Reinforcement learning

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Multi-Armed Bandit

[image from wikipedia]

A row of N old-style slot machines

- Basic definitions

  N arms or *bandits*

  Each arm i yields a random reward r with probability distribution $P_i(r)$

  For simplicity, only Bernoullian rewards (i.e. either 0 or 1) will be considered here

  Each time t in a sequence, the player (i.e. the agent) selects the arm $\pi(t)$

  In other words, $\pi(t)$ is the *policy* adopted by the agent

- Problem

  Find a strategy $\pi(t)$ that maximizes the *total reward* over time

  The strategy will include random choices i.e. it will be *stochastic*

Multi-Armed Bandit: strategies

- Informed (i.e. *optimal*) strategy

  At all times, select the bandit with higher probability of reward:

  $$\pi^*(t) = \mathrm{argmax}_i P_i(r = 1)$$

  Clearly, this strategy is optimal but requires knowing all distributions $P_i(r)$

  With enough data (*e.g. from other players*), these distributions can be learnt

- Random strategy

  At all times, select a bandit i at random, with *uniform probability*

  How does the Random strategy compare with the optimal, informed strategy?

Multi-Armed Bandit: evaluating strategies

- *Total Expected Regret*

How far from optimality a policy is, considering the total reward over T trials

For just <u>one</u> sequence of T trials, the *Total Regret with expected rewards* is

decision taken at step t

$$R(T) := T\mu^* - \sum_{t=1}^{T} \mu_{\pi(t)}$$

where

expected (i.e. *mean*) reward of bandit k

$$\mu_k := \mathbb{E}[r] = \textstyle\sum_r r P_k(r) \qquad \mu^* := \max_k \ \mu_k$$

In a more general definition, the *Total <u>Expected</u> Regret* is

$$\overline{R}(T) := T\mu^* - \sum_{k=1}^{N} \mathbb{E}[T_k(T)]\mu_k = \sum_{k=1}^{N} \mathbb{E}[T_k(T)]\Delta_k \quad \text{where} \quad \Delta_k := \mu^* - \mu_k$$

number of times bandit *k* is selected in T trials (i.e. *a random variable*)

With the optimal policy π^* the total expected regret is 0.
Whereas, with the *random policy* the total expected regret grows linearly over time:

$$\overline{R}(T) = \frac{T}{N} \sum_{k=1}^{N} \Delta_k$$

…since, with a random strategy $\mathbb{E}[T_k(T)] = \frac{T}{N}$

Multi-Armed Bandit: *Online learning*

Adaptive policy: *exploration vs. exploitation*

exploration: make trials over the set of *N* arms to learn about the expected reward μ_k

exploitation: make use of the current best guess about the expected rewards μ_k

- Greedy policy

  Initialize all the estimated values μ_k at random

  Repeat:

  1) select the bandit with the current best estimated reward $i = \text{argmax}_k \hat{\mu}_k$
  2) and update the current estimate about *i* as the *average* reward

$$\hat{\mu}_i := \frac{\sum_{t=1}^{T_i} r_{i,t}}{T_i}$$

reward of arm *i* at trial *t*

Total number of times the arm *i* has been played

- ε-greedy policy $(0 < \varepsilon < 1)$

  Initialize all the estimated values μ_k at random

  Repeat:

  1) with probability $(1 - \varepsilon)$ select the bandit with the best estimated reward else (*i.e. with probability* ε) select one arm at random
  2) update the current estimate about *i* as the *average* reward (see above)

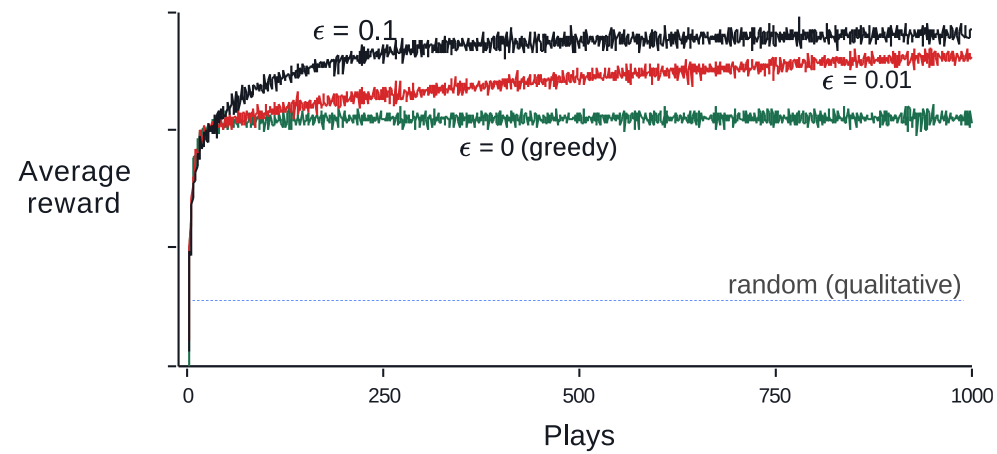
Multi-Armed Bandit: *Online learning*

- Experimental comparison of different strategies

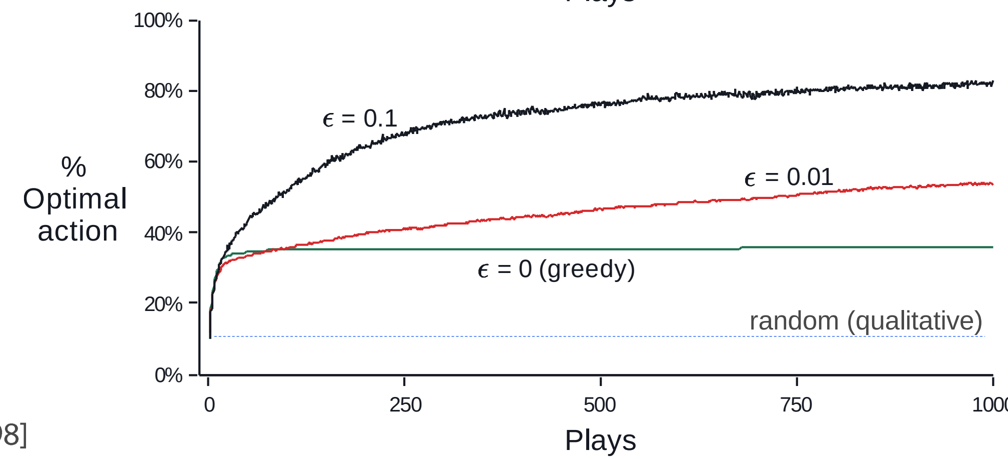
  10 arms bandit with different rewards (10-arms testbed)
  Averaged over 2000 runs (i.e. sequences of trials)

After a certain period of time, the *greedy* strategy stops exploring and exploits its estimates

whereas, the ε-greedy strategy keeps exploring and approaches optimality



The random strategy never improves its performances, as expected



[images adapted from: Sutton, Barto, *Reinforcement Learning*. 1998]

Multi-Armed Bandit: evaluating strategies

- *From a theoretical standpoint*

  All *greedy* strategies are *biased*: they depend on the initial random distribution

  Optimistic variant: initially, set all $\hat{\mu}_k := 1$

  The average total regret always grows linearly, in the long run

  In fact:

  - on the average, the *greedy* strategy will get stuck in a suboptimal choice
  - the ε-greedy strategy will continue to choose an arm at random (with probability ε)

  Can we do any better?

Multi-Armed Bandit: Optimal *online learning*

- Lower bound theorem [Lai & Robbins 1985]

Consider a generic, adaptive (i.e. learning) strategy for the multi-armed bandit problem with Bernoulli reward (i.e. $r \in \{0, 1\}$)

$$\lim_{T\to\infty} \overline{R}(T) \geq \ln T \sum_{k|\Delta_k>0} \frac{\Delta_k}{\mathrm{kl}(\mu_k, \mu^*)}$$

$\Delta_k := \mu^* - \mu_k$

where

$$\mathrm{kl}(\mu_k, \mu^*) := \mu_k \ln \frac{\mu_k}{\mu^*} + (1 - \mu_k) \ln \frac{(1 - \mu_k)}{(1 - \mu^*)}$$

a special case of the *Kullback-Leibler divergence* :
in this case, it measures of the difference between two (Bernoulli) distributions

In other words, we can achieve logarithmic growth for the total expected regret, but not better: on average, any adaptive strategy will choose suboptimal bandits a minimum number of times

$$\lim_{T\to\infty} \mathbb{E}[T_k(T)] \geq \frac{\ln T}{\mathrm{kl}(\mu_k, \mu^*)}$$

Multi-Armed Bandit: UCB strategy

- Upper confidence bound (UCB) strategy [Auer, Cesa-Bianchi and Fisher 2002]

Initialize all the estimates of the expected reward $\hat{\mu}_k := 0$
Play each arm once *(to avoid zeroes in the formula below)*

total number of trials

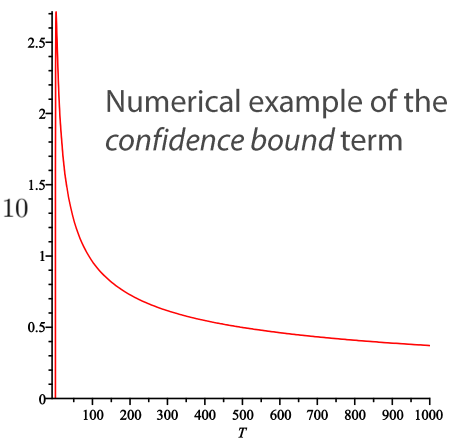
number of times
the arm k has been played

Repeat:

1) select the bandit $i = \mathrm{argmax}_k \left(\hat{\mu}_k + \sqrt{\dfrac{2 \ln T}{T_k}} \right)$

2) update the current estimate about i
as the *average* reward

$\sqrt{\dfrac{2 \ln T}{(T/N)}}, \quad N = 10$



Numerical example of the *confidence bound* term

Theorem

With the UCB strategy, $\lim\limits_{T \to \infty} \mathbb{E}[T_k(T)] \leq \dfrac{8 \ln T}{\Delta_k^2} + c$

i.e. a (small) constant

where it can be shown that $\dfrac{8}{\Delta_k^2} \geq \dfrac{1}{\mathrm{kl}(\mu_k, \mu^*)}$

(i.e. there is a reasonably small gap between the two bounds – near optimality)

Multi-Armed Bandit: Thompson Sampling

- Thompson Sampling strategy (*also 'Bayesian Bandit'*) [Thompson, 1933]

  Initialize all the expected reward $\hat{\mu}_k :\sim \mathrm{Beta}(1, 1)$

  i.e. assume that this is a random variable with this (*prior*) distribution

  Repeat:

  1) <u>sample</u> each of the N distributions to obtain an estimate $\hat{\mu}_k$
  2) select the bandit $i = \mathrm{argmax}_k \hat{\mu}_k$
  3) update the *posterior* distribution

     $$\hat{\mu}_i :\sim \mathrm{Beta}(R_i + 1,\ T_i - R_i + 1)$$

     total number of times the arm has been played

     total (*Bernoulli*) reward from this arm (*i.e. number of wins*)

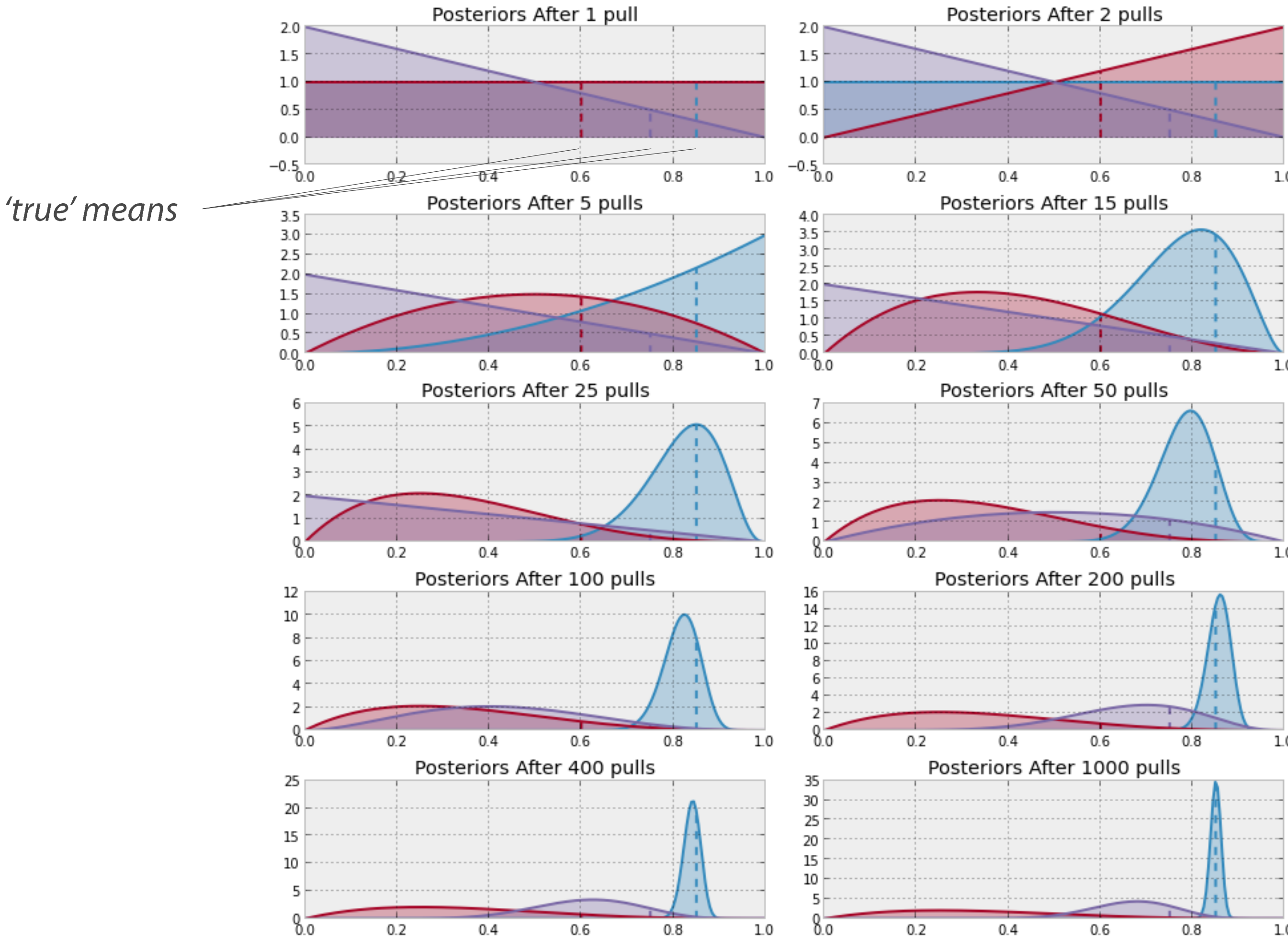
  Theorem [Kaufmann et al., 2012]

  The Thompson Sampling strategy has essentially the same theoretical bounds of the UCB strategy

Multi-Armed Bandit: Thompson Sampling

- Thompson Sampling strategy (*also 'Bayesian Bandit'*) [Thompson, 1933]

Example run with 3 arms: trace of the posterior probabilities for each μ_k

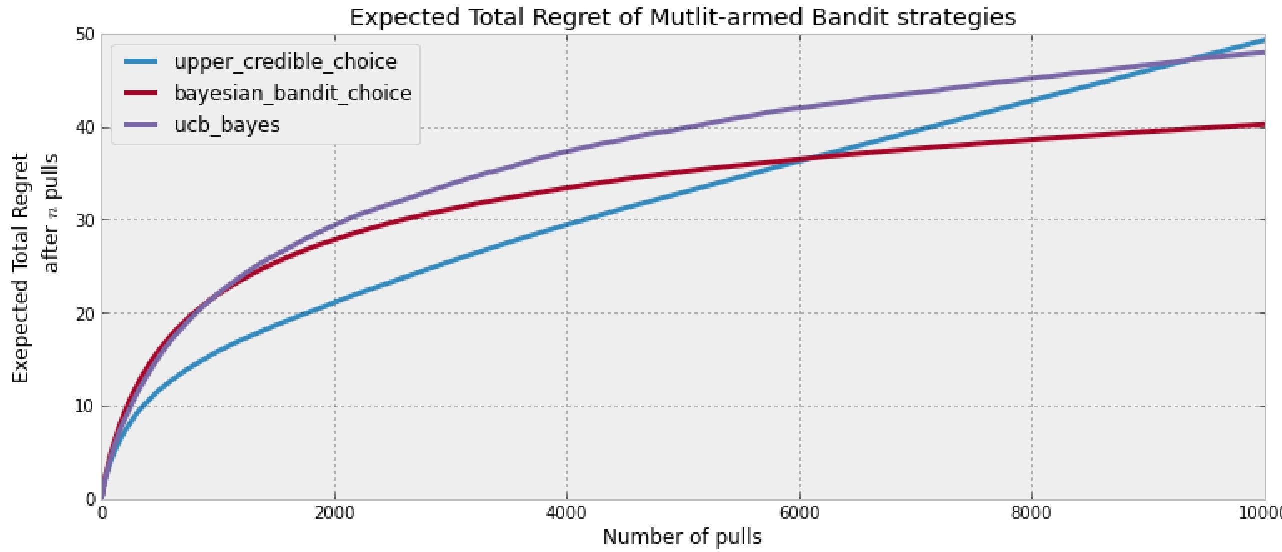


[image from: http://camdp.com/blogs/multi-armed-bandits]

Multi-Armed Bandit: Thompson Sampling

- Thompson Sampling strategy (*also 'Bayesian Bandit'*) [Thompson, 1933]

  In practical experiments, this strategy shows better performances in the long run
  [Chapelle & Li, 2011]



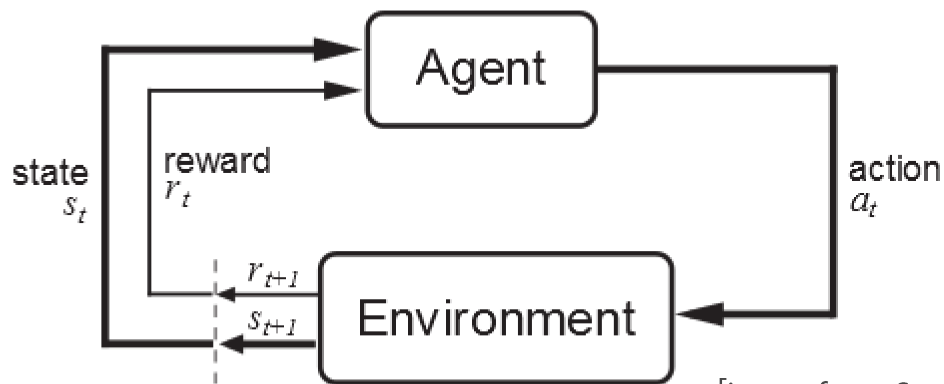
Actually, Thompson Sampling is a preferred strategy at Google Inc.
(see https://support.google.com/analytics/answer/2846882?hl=en)

[image from: http://camdp.com/blogs/multi-armed-bandits]

Agent/Environment Interactions

With multi-armed bandits, the context never changes
*in the sense that the optimal choice does **not** depend on the current state*

What if the actions of the agent change the state of its interaction with the environment?



[image from: Sutton, Barto, *Reinforcement Learning.* 1998]

Examples:

- a_t could be a *move in a game*, whereby the agent changes the state of the game
- a_t could be a *movement*, whereby the agent changes its position in the environment

The agent could be wanting to learn an *optimal strategy* towards a given goal…

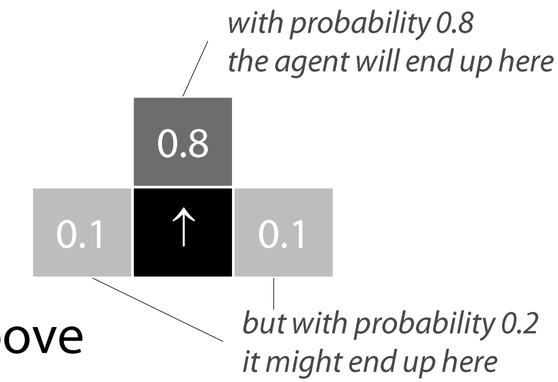
An example: *gridworld*

	1	2	3	4
1	-0.02	-0.02	-0.02	1
2	-0.02		-0.02	-1
3	-0.02	-0.02	-0.02	-0.02

The *state* of the agent is the position on the grid: e.g. (1,1), (3,4), (2,3)

At each time step, the agent can *move* one box in the directions ←↑↓→

The effect of each move is somewhat stochastic, however: for example, a move ↑ has a slight probability of producing a different (*and perhaps unwanted*) effect



Entering each state yields the *reward* shown in each box above

There are two *absorbing states*: entering either the green or the red box means exiting the *gridworld* and completing the game

- What is the best (*i.e. maximally rewarding*) movement policy?

Markov Decision Process (MDP)

	1	2	3	4
1	-0.02	-0.02	-0.02	1
2	-0.02		-0.02	-1
3	-0.02	-0.02	-0.02	-0.02

Formalization and abstraction of the gridworld example

Markov Decision Process: $< S, A, r, P, \gamma >$

A set of *states* : $S = \{s_1, s_2, \dots\}$

A set of *actions* : $A = \{a_1, a_2, \dots\}$

A *reward function* : $r : S \to \mathbb{R}$

A *transition probability distribution* : $P(S_{t+1} \mid S_t, A_t)$ (also called a *model*)

Markov property: the transition probability depends only the previous state and action

$$P(S_{t+1} \mid S_t, A_t) = P(S_{t+1} \mid S_t, A_t, S_{t-1}, A_{t-1}, S_{t-2}, A_{t-2}, \dots)$$

A *discount factor* : $0 \leq \gamma \leq 1$

Markov Decision Process (MDP): policies and values

The agent is supposed to adopt a *deterministic* policy : $\pi : S \to A$

In other words, the agent always chooses its *action* depending on the *state* alone

Given a policy π , the ***state value function*** is defined, for each state s as:

$$V^{\pi}(s) := \mathbb{E}[r(S_t) + \gamma r(S_{t+1}) + \gamma^2 r(S_{t+2}) + \dots \mid \pi, S_t = s]$$

Note the role of the *discount factor*: a value $\gamma \leq 1$ means that that future rewards could be weighted less (by the agent) than immediate ones

Note also that all states S_t must be described by *random variables* :
i.e. the policy is deterministic but the state transition is not

In the *gridworld* example:

- The set of states is finite
- The set of actions is finite
- For every policy, each entire story is finite

  Sooner or later the agent will fall into one of the absorbing states

Bellman equations

By working on the definition of value function:

$$
\begin{aligned}
V^\pi(s) &:= \mathbb{E}[r(S_t) + \gamma r(S_{t+1}) + \gamma^2 r(S_{t+2}) + \ldots \mid \pi, S_t = s] \\
&= \mathbb{E}[r(S_t) + \gamma(r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots) \mid \pi, S_t = s] \\
&= r(s) + \gamma \mathbb{E}[r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots \mid \pi, S_t = s] \\
&= r(s) + \gamma \textstyle\sum_{s'} P(s' \mid s, \pi(s)) \cdot \mathbb{E}[r(S_{t+1}) + \gamma r(S_{t+2}) + \ldots \mid \pi, S_{t+1} = s'] \\
&= r(s) + \gamma \textstyle\sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^\pi(S_{t+1})
\end{aligned}
$$

This means that in a Markov Decision Process:

$$V^\pi(s) = r(s) + \gamma \textstyle\sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^\pi(S_{t+1})$$

This is true for any *state*, so there is one such equation for each of those

If the set of states is *<u>finite</u>*, *there are exactly* $|S|$ *(linear) Bellman equations for* $|S|$ *variables:*
in general, given π, V^π *can be computed in closed form*

Optimal policy – Optimal value function

- Basic definitions

$$\pi^*(s) := \text{argmax}_\pi V^\pi(s),\ \forall s \in S$$

$$V^*(s) := \max_\pi V^\pi(s),\ \forall s \in S$$

Property: for every MDP, there exists such an optimal deterministic policy (*possibly non-unique*)

With Bellman Equations:

$$\max_\pi V^\pi(s) = r(s) + \gamma \max_\pi \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^\pi(S_{t+1}) \right)$$

$$V^*(s) = r(s) + \gamma \max_\pi \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, \pi(s)) \cdot V^*(S_{t+1}) \right)$$

$$= r(s) + \gamma \max_a \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, a) \cdot V^*(S_{t+1}) \right)$$

Therefore:

$$\pi^*(s) = \text{argmax}_a \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, a) V^*(S_{t+1}) \right)$$

Computing V^* *directly from these equations is unfeasible, however*

There are in fact $|S|^{|A|}$ *possible strategies*

However, once V^* *has been determined,* π^* *can be determined as well*

Optimal value function: value iteration

- Value iteration algorithm

  Initialize: $V(s) := r(s),\ \forall s \in S$
  Repeat:

  Note that there is no policy: all actions must be explored

  1) For every state, update: $V(s) := r(s) + \gamma \max_a \sum_{s'} P(s' \mid s, a) V(s')$

  Theorem: for every fair way (*i.e. giving an equal chance*) of visiting the states in S, this algorithm converges to V^*

Value iteration and optimal policy

	1	2	3	4
1	-0.02	-0.02	-0.02	1
2	-0.02		-0.02	-1
3	-0.02	-0.02	-0.02	-0.02

Initialize states
(e.g. using rewards as initial values)

Iterate and compute

$\overset{V^*}{\rightarrow}$

	1	2	3	4
1	0.86	0.90	0.93	1
2	0.82		0.69	-1
3	0.78	0.75	0.71	0.49

V^*

$\downarrow$

Define the optimal policy as:

$$\pi^*(s) := \text{argmax}_a \left(\sum_{S_{t+1}} P(S_{t+1} \mid s, a) \cdot V^*(S_{t+1}) \right)$$

	1	2	3	4
1	$\rightarrow$	$\rightarrow$	$\rightarrow$	1
2	$\uparrow$		$\uparrow$	-1
3	$\uparrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$

π^*

Optimal policy: policy iteration

- Policy iteration algorithm

  Initialize $\pi(s), \forall s \in S$ at random
  Repeat:

  This step is computationally expensive: either solve the equations or use value iteration (with fixed policy π)

  1) For each state, compute: $V(s) := V^{\pi}(s)$
  2) For each state, define: $\pi(s) := \mathrm{argmax}_a \sum_{s'} P(s' \mid s, a) V(s')$

  Theorem: for every fair way (*i.e. giving an equal chance*) of visiting the states in S, this algorithm converges to π^*

  As with the value iteration algorithm, this algorithm uses partial estimates to compute new estimates.
  It is also <u>greedy</u>, in the sense that it exploits its current estimate $V^{\pi}(s)$

  Policy iteration converges with very few number of iterations,
  but every iteration takes much longer time than that of *value iteration*

  The tradeoff with value iteration is the <u>action space</u>:
  when action space is large and state space is small, policy iteration could be better

Offline vs. *Online* learning

- *Value iteration* and *policy iteration* are offline algorithms

  The *model*, i.e. the Markov Decision Process is known

  What needs to be learn is the optimal policy π^*

  In the algorithms, *visiting states* just means considering: there is no agent actually playing the game.

- Different conditions: learn by doing

  Suppose the *model* (i.e. the MDP) is NOT known, or perhaps known only in part

  Then the agent must learn by doing…

Q-Learning

An analogous of the value function V^π

Given a policy π, the ***action value function*** is defined, for each pair $< s, a >$ as:

$Q^\pi(s,a) := \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot V^\pi(S_{t+1})$ *i.e. choose* a *in* s *and then follow* π *afterwards*

Following a similar line of reasoning as before, the optimal *action value function* is

$$Q^*(s,a) = \sum_{S_{t+1}} P(S_{t+1} \mid s,a) \cdot [r(S_{t+1}) + \gamma \max_{a'} Q^*(S_{t+1}, a')]$$

- Q-learning algorithm (*ε-greedy version*)

Initialize $\hat{Q}(s,a)$ at random, put the agent is in a random state s
Repeat:

1) Select the action $\mathrm{argmax}_a \hat{Q}(s,a)$ with probability $(1-\varepsilon)$
   otherwise, select a at random
2) The agent is now in state s' and has received the reward r
3) Update $\hat{Q}(s,a)$ by

$$\Delta\hat{Q}(s,a) = \alpha(r + \gamma \max_{a'} \hat{Q}(s',a') - \hat{Q}(s,a))$$

Exponential Moving Average
(we will see this again…)

Q-Learning

- Q-learning algorithm

  Theorem (Watkins, 1989): in the limit of that each action is played infinitely often and each state is visited infinitely often and $\alpha \to 0$ as experience progresses, then

$$\hat{Q}(s,a) \to Q^*(s,a)$$

  with probability 1

The Q-learning algorithm bypasses the MDP entirely,
in the sense that the optimal strategy is learnt without learning the model $P(S_{t+1} \mid S_t, A_t)$