutions are possible – *fixed point* (*failure*)

1) Refutation: $\Gamma \cup \{\neg\varphi\}$

2) Translation into *prenex normal form* (PNF):

   All wff are now in the form:

   $$Qx_1 Qx_2 \dots Qx_n \psi \quad \text{(the } matrix\ \psi \text{ does not contain quantifiers)}$$

3) Removal of all existential quantifiers - *skolemization*:

   All wff are now in the form:

   $$\forall x_1 \forall x_2 \dots \forall x_m \chi \quad \text{(the } skolemized\ matrix\ \chi \text{ does not contain quantifiers)}$$

   Given that all wffs are universal sentences, the universal quantifiers can just be omitted

Example:

1: $\forall x\ (P(x) \rightarrow (\exists y\ Q(x,y) \wedge R(y)))$

2: $\forall x\ (\neg P(x) \vee (\exists y\ Q(x,y) \wedge R(y)))$ (removing $\rightarrow$)

2: $\forall x\ \exists y\ (\neg P(x) \vee (Q(x,y) \wedge R(y)))$ (PNF)

3: $\forall x\ (\neg P(x) \vee (Q(x, k(x)) \wedge R(k(x))))$ (Skolemization, with a <u>new</u> function k/1)

4: $\neg P(x) \vee (Q(x, k(x)) \wedge R(k(x)))$ (omitting universal quantifiers)

Just atoms, connectives and parentheses…

Clausal Form in L_{FO}

1) Refutation: $\Gamma \cup \{\neg\varphi\}$

2) Translation into PNF :

   All wff are now in the form:

   $$Qx_1 Qx_2 \ldots Qx_n \psi \quad \text{(the \textit{matrix} } \psi \text{ does not contain quantifiers)}$$

3) Removal of all existential quantifiers - *skolemization*:

   All wff are now in the form:

   $$\forall x_1 \forall x_2 \ldots \forall x_m \chi \quad \text{(the \textit{skolemized matrix} } \chi \text{ does not contain quantifiers)}$$

   Given that all wffs are universal sentences, the universal quantifiers can just be omitted

4) The *clausal form* can be obtained by just treating *atoms* as *propositions* and applying the rules of propositional logic

   First translate in *conjunctive normal form* (CNF) and then in *clausal form* (CF)

   Example:

   5: $\neg P(x) \lor (Q(x, k(x)) \land R(k(x)))$ (from before)

   6: $(\neg P(x) \lor Q(x, k(x))) \land (\neg P(x) \lor R(k(x)))$ (CNF, by distributing $\lor$)

   7: $\{\neg P(x), Q(x, k(x))\}, \{\neg P(x), R(k(x))\}$ (*Clausal Form*)

Unificare necesse est, for resolution

- Problem: $\Gamma \models \varphi$?

  $\Gamma \equiv \{\forall x\,(Greek(x) \rightarrow Human(x)), \forall x\,(Human(x) \rightarrow Mortal(x)), Greek(socrates)\}$

  $\varphi \equiv Mortal(socrates)$

  Refutation, translation, clausal form:

  1: $\{\forall x\,(Greek(x) \rightarrow Human(x)), \forall x\,(Human(x) \rightarrow Mortal(x)), Greek(socrates),$
     $\neg Mortal(socrates)\}$
                     ($\Gamma \cup \{\neg\varphi\}$ is already in PNF, no skolemization is needed)

  2: $\{\{Human(x), \neg Greek(x)\}, \{Mortal(x), \neg Human(x)\}, \{Greek(socrates)\},$
     $\{\neg Mortal(socrates)\}\}$
                 (Clausal Form)

  Resolution method (attempt):

  3: Try resolving: $\{\neg Mortal(socrates)\}, \{Mortal(x), \neg Human(x)\}$

              Technically, no resolution is applicable: no pairs of complementary literals

  Intuitively though,
  the two literals $\neg Mortal(socrates)$ and $Mortal(x)$ <u>are</u> complementary, somehow…

Unification

Replacing variables with terms to render two atoms <u>identical</u>

- **Unifier**

  A substitution of variables with terms $\sigma = [x_1 = t_1, x_2 = t_2 \dots x_n = t_n]$
  that makes two complementary literals α and $\neg\beta$ *resolvable*

  That is, it makes the two atoms *identical*: $\sigma(\alpha) = \sigma(\beta)$

  - Obviously, a unifier does not necessarily exist:
    for instance $P(g(x, f(a)), a)$ and $\neg P(g(b, f(w)), k(w))$ are not unifiable

- MGU - *most general unifier*

  It is the minimal *unifier* of α and $\neg\beta$

  $$\text{MGU } \mu \Leftrightarrow \forall\, \sigma\, \exists\, \sigma' : \sigma = \mu \cdot \sigma'$$

  Any other unifier can be obtained as a composition of μ

- ■ **Martelli and Montanari's algorithm**

  Input: $[s_1 = t_1, s_2 = t_2 \ldots s_n = t_n]$ (a system of *symbolic* equations)

  Procedure:

  Exhaustive application of the following rules to the system of symbolic equations
  (each rule *transforms* the original system)

  (1) $f(s_1, ..., s_n) = f(t_1, ..., t_n)$ *replace by the equations*
                                                 $s_1 = t_1, ..., s_n = t_n,$

  (2) $f(s_1, ..., s_n) = g(t_1, ..., t_m)$ where $f \neq g$ *halt with failure,* ⟵———— Applies even when
                                                                                     either *m* or *n* are 0

  (3) $x = x$ *delete the equation,*   (i.e. with *constants*)

  (4) $t = x$ where t is not a variable   *replace by the equation $x = t$,*

  (5) $x = t$ where x does not occur in t *apply the substitution $\{x/t\}$*
        and x occurs elsewhere   *to all other equations*

  (6) $x = t$ where x occurs in t and x differs from t *halt with failure.*

  Unless an explicit failure occurs (i.e. by rules (2) or (6)),
  the procedure terminates with success when no further rule is applicable

Example: $[f(x, a) = f(g(z), y), h(u) = h(d)]$

    $[x = g(z), y = a, h(u) = h(d)]$ Rule (1) on $f(x, a) = f(g(z), y)$

    $[x = g(z), y = a, u = d]$ Rule (1) on $h(u) = h(d)$, MGU

Example: $[f(x, a) = f(g(z), y), h(x, z) = h(u, d)]$

    $[x = g(z), y = a, h(x, z) = h(u, d)]$ Rule (1) on $f(x, a) = f(g(z), y)$

    $[x = g(z), y = a, h(g(z), z) = h(u, d)]$ Rule (5) on $x = g(z)$

    $[x = g(z), y = a, u = g(z), z = d]$ Rule (1) on $h(g(z), z) = h(u, d)$

    $[x = g(d), y = a, u = g(d), z = d]$ Rule (5) on $z = d$, MGU

Example: $[f(x, a) = f(g(z), y), h(x, z) = h(d, u)]$

    $[x = g(z), y = a, h(x, z) = h(d, u)]$ Rule (1) on $f(x, a) = f(g(z), y)$

    $[x = g(z), y = a, h(g(z), z) = h(d, u)]$ Rule (5) on $x = g(z)$

    $[x = g(z), y = a, g(z) = d, z = u]$ Rule (2) on $g(z) = d$ FAILURE

- Example: $\Gamma \models \varphi$? (transitive property - in clausal form)

    $\Gamma \equiv \{\{\neg C(x,y), \neg C(y,z), C(x,z)\}, \{C(a,b)\}, \{C(b,c)\}, \{C(c,d)\}\}$

    $\varphi \equiv \{C(a,d)\}$

    Refutation and resolution:

    1: $\{\{\neg C(x,y), \neg C(y,z), C(x,z)\}, \{C(a,b)\}, \{C(b,c)\}, \{C(c,d)\}, \{\neg C(a,d)\}\}$

    2: Unify and resolve $\{\neg C(x,y), \neg C(y,z), C(x,z)\}$ and $\{\neg C(a,d)\}$:
    $[x=a, z=d]$ with resolvent $\{\neg C(a,y), \neg C(y,d)\}$

    3: Unify and resolve $\{\neg C(x,y), \neg C(y,z), C(x,z)\}$ and $\{\neg C(a,y), \neg C(y,d)\}$:
    $[x=a, z=y]$ with resolvent $\{\neg C(a,y), \neg C(y,y), \neg C(y,d)\}$

    4: *This way seems to lead nowhere:* $\neg C(y,y)$ *will never be resolved in* $\Gamma \cup \{\neg \varphi\}$

    Why is this??

- Example: $\Gamma \models \varphi$? (transitive property - in clausal form)

  $\Gamma \equiv \{\{\neg C(x,y), \neg C(y,z), C(x,z)\}, \{C(a,b)\}, \{C(b,c)\}, \{C(c,d)\}\}$

  $\varphi \equiv \{C(a,d)\}$

  Refutation and resolution, <u>standardize</u> variables before each resolution
  (i.e. rename all variables with new, unique names)

  1: $\{\{\neg C(x,y), \neg C(y,z), C(x,z)\}, \{C(a,b)\}, \{C(b,c)\}, \{C(c,d)\}, \{\neg C(a,d)\}\}$

  2: Unify and resolve $\{\neg C(x_1,y_1), \neg C(y_1,z_1), C(x_1,z_1)\}$ and $\{\neg C(a,d)\}$:
  $[x_1=a, z_1=d]$ with resolvent $\{\neg C(a, y_1), \neg C(y_1,d)\}$

  3: Unify and resolve $\{\neg C(x_2,y_2), \neg C(y_2,z_2), C(x_2,z_2)\}$ and $\{\neg C(a,y_3), \neg C(y_3,d)\}$:
  $[x_2=a, z_2=y_3]$ with resolvent $\{\neg C(a, y_2), \neg C(y_2, y_3), \neg C(y_3,d)\}$

  4: Unify and resolve $\{\neg C(a, y_4), \neg C(y_4, y_5), \neg C(y_5,d)\}$ and $\{C(a,b)\}$:
  $[y_4=b]$ with resolvent $\{\neg C(b, y_5), \neg C(y_5,d)\}$

  5: Unify and resolve $\{\neg C(b, y_5), \neg C(y_5,d)\}$ and $\{C(b,c)\}$:
  $[y_4=c]$ with resolvent $\{\neg C(c,d)\}$

  5: Resolve $\{\neg C(c,d)\}$ and $\{C(c,d)\}$:
  resolvent $\{\}$ (*success*)

A <u>correct</u> procedure for $\Gamma \vdash \varphi$ in L_{FO}

a) Refutation $\Gamma \cup \{\neg\varphi\}$,

b) Prenex normal form and skolemization $sko(\Gamma \cup \{\neg\varphi\})$

c) Translation of $sko(\Gamma \cup \{\neg\varphi\})$ into CNF hence into CF

d) Repeat application of the resolution method:

    1) Selection of two clauses $\{\beta_1, \beta_2, \dots, \beta_n, \alpha\}$, $\{\neg\alpha', \gamma_1, \gamma_2, \dots, \gamma_m\}$

    2) Standardization of variables
    (i.e. create new copies of the two clauses having <u>new</u> and <u>unique</u> variables)

    3) Construction of the MGU μ (if it exists) for the two literals α e α'

    4) Generation of the resolvent by applying of μ
    $\{\beta_1, \beta_2, \dots, \beta_n, \alpha\}[\mu]$, $\{\neg\alpha', \gamma_1, \gamma_2, \dots, \gamma_m\}[\mu] \vdash \{\beta_1, \beta_2, \dots, \beta_n, \gamma_1, \gamma_2, \dots, \gamma_m\}[\mu]$

e) Until

    1) The empty clause has been derived (*success*)

    2) No further resolutions are possible – *fixed point* (*failure*)

    Note: the method is not guaranteed to <u>terminate</u> (i.e. it might *diverge*)

Problem: $\forall x\,(Q(f(x)) \rightarrow P(x)) \models \exists x\,(Q(f(x)) \rightarrow P(f(x)))$?

(The answer is <u>negative</u>: there is no entailment)

Refutation:

$\{\forall x\,(Q(f(x)) \rightarrow P(x))\} \cup \{\neg\exists x\,(Q(f(x)) \rightarrow P(f(x)))\}$

Prenex normal form:

$\{\forall x\,(Q(f(x)) \rightarrow P(x))\} \cup \{\forall x\,\neg(Q(f(x)) \rightarrow P(f(x)))\}$

(no *skolemization* required)

Clausal form:

$\{Q(f(x)) \rightarrow P(x)\} \cup \{\neg(P(f(x)) \wedge \neg Q(f(x)))\}$

$\{\neg Q(f(x)) \vee P(x)\} \cup \{\neg P(f(x)) \vee Q(f(x))\}$

$\{\{\neg Q(f(x)), P(x)\}, \{\neg P(f(x)), Q(f(x))\}\}$

Resolution:

1: $\{\neg Q(f(x_1)), P(x_1)\}, \{\neg P(f(x_2)), Q(f(x_2))\}, [x_1/f(x_2)] \vdash \{\neg Q(f(f(x_2))), Q(f(x_2))\}$
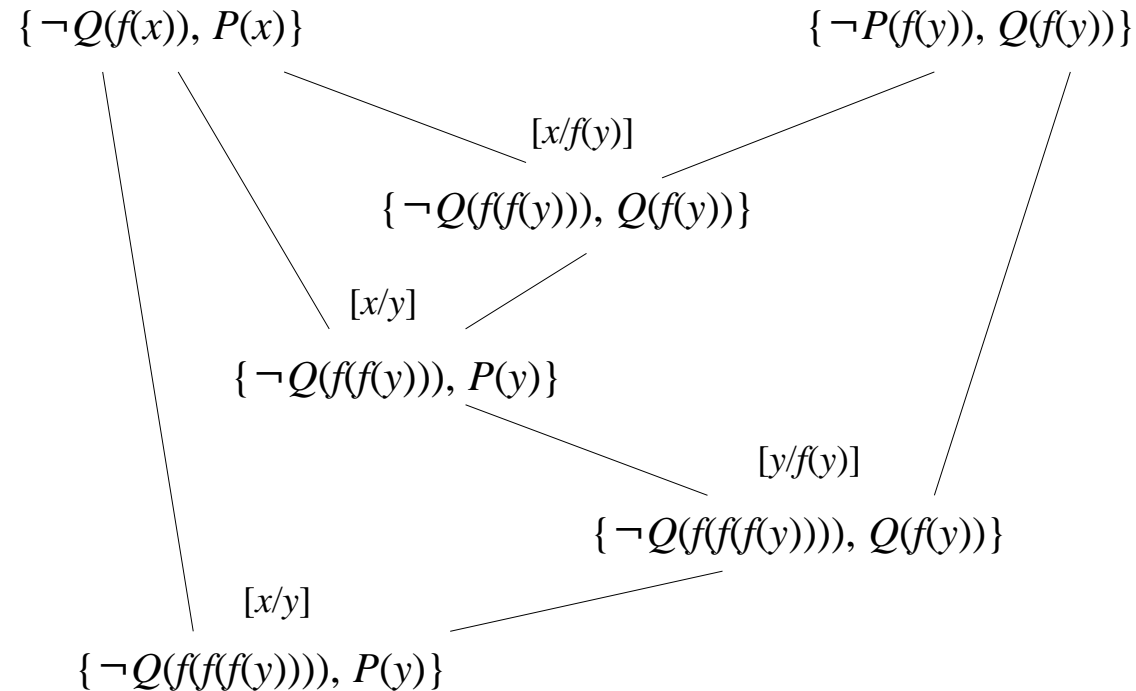
2: $\{\neg Q(f(x_3)), P(x_3)\}, \{\neg Q(f(f(x_4))), Q(f(x_4))\}, [x_3/x_4] \vdash \{\neg Q(f(f(x_4))), P(x_4)\}$

3: $\{\neg Q(f(f(x_5))), P(x_5)\}, \{\neg P(f(x_6)), Q(f(x_6))\}, [x_5/f(x_6)] \vdash \{\neg Q(f(f(f(x_6)))), Q(f(x_6))\}$

4: $\{\neg Q(f(x_7)), P(x_7)\}, , \{\neg Q(f(f(f(x_8)))), Q(f(x_8))\}, [x_7/x_8] \vdash \{\neg Q(f(f(f(x_8)))), P(x_8)\}$

  …

{¬Q(f(x)), P(x)}   {¬P(f(y)), Q(f(y))}

[x/f(y)]

{¬Q(f(f(y))), Q(f(y))}

[x/y]

{¬Q(f(f(y))), P(y)}

[y/f(y)]

{¬Q(f(f(f(y)))), Q(f(y))}

[x/y]

{¬Q(f(f(f(y)))), P(y)}

.
.
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Standardization of variables not applied here, for simplicity

- **The method is *correct* in L_{FO}**

  If the method finds the empty clause for $sko(\Gamma \cup \{\neg\varphi\})$ then $\Gamma \models \varphi$

- **Is the method *complete* in L_{FO} ?**

  Within the limits of semi-decidability, yes (Robinson, 1963)

  When $\Gamma \models \varphi$, the method will eventually find the empty clause for $sko(\Gamma \cup \{\neg\varphi\})$

  Very often (but not in the worst case) the method is more efficient than the one in the corollary of Herbrand's theorem

  The advantage is due to *lifting*
  (the method can resolve also non-ground clauses)

  When $\Gamma \not\models \varphi$, the method might diverge

  CAUTION: Unless the selection procedure is *fair* (more on this topic to follow)
  the method might diverge even when $\Gamma \models \varphi$

  Critical element:
  - Selecting the clauses and literals to be resolved