Entailment and Algorithms

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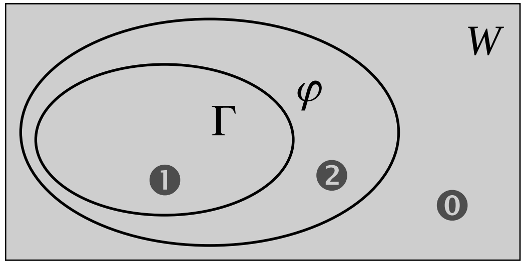
Entailment as *satisfiability*

Transforming problems: entailment as satisfiability

- Step 1: the problem " $\Gamma \models \varphi$? "
  can be transformed into a *satisfiability* problem

  In fact, $\Gamma \models \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is *not* satisfiable



($w(\Gamma)$ is the set of possible worlds that satisfy Γ)

$\Gamma \models \varphi \;\Rightarrow\; w(\Gamma) \subseteq w(\{\varphi\})$ ❶ ⊆ {❶, ❷}

$w(\{\neg\varphi\})$ = ⓿

$w(\Gamma \cup \{\neg\varphi\}) = w(\Gamma) \cap w(\{\neg\varphi\})$

$w(\Gamma \cup \{\neg\varphi\}) = \varnothing$ ❶ ∩ ⓿ = ∅

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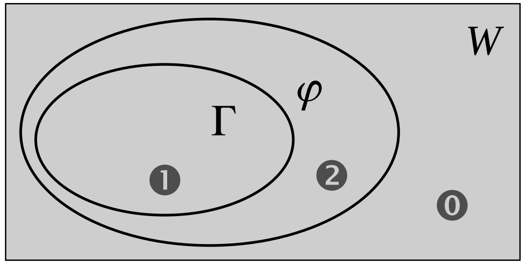

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- Step 2: the problem “is $\Gamma \cup \{\neg\varphi\}$ satisfiable?”
  can be transformed into a wff *satisfiability* problem (a.k.a. ‘SAT problem’)

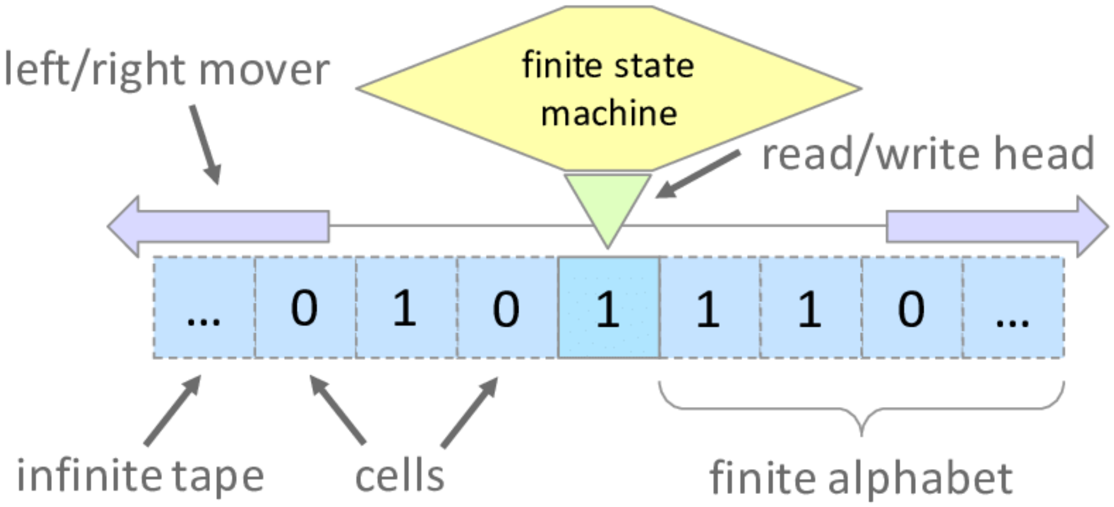
  Taking this one step further, we can transform $\Gamma \cup \{\neg\varphi\}$ into *just one formula*:

  $$\bigwedge (\Gamma \cup \{\neg\varphi\})$$

  This is the wff obtained by combing all the wffs in $\Gamma \cup \{\neg\varphi\}$ with $\wedge$,
  it is called the *conjunctive closure* of the set $\Gamma \cup \{\neg\varphi\}$

Computational Complexity Theory
(in a quick ride)

Turing Machine (A. Turing, 1937)



- A more precise definition

  A non-empty and finite set of *states* S

  At each instant the machine is in a state $s \in S$

  A non-empty and finite alphabet of *symbols* Q

  The alphabet Q includes a *blank*, default symbol b

  Each cell in the tape contains a symbol $q \in Q$

  A partial *transition* function *(i.e., the 'program')*

$$\tau : \underset{\text{current state}}{S} \times \underset{\text{input symbol}}{Q} \rightarrow \underset{\text{next state}}{S} \times \underset{\text{output symbol}}{Q} \times \underset{\text{head move}}{\{\text{Left}, \text{None}, \text{Right}\}}$$

  It is partial in the sense it needs not be defined on any input tuple

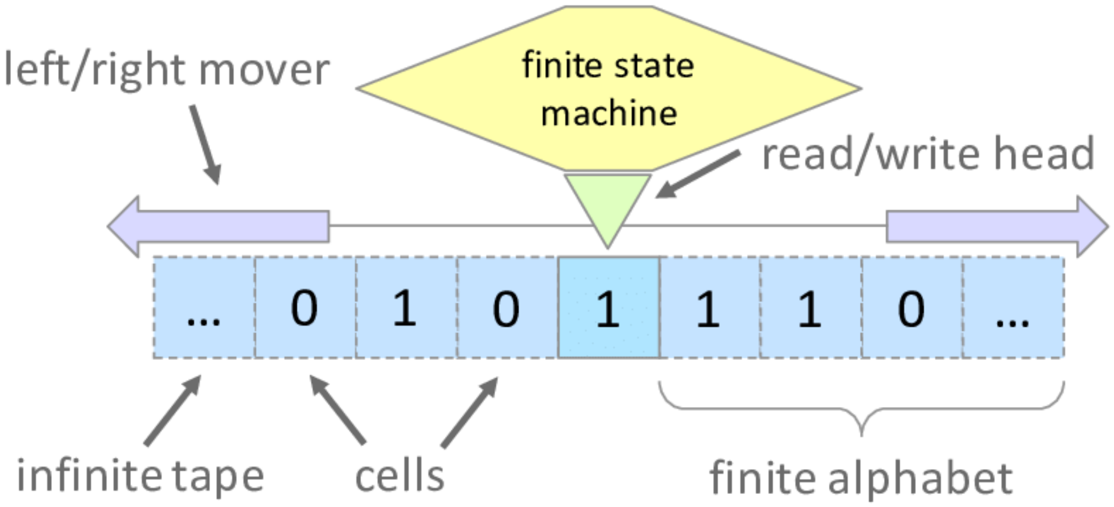
  A subset of *terminal* states $T \subseteq S$

  An *initial* state $s_0 \in S$

Turing Machine (A. Turing, 1937)

- A ***busy beaver*** example (3 states)

$$S = \{A, B, C, \mathrm{HALT}\}$$
$$s_0 = A \qquad T = \{\mathrm{HALT}\}$$
$$Q = \{0, 1\} \quad b = 0$$
$$\begin{aligned} \tau = & \\ & < A, 0 > \rightarrow < B, 1, \mathrm{Right} > \\ & < A, 1 > \rightarrow < C, 1, \mathrm{Left} > \\ & < B, 0 > \rightarrow < A, 1, \mathrm{Left} > \\ & < B, 1 > \rightarrow < B, 1, \mathrm{Right} > \\ & < C, 0 > \rightarrow < B, 1, \mathrm{Left} > \\ & < C, 1 > \rightarrow < \mathrm{HALT}, 1, \mathrm{Right} > \end{aligned}$$



Assume that the tape is infinite and plenty of blank symbols 0

What does this machine do?

Decisions and decidability (automation)

- What is a <u>*problem*</u>?

  A *problem* is an association, i.e. a **relation** between *inputs* and *outputs,* a.k.a. *solutions*

  $$K = I \times S$$

- *Search* problem

  Typically, K associates *one* input to *many* solutions

  Optimization problems

  A *search problem* plus an *objective* or *cost* function

  $c : S \to \mathbb{R}$ (i.e. from S to the set of real numbers)

  In general, the task in a search problem is finding the solution(s) having maximal or minimal cost

- ***Decision*** problem

  The solution space S is $\{0, 1\}$
  and K associates each input to a <u>*unique*</u> solution: $K : I \to \{0, 1\}$

  Remarkable example of decision problem: $\Gamma \models \varphi$?

  The input space I contains all possible combinations of set Γ of wffs with individual wffs φ
  The solution is uniquely defined for any instance of such problems in I

Decisions and decidability (automation)

- ***Decidable*** problem

  A decision problem K for which there exists an algorithm, i.e a *Turing machine*,

  (there are other ways of defining an algorithm or an *effective procedure*: they are all equivalent)

  that ***always terminates*** and produces the right answer in ***finite time***.

Example of an *undecidable* problem: The *Halting Problem*

Given the formal description of a particular Turing machine and a specific input,
is it possible to tell if whether it will either halt eventually or run forever?

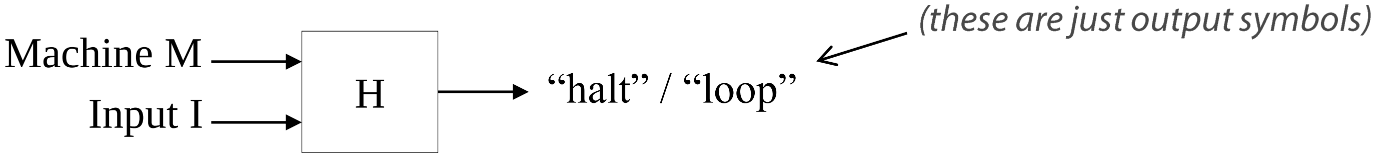
In other words, does it exist a Turing machine that, given in input the description of *another* Turing machine, will always produce the answer desired?

The answer is **no** (such a Turing machine *cannot* exist)

An aside: The *Halting Problem*

- Intuitive ideas behind the proof (i.e. of the *undecidability* of this problem)

  Let’s assume there exists a Turing machine H that, given the description of a Turing machine M with input I always terminates producing an output “halt” or “loop” depending on whether M with input I will terminate or not



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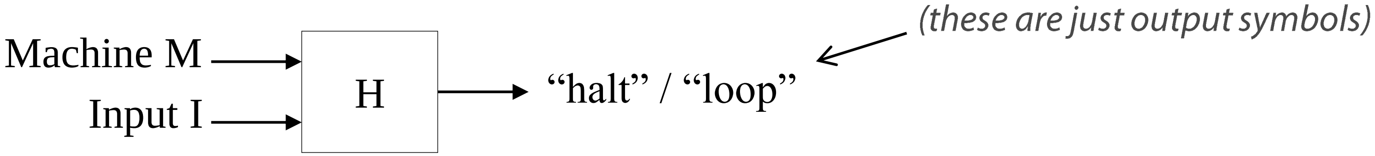
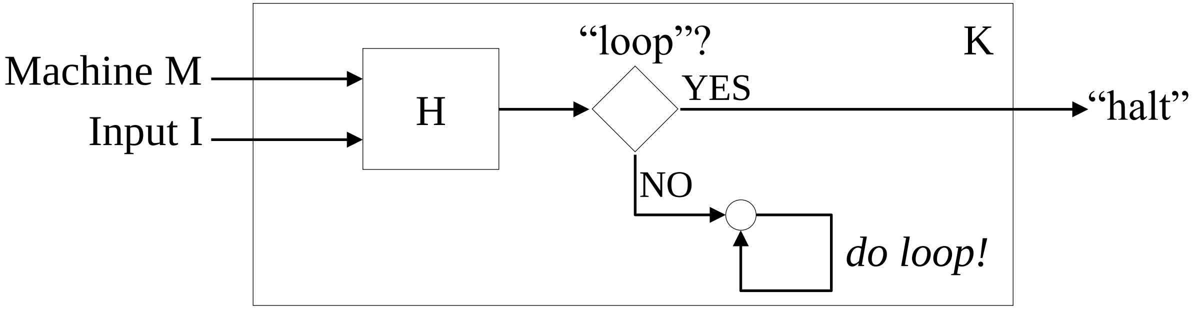


<u>*Assume* H *existed*</u>

We could build another Turing machine K that enters an infinite loop whenever the output of H is "halt" and that terminates, with output "halt", when H outputs "loop"



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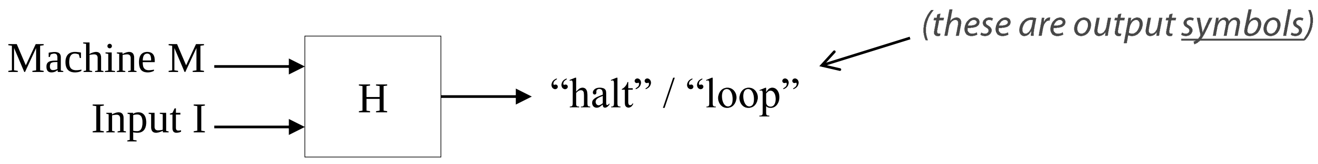

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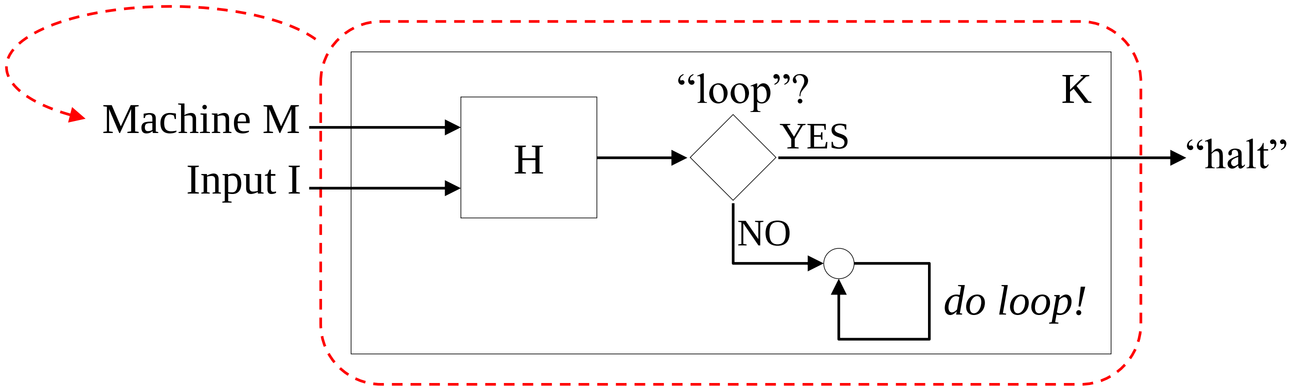


What will be the output of K when given K *itself* as the input?

K should *diverge* when K *terminates* and vice-versa: i.e. we have an absurdity

Computational complexity

These notions apply to *decidable problems* only

It is based on the performances of a (known) Turing machine that gives the answer with respect to the *worst case* (i.e. the less favorable input)

- *Time* complexity

  The number of *steps* that the Turing machine requires for computing the answer, as a function of some numerical dimension of the input (e.g. the number of atoms in a wff)

- *Memory* complexity

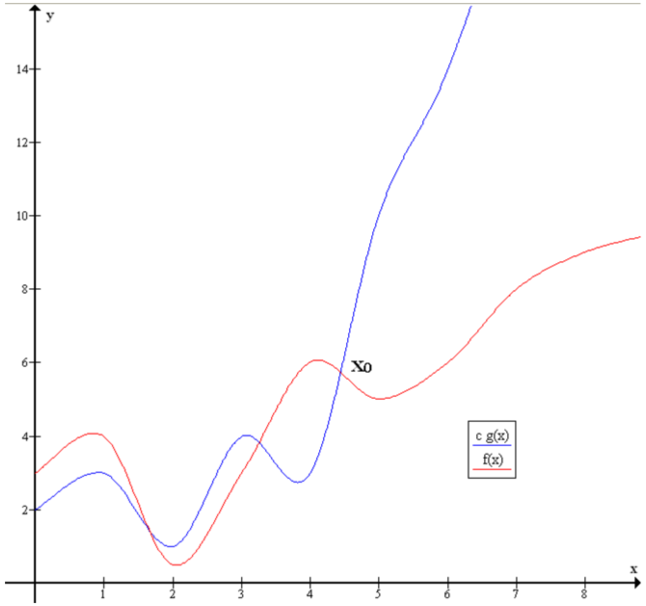
  The number of tape *cells* that the Turing machine requires for computing the answer, as a function of some numerical dimension of the input

- Big-O notation

$$f(x) = O(g(x))$$

means that

$\exists c > 0, \; \exists x_0 > 0$ such that $|f(x)| \leq c|g(x)|, \;\; \forall x > x_0$



Classes P, NP and NP-complete - The SAT problem

- Class P

  The class of problems for which there is a Turing machine that requires $O(\text{poly}(n))$ time

  where $\text{poly}(\ )$ is a polynomial of finite degree and n is the dimension of the (*worst-case*) input

- Class NP

  The class of all problems:

  a) A method for *enumerating* all possible answers (i.e. *recursive enumerability*)

  b) An algorithm in class P that *verifies* if a possible answer is also a *solution*

  It includes all problems in class P (that is, $\text{P} \subseteq \text{NP}$)

Entailment as a Decision Problem

Classes P, NP and NP-complete – The SAT problem

- Class NP-complete

  It is a subclass of NP (NP-complete $\subseteq$ NP)

  A problem K is NP-complete if every problem in class NP is *reducible* to K

- Reducibility

  For class NP-complete

  Consider a problem K for which a decision algorithm $M(K)$ is known

  A problem J is *reducible* to K if there exist a decision algorithm $M(J)$ such that:

  a) algorithm $M(K)$ is called just once, as a "subroutine", at the end of $M(J)$

  b) apart from $M(K)$, $M(J)$ has polynomial complexity

- The SAT problem

  Is NP-complete (*historically, it is the first one to be known*)

  Moral: if we had a polynomial decision algorithm for SAT, we would also have that

  P = NP

  This fact is not known, it is believed that: P $\neq$ NP

  (*and a lot will change in the digital world, if this proves to be false*)

Satisfiability and decidability (in L_P)

- Is the decision problem "is the wff φ satisfiable?" *decidable*?

  It can be transformed into a *search* problem

  i.e. finding a possible world (in the set of all possible worlds) that satisfies φ

  In the scientific literature, this problem is called "SAT"

  Intuition: we can try every possible value assignment for the atoms in φ

  Hint: the problem is NP-complete

Exhaustive (Tree) Search

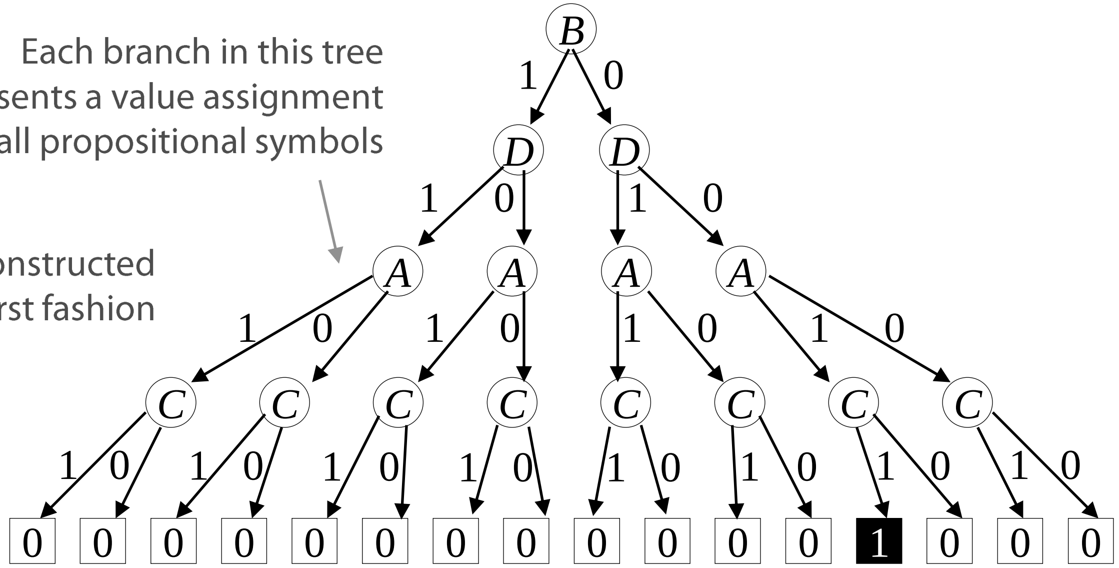
Satisfiability and decidability (in L_P)

Example: is this wff *satisfiable*?

$$\neg(B \vee D \vee \neg(A \wedge C))$$

Each branch in this tree represents a value assignment to all propositional symbols

The tree can be constructed in a depth-first fashion



Each leaf in the tree is the value of the wff with the corresponding value assignments

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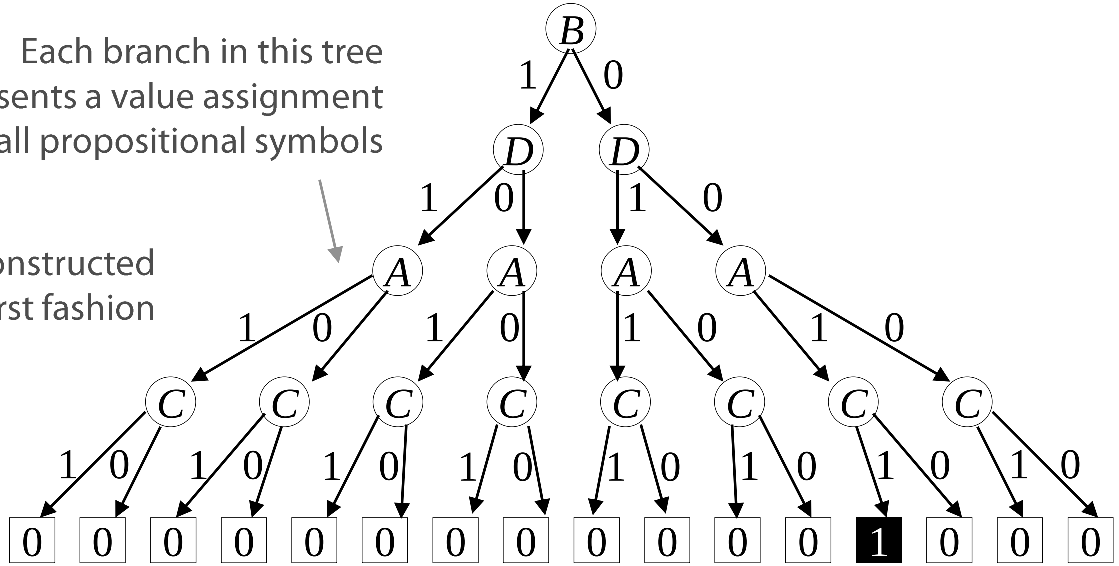

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In this case, a depth-first algorithm stops here

But the algorithm is forced to try all possible assignments when ψ is *not* satisfiable, for example with: $(\neg B \wedge \neg D \wedge \neg A \wedge \neg C)$

Satisfiability and decidability (in L_P)

Example: is this wff *satisfiable*?

$\neg(B \lor D \lor \neg(A \land C))$

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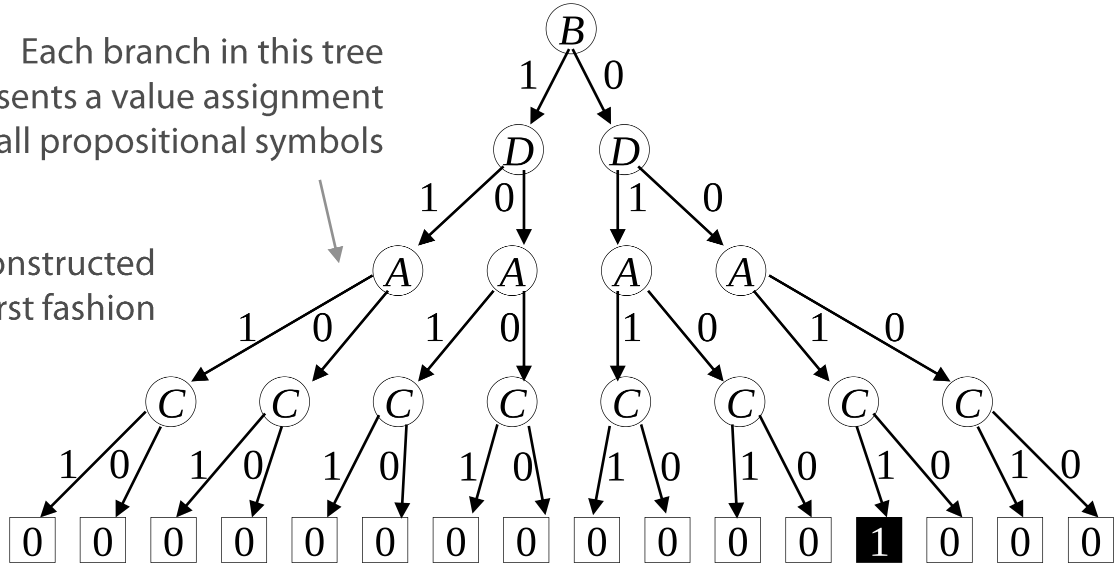

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This method has $O(2^n)$ time complexity, where n is the number of propositional symbols