# First-Order Resolution

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# Propositional Resolution

*A decision method for* $\Gamma \models \varphi$

a) Refutation $\Gamma \cup \{\neg\varphi\}$ and translation into *conjunctive normal form* (CNF)
   $\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_n$ where each $\beta_i$ is a disjunction of literals (i.e. $A$ or $\neg A$)

b) Translation of $\Gamma \cup \{\neg\varphi\}$ in *clausal form* (CF)
   $\{\beta_1, \beta_2, \ldots, \beta_n\}$ where each $\beta_i$ is a *clause* (i.e. a set of literals, representing a disjunction)

c) Exhaustive application of the resolution rule
   1) Selection of two clauses $\{\beta_1, \beta_2, \ldots, \beta_n, \alpha\}, \{\neg\alpha, \gamma_1, \gamma_2, \ldots, \gamma_m\}$
   2) Generation of the *resolvent*
      $\{\beta_1, \beta_2, \ldots, \beta_n, \alpha\}, \{\neg\alpha, \gamma_1, \gamma_2, \ldots, \gamma_m\} \vdash \{\beta_1, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2, \ldots, \gamma_m\}$

Termination conditions:

1) The empty clause has been derived (*success*)
2) No further resolutions are possible – *fixed point* (*failure*)

# Clausal Form in $L_{FO}$

1) Refutation: $\Gamma \cup \{\neg\varphi\}$
2) Translation into *prenex normal form* (PNF):

   All wff are now in the form:

   $Qx_1 Qx_2 \ldots Qx_n \psi$ (the *matrix* $\psi$ does not contain quantifiers)
3) Removal of all existential quantifiers - *skolemization*:

   All wff are now in the form:

   $\forall x_1 \forall x_2 \ldots \forall x_m \chi$ (the *skolemized matrix* $\chi$ does not contain quantifiers)

   Given that all wffs are universal sentences, the universal quantifiers can just be omitted

Example:

1: $\forall x\, (P(x) \rightarrow (\exists y\, Q(x,y) \wedge R(y)))$

2: $\forall x\, (\neg P(x) \vee (\exists y\, Q(x,y) \wedge R(y)))$ (removing $\rightarrow$ )

2: $\forall x\, \exists y\, (\neg P(x) \vee (Q(x,y) \wedge R(y)))$ (PNF)

3: $\forall x\, (\neg P(x) \vee (Q(x, k(x)) \wedge R(k(x))))$ (Skolemization, with a new function $k/1$)

4: $\neg P(x) \vee (Q(x, k(x)) \wedge R(k(x)))$ (omitting universal quantifiers)

Just atoms, connectives and parentheses…

# *Clausal Form* in $L_{FO}$

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3) Removal of all existential quantifiers - *skolemization*:

   All wff are now in the form:

   $\forall x_1 \, \forall x_2 \ldots \forall x_m \chi$ (the *skolemized matrix* $\chi$ does not contain quantifiers)

   Given that all wffs are universal sentences, the universal quantifiers can just be omitted

4) The *clausal form* can be obtained by just treating *atoms* as *propositions* and applying the rules of propositional logic

   First translate in *conjunctive normal form* (CNF) and then in *clausal form* (CF)

Example:

5: $\neg P(x) \vee (Q(x, k(x)) \wedge R(k(x)))$ (from before)

6: $(\neg P(x) \vee Q(x, k(x))) \wedge (\neg P(x) \vee R(k(x)))$ (CNF, by distributing $\vee$)

7: $\{\neg P(x), Q(x, k(x))\}, \ \{\neg P(x), R(k(x))\}$ (*Clausal Form*)

# *Unificare necesse est*, for resolution

- Problem: $\Gamma \models \varphi$ ?

  $\Gamma \equiv \{\forall x\ (Philosopher(x) \rightarrow Uman(x)), \forall x\ (Uman(x) \rightarrow Mortal(x)), Philosopher(socrates)\}$

  $\varphi \equiv Mortal(socrates)$

  *Refutation, translation, clausal form*:

  1: $\{\forall x\ (Philosopher(x) \rightarrow Uman(x)), \forall x\ (Uman(x) \rightarrow Mortal(x)),$
  $Philosopher(socrates), \neg Mortal(socrates)\}$

  ($\Gamma \cup \{\neg\varphi\}$ is already in PNF, no skolemization is needed)

  2: $\{\{Uman(x), \neg Philosopher(x)\}, \{Mortal(x), \neg Uman(x)\}, \{Philosopher(socrates)\},$
  $\{\neg Mortal(socrates)\}\}$

  (*Clausal Form*)

  *Resolution method (first attempt):*

  3: $\{Uman(x), \neg Philosopher(x)\}, \{Mortal(x), \neg Uman(x)\} \{\neg Philosopher(x)\ , Mortal(x)\}$

  4: Try resolving: $\{Uman(socrates)\}, \{Mortal(x), \neg Uman(x)\}$

  ???

  *Intuitively, the two literals* $Uman(socrates)$ *and* $\neg Uman(x)$ *are complementary, somehow…*

# Unification

*Replacing variables with terms to render two atoms identical*

- **Unifier**

  A substitution of variables with terms $\sigma = [x_1 = t_1, x_2 = t_2 \ldots x_n = t_n]$ that makes two complementary literals $\alpha$ and $\neg\beta$ *resolvable*

  That is, it makes the two atoms *identical*: $\sigma(\alpha) = \sigma(\beta)$

  - Obviously, a unifier does not necessarily exist: for instance $P(g(x, f(a)), a)$ and $\neg P(g(b, f(w)), k(w))$ are not unifiable

- MGU - *most general unifier*

  It is the minimal *unifier* of $\alpha$ and $\neg\beta$

  $$\text{MGU } \mu \Leftrightarrow \forall\, \sigma\, \exists\, \sigma' : \sigma = \mu \cdot \sigma'$$

  Any other unifier can be obtained as a composition of $\mu$

# Constructing the MGU

- Martelli and Montanari's algorithm

  Input: $[s_1 = t_1, s_2 = t_2 \ldots s_n = t_n]$ (a system of *symbolic* equations)

  Procedure:

  Exhaustive application of the following rules to the system of symbolic equations (each rule *transforms* the original system)

| | | |
|---|---|---|
| (1) | $f(s_1, ..., s_n) = f(t_1, ..., t_n)$ | *replace by the equations* $s_1 = t_1, ..., s_n = t_n$, |
| (2) | $f(s_1, ..., s_n) = g(t_1, ..., t_m)$ where $f \neq g$ | *halt with failure*, ⟵ Applies even when either *m* or *n* are 0 (i.e. with *constants*) |
| (3) | $x = x$ | *delete the equation*, |
| (4) | $t = x$ where $t$ is not a variable | *replace by the equation* $x = t$, |
| (5) | $x = t$ where $x$ does not occur in $t$ and $x$ occurs elsewhere | *apply the substitution* $\{x/t\}$ *to all other equations* |
| (6) | $x = t$ where $x$ occurs in $t$ and $x$ differs from $t$ | *halt with failure*. |

  Unless an explicit failure occurs (i.e. by rules (2) or (6)),
  the procedure terminates with success when no further rule is applicable

# Constructing the MGU: examples

Example: $[f(x, a) = f(g(z), y), h(u) = h(d)]$

| | |
|---|---|
| $[x = g(z), y = a, h(u) = h(d)]$ | Rule (1) on $f(x, a) = f(g(z), y)$ |
| $[x = g(z), y = a, u = d]$ | Rule (1) on $h(u) = h(d)$, MGU |

Example: $[f(x, a) = f(g(z), y), h(x, z) = h(u, d)]$

| | |
|---|---|
| $[x = g(z), y = a, h(x, z) = h(u, d)]$ | Rule (1) on $f(x, a) = f(g(z), y)$ |
| $[x = g(z), y = a, h(g(z), z) = h(u, d)]$ | Rule (5) on $x = g(z)$ |
| $[x = g(z), y = a, u = g(z), z = d]$ | Rule (1) on $h(g(z), z) = h(u, d)$ |
| $[x = g(d), y = a, u = g(d), z = d]$ | Rule (5) on $z = d$, MGU |

Example: $[f(x, a) = f(g(z), y), h(x, z) = h(d, u)]$

| | |
|---|---|
| $[x = g(z), y = a, h(x, z) = h(d, u)]$ | Rule (1) on $f(x, a) = f(g(z), y)$ |
| $[x = g(z), y = a, h(g(z), z) = h(d, u)]$ | Rule (5) on $x = g(z)$ |
| $[x = g(z), y = a, g(z) = d, z = u]$ | Rule (2) on $g(z) = d$ FAILURE |

# *Standardization* of variables is also necessary

- Example: $\Gamma \models \varphi$ ? (transitive property - in clausal form)

  $\Gamma \equiv \{\{\neg C(x,y), \neg C(y,z), C(x,z)\}, \{C(a,b)\}, \{C(b,c)\}, \{C(c,d)\}\}$

  $\varphi \equiv \{C(a,d)\}$

  *Refutation and resolution:*

  1: $\{\{\neg C(x,y), \neg C(y,z), C(x,z)\}, \{C(a,b)\}, \{C(b,c)\}, \{C(c,d)\}, \{\neg C(a,d)\}\}$

  2: Unify and resolve $\{\neg C(x,y), \neg C(y,z), C(x,z)\}$ and $\{\neg C(a,d)\}$ :
  $[x=a, z=d]$ with resolvent $\{\neg C(a,y), \neg C(y,d)\}$

  3: Unify and resolve $\{\neg C(x,y), \neg C(y,z), C(x,z)\}$ and $\{\neg C(a,y), \neg C(y,d)\}$ :
  $[x=a, z=y]$ with resolvent $\{\neg C(a,y), \neg C(y,y), \neg C(y,d)\}$

  4: *This way seems to lead nowhere*: $\neg C(y,y)$ *will never be resolved in* $\Gamma \cup \{\neg\varphi\}$

  *Why is this??*

# *Standardization* of variables is also necessary

- Example: $\Gamma \models \varphi$ ? (transitive property - in clausal form)

  $\Gamma \equiv \{\{\neg C(x,y), \neg C(y,z), C(x,z)\}, \{C(a,b)\}, \{C(b,c)\}, \{C(c,d)\}\}$

  $\varphi \equiv \{C(a,d)\}$

  *Refutation and resolution, <u>standardize</u> variables before each resolution*
  (i.e. rename all variables with new, unique names)

  1: $\{\{\neg C(x,y), \neg C(y,z), C(x,z)\}, \{C(a,b)\}, \{C(b,c)\}, \{C(c,d)\}, \{\neg C(a,d)\}\}$

  2: Unify and resolve $\{\neg C(x_1,y_1), \neg C(y_1,z_1), C(x_1,z_1)\}$ and $\{\neg C(a,d)\}$ :
  $[x_1=a, z_1=d]$ with resolvent $\{\neg C(a, y_1), \neg C(y_1,d)\}$

  3: Unify and resolve $\{\neg C(x_2,y_2), \neg C(y_2,z_2), C(x_2,z_2)\}$ and $\{\neg C(a,y_3), \neg C(y_3,d)\}$ :
  $[x_2=a, z_2=y_3]$ with resolvent $\{\neg C(a, y_2), \neg C(y_2, y_3), \neg C(y_3,d)\}$

  4: Unify and resolve $\{\neg C(a, y_4), \neg C(y_4, y_5), \neg C(y_5,d)\}$ and $\{C(a,b)\}$ :
  $[y_4=b]$ with resolvent $\{\neg C(b, y_5), \neg C(y_5,d)\}$

  5: Unify and resolve $\{\neg C(b, y_5), \neg C(y_5,d)\}$ and $\{C(b,c)\}$ :
  $[y_4=c]$ with resolvent $\{\neg C(c,d)\}$

  5: Resolve $\{\neg C(c,d)\}$ and $\{C(c,d)\}$ :
  resolvent $\{\}$ (*success*)

# Resolution with unification for $L_{FO}$

*A correct procedure for* $\Gamma \models \varphi$ in $L_{FO}$

a) Refutation $\Gamma \cup \{\neg\varphi\}$,

b) Prenex normal form and skolemization $sko(\Gamma \cup \{\neg\varphi\})$

c) Translation of $sko(\Gamma \cup \{\neg\varphi\})$ into CNF hence into CF

d) Repeat application of the resolution method:

1) Selection of two clauses $\{\beta_1, \beta_2, \dots, \beta_n, \alpha\}, \{\neg\alpha', \gamma_1, \gamma_2, \dots, \gamma_m\}$
2) *Standardization* of variables
(i.e. create new copies of the two clauses having new and unique variables)
3) Construction of the $\mathrm{MGU}$ $\mu$ (if it exists) for the two literals $\alpha$ e $\alpha'$
4) Application generation of the resolvent with the application of $\mu$
$\{\beta_1, \beta_2, \dots, \beta_n, \alpha\}[\mu], \{\neg\alpha', \gamma_1, \gamma_2, \dots, \gamma_m\}[\mu] \vdash \{\beta_1, \beta_2, \dots, \beta_n, \gamma_1, \gamma_2, \dots, \gamma_m\}[\mu]$

e) Until

1) The empty clause has been derived (*success*)
2) No further resolutions are possible – *fixed point* (*failure*)

But the method is not guaranteed to terminate (i.e. it might *diverge*)

# The method might diverge…

Problem: $\forall x\,(Q(f(x)) \rightarrow P(x)) \models \exists x\,(P(f(x)) \wedge \neg Q(f(x)))$ ?

(*The answer is* *<u>negative</u>*: *there is no entailment*)

Refutation:

$\{\ \forall x\,(Q(f(x)) \rightarrow P(x))\ \} \cup \{\neg \exists x\,(P(f(x)) \wedge \neg Q(f(x)))\ \}$

Prenex normal form:

$\{\ \forall x\,(Q(f(x)) \rightarrow P(x))\ \} \cup \{\ \forall x\ \neg(P(f(x)) \wedge \neg Q(f(x)))\ \}$

(no *skolemization* required)

Clausal form:

$\{\ Q(f(x)) \rightarrow P(x)\ \} \cup \{\neg(P(f(x)) \wedge \neg Q(f(x)))\ \}$

$\{\neg Q(f(x)) \vee P(x)\ \} \cup \{\ \neg P(f(x)) \vee Q(f(x))\ \}$

$\{\{\neg Q(f(x)) \vee P(x)\}, \{\neg P(f(x)) \vee Q(f(x))\}\}$

Resolution:

1: $\{\neg Q(f(x_1)), P(x_1)\}, \{\neg P(f(x_2)), Q(f(x_2))\}, [x_1/f(x_2)] \vdash \{\neg Q(f(f(x_2))), Q(f(x_2))\}$

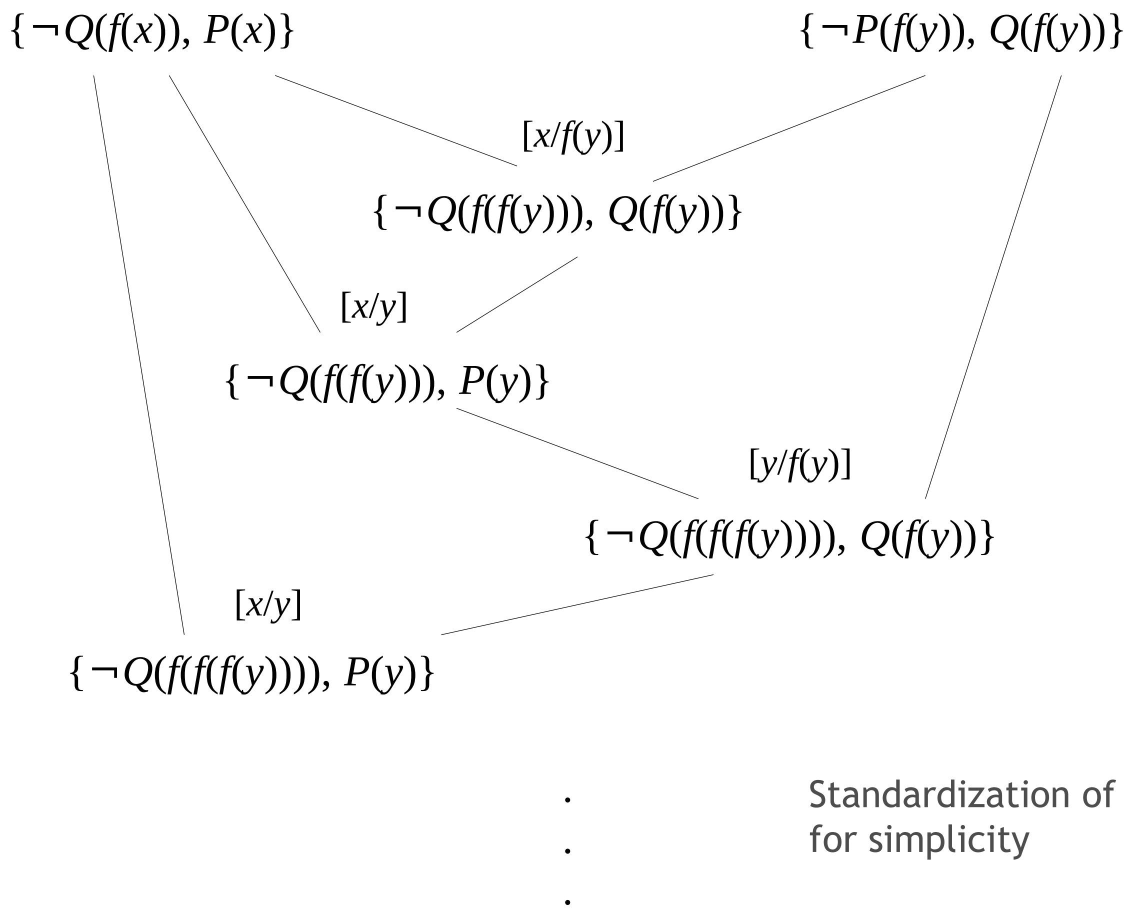
2: $\{\neg Q(f(x_3)), P(x_3)\}, \{\neg Q(f(f(x_4))), Q(f(x_4))\}, [x_3/x_4] \vdash \{\neg Q(f(f(x_4))), P(x_4)\}$

3: $\{\neg Q(f(f(x_5))), P(x_5)\}, \{\neg P(f(x_6)), Q(f(x_6))\}, [x_5/f(x_6)] \vdash \{\neg Q(f(f(f(x_6)))), Q(f(x_6))\}$

4: $\{\neg Q(f(x_7)), P(x_7)\}, , \{\neg Q(f(f(f(x_8)))), Q(f(x_8))\}, [x_7/x_8] \vdash \{\neg Q(f(f(f(x_8)))), P(x_8)\}$

...

# The method might diverge…




Standardization of variables not applied here, for simplicity

# Properties of resolution with unification

- The method is *correct* in $L_{FO}$

  If the method finds the empty clause for $sko(\Gamma \cup \{\neg\varphi\})$ then $\Gamma \models \varphi$

- Is the method *complete* in $L_{FO}$ ?

  Within the limits of semi-decidability, yes (Robinson, 1963)

  When $\Gamma \models \varphi$, the method will eventually find the empty clause for $sko(\Gamma \cup \{\neg\varphi\})$

  Very often (but not in the worst case) the method is more efficient than the one in the corollary of Herbrand's theorem

  The advantage is due to *lifting*
  (the method can resolve also non-ground clauses)

  When $\Gamma \not\models \varphi$, the method might diverge

  CAUTION: Unless the selection procedure is *fair* (more on this topic to follow)
  the method might diverge even when $\Gamma \models \varphi$

  Critical element:

  - Selecting the clauses and literals to be resolved