# First-Order Logic

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# Propositional possible worlds

Each possible world is a structure $<\{0,1\}, \boldsymbol{P}, v>$

$\{0,1\}$ are the *truth values*

$\boldsymbol{P}$ is the ***signature*** of the formal language: a set of propositional symbols

$v$ is a *function* : $\boldsymbol{P} \rightarrow \{0,1\}$ assigning truth values to the symbols in $\boldsymbol{P}$

## Propositional symbols (*signature*)

Each symbol in $\boldsymbol{P}$ stands for an actual *proposition* (in natural language)

In the simple convention, we use the symbols $A, B, C, D$, ...

Caution: $\boldsymbol{P}$ is not necessarily *finite*

## Possible worlds

The class of structures contains all possible worlds:

$<\{0,1\}, \boldsymbol{P}, v>$
$<\{0,1\}, \boldsymbol{P}, v'>$
$<\{0,1\}, \boldsymbol{P}, v''>$
...

Each class of structure shares $\boldsymbol{P}$ and $\{0,1\}$

The functions $v$ are different: the assignment of truth values varies, depending on the possible world

If $\boldsymbol{P}$ is finite, there are only *finitely* many distinct possible worlds (actually $2^{|\boldsymbol{P}|}$)

# An aside: tuples, relations and functions

- **Tuple**

  Consider a generic set of objects $\mathbf{U}$

  An example *set* of objects from $\mathbf{U}$ is denoted as $\{u_1, u_2\}$, where $u_1, u_2 \in \mathbf{U}$

  *In a set, the order of elements is not relevant*

  An example of *tuple* of objects from $\mathbf{U}$ is denoted as $\langle u_1, u_2 \rangle$, where $u_1, u_2 \in \mathbf{U}$

  *In a tuple, the order is relevant, i.e.* $\langle u_1, u_2 \rangle \neq \langle u_2, u_1 \rangle$

- **Cartesian product**

  The cartesian product $\mathbf{U} \times \mathbf{U} =: \mathbf{U}^2$ is the set of all tuples $\langle u_1, u_2 \rangle,\ u_1, u_2 \in \mathbf{U}$

  Analogously, $\mathbf{U}^3$ is the set of all tuples $\langle u_1, u_2, u_3 \rangle,\ u_1, u_2, u_3 \in \mathbf{U}$
  $\mathbf{U}^4$ is the set of all tuples $\langle u_1, u_2, u_3, u_4 \rangle,\ u_1, u_2, u_3, u_4 \in \mathbf{U}$ and so on …

- **Relation**

  *arity is always an integer*

  A relation of *arity* $n$ is a subset of $\mathbf{U}^n$

- **Function**

  A function of type $\mathbf{U}^n \to \mathbf{U}$ is a relation of arity $n+1$ such that each tuple is constructed by associating each tuple of $\mathbf{U}^n$ with exactly one object from $\mathbf{U}$

# First-order possible worlds

*Possible worlds made of objects, functions and relations*

Each possible world is a structure <$\mathbf{U}$, $\Sigma$, $v$>

**U** is a set of object, called ***domain*** (also *universe of discourse*)

$\Sigma$ is a set of symbols, called ***signature***

$v$ is a *function* that gives a *meaning* to the symbols in $\Sigma$ with respect to **U**

## Signature $\Sigma$

- *individual constants*: $a, b, c, d$, ...
- *function symbols (with arity)*: $f$ /n, $g$ /p, $h$ /q, ...
- *predicate symbols* (with arity): $P$ /k, $Q$ /l, $R$ /m, ...

*Arity is an integer that describes the expected number of arguments*

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Each possible world is a structure $<\mathbf{U}, \Sigma, v>$

- $\mathbf{U}$ is a set of object, called ***domain*** (also *universe of discourse*)
- $\Sigma$ is a set of symbols, called ***signature***
- $v$ is a *function* that gives a *meaning* to the symbol in $\Sigma$ with respect to $\mathbf{U}$

**Term**

- A single *individual constant* is a **term**
- If $f/n$ is a *functional symbol* (with arity $n$) and $t_1, \ldots, t_n$ are **terms**, then $f(t_1, \ldots, t_n)$ is a **term**

**Atom**

- If $P/n$ is a *predicate symbol* (with arity $n$) and $t_1, \ldots, t_n$ are **terms**, then $P(t_1, \ldots, t_n)$ is an **atom** (i.e a first-order well-formed formula – wff)

# First-order possible worlds

*Possible worlds made of objects, functions and relations*

Each possible world is a structure $<\mathbf{U}, \Sigma, v>$

$\mathbf{U}$ is a set of object, called ***domain*** (also *universe of discourse*)

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$v$ is a *function* that gives a *meaning* to the symbol in $\Sigma$ with respect to $\mathbf{U}$

Function $v$ (***interpretation***)

- $v$ assigns each *individual constant* to an *object* in $\mathbf{U}$

  $v(a) \in \mathbf{U}$ ($a$ individual constant)

- $v$ assigns each *functional symbol* a *function* defined on $\mathbf{U}$

  $v(f/n) : \mathbf{U}^n \rightarrow \mathbf{U}$ ($f/n$ functional symbol)

- $v$ assigns each *predicate symbol* a *relation* defined on $\mathbf{U}$

  $v(P/m) \subseteq \mathbf{U}^m$ ($P/m$ predicate symbol)

# First-order language (*without variables*)

- Well-formed formulae (wff)

  All symbols in the *signature* $\Sigma$ (i.e. *constants*, *function* and *predicate symbols*)

  Two (primary) ***logical connectives***: $\neg$, $\rightarrow$

  Three (derived) ***logical connectives***: $\wedge$, $\vee$, $\leftrightarrow$

  Parenthesis: (, ) (there are no *precedence rules* in this language)

  The definition of *terms* and *atoms* (see before)

  A set of syntactic rules

  The set of all the **wff** of $L_{FO}$ is denoted as $\text{wff}(L_{FO})$

$$\varphi \text{ is an } \textit{atom} \Rightarrow \varphi \in \text{wff}(L_{FO})$$

$$\varphi \in \text{wff}(L_{FO}) \Rightarrow (\neg\varphi) \in \text{wff}(L_{FO})$$

$$\varphi, \psi \in \text{wff}(L_{FO}) \Rightarrow (\varphi \rightarrow \psi) \in \text{wff}(L_{FO})$$

$$\varphi, \psi \in \text{wff}(L_{FO}) \Rightarrow (\varphi \vee \psi) \in \text{wff}(L_{FO}), \quad (\varphi \vee \psi) \Leftrightarrow ((\neg\varphi) \rightarrow \psi)$$

$$\varphi, \psi \in \text{wff}(L_{FO}) \Rightarrow (\varphi \wedge \psi) \in \text{wff}(L_{FO}), \quad (\varphi \wedge \psi) \Leftrightarrow (\neg(\varphi \rightarrow (\neg\psi)))$$

$$\varphi, \psi \in \text{wff}(L_{FO}) \Rightarrow (\varphi \leftrightarrow \psi) \in \text{wff}(L_{FO}), \quad (\varphi \leftrightarrow \psi) \Leftrightarrow ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$$

*Note that rules are identical to the propositional ones!*

# Satisfaction (*without* *variables*)

- Given a possible world $<\mathbf{U}, \Sigma, v>$

If $\varphi$ is an *atom* (i.e. $\varphi$ has the form $P(t_1, \ldots, t_n)$)

$<\mathbf{U}, \Sigma, v> \models \varphi$ iff $<v(t_1), \ldots, v(t_n)> \in v(P)$

If $\varphi$ e $\psi$ are wffs

$<\mathbf{U}, \Sigma, v> \models (\neg\varphi)$ iff $<\mathbf{U}, \Sigma, v> \not\models \varphi$

$<\mathbf{U}, \Sigma, v> \models (\varphi \rightarrow \psi)$ iff NOT $<\mathbf{U}, \Sigma, v> \models \varphi$ OR $<\mathbf{U}, \Sigma, v> \models \psi$

$<\mathbf{U}, \Sigma, v> \models (\varphi \land \psi)$ iff $<\mathbf{U}, \Sigma, v> \models \varphi$ AND $<\mathbf{U}, \Sigma, v> [s] \models \psi$

$<\mathbf{U}, \Sigma, v> \models (\varphi \lor \psi)$ iff $<\mathbf{U}, \Sigma, v> \models \varphi$ OR $<\mathbf{U}, \Sigma, v> [s] \models \psi$

# What is *true*?

- **A world of *cats***

| *Likes* | tom | spot | kitty | felix |
|---|---|---|---|---|
| tom | x | | | |
| spot | x | | x | |
| kitty | | x | x | |
| felix | | | x | |

***translates into*** $\langle \mathbf{U}, \Sigma, v \rangle$

- ***Universe***

$\mathbf{U} := \{\underline{\text{tom}}, \underline{\text{spot}}, \underline{\text{kitty}}, \underline{\text{felix}}\}$ ← *Could not put real cats in* **U**: *underlined names represent real objects*

- ***Signature***

$\Sigma := \{tom, spot, kitty, felix, Likes/2\}$ *i.e. four constants and one predicate symbol*

- ***Interpretation***

$v(tom) = \underline{\text{tom}}$, $v(spot) = \underline{\text{spot}}$, $v(kitty) = \underline{\text{kitty}}$, $v(felix) = \underline{\text{felix}}$,

$v(Likes/2) =$ *a subset of* $\mathbf{U} \times \mathbf{U}$

$\{\langle\underline{\text{tom}}, \underline{\text{tom}}\rangle, \langle\underline{\text{spot}}, \underline{\text{tom}}\rangle, \langle\underline{\text{spot}}, \underline{\text{kitty}}\rangle, \langle\underline{\text{kitty}}, \underline{\text{spot}}\rangle, \langle\underline{\text{kitty}}, \underline{\text{kitty}}\rangle, \langle\underline{\text{felix}}, \underline{\text{kitty}}\rangle\}$

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***translates into*** $<\mathbf{U}, \Sigma, v>$

- ***Sentences***

$<\mathbf{U}, \Sigma, v> \models Likes(spot, tom)$ because $<v(spot), v(tom)> \in v(Likes/2)$

$<\mathbf{U}, \Sigma, v> \models Likes(tom, tom)$ because $<v(tom), v(tom)> \in v(Likes/2)$

$<\mathbf{U}, \Sigma, v> \models \neg Likes(kitty, felix)$ because $<v(kitty), v(felix)> \notin v(Likes/2)$

$<\mathbf{U}, \Sigma, v> \not\models Likes(tom, kitty)$ because $<v(tom), v(kitty)> \notin v(Likes/2)$

$<\mathbf{U}, \Sigma, v> \not\models \neg Likes(felix, kitty)$ because $<v(felix), v(kitty)> \in v(Likes/2)$

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- **A world of *cats***

| *Likes* | tom | spot | kitty | felix |
|---|---|---|---|---|
| tom | x | | | |
| spot | x | | x | |
| kitty | | x | x | |
| felix | | | x | |

***translates into*** $<\mathbf{U}, \Sigma, v>$

- ***Sentences***

$$<\mathbf{U}, \Sigma, v> \models (Likes(spot, tom) \wedge Likes(felix, kitty))$$

$$<\mathbf{U}, \Sigma, v> \models (Likes(tom, kitty) \vee Likes(tom, tom))$$

$$<\mathbf{U}, \Sigma, v> \models (Likes(spot, tom) \vee \neg Likes(spot, tom))$$

*is satisfied in this possible world but also in <u>any</u> possible world*

# First-order language

- Well-formed formulae (wff)
  - All symbols in the *signature* $\Sigma$ (i.e. *constants*, *function* and *predicate symbols*)
  - A set of **variables**: $x, y, z$
  - Two (primary) ***logical connectives***: $\neg, \rightarrow$
  - Three (derived) ***logical connectives***: $\wedge, \vee, \leftrightarrow$
  - Two **quantifiers**: $\forall, \exists$
  - Parenthesis: (, ) (there are no *precedence rules* in this language)

An extended definition of *terms* and *atoms*

**Term**

A single *individual constant* or a variable is a **term**

If $f/n$ is a *functional symbol* (with arity $n$) and $t_1, \dots, t_n$ are **terms**, then $f(t_1, \dots, t_n)$ is a **term**

**Atom**

If $P/n$ is a *predicate symbol* (with arity $n$) and $t_1, \dots, t_n$ are **terms**, then $P(t_1, \dots, t_n)$ is an **atom** (i.e a first-order well-formed formula – wff)

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An extended definition of *terms* and *atoms (see before)*

A set of syntactic rules

$\varphi$ is an *atom* $\Rightarrow \varphi \in \mathrm{wff}(L_{FO})$

$\varphi \in \mathrm{wff}(L_{FO}) \Rightarrow (\neg\varphi) \in \mathrm{wff}(L_{FO})$

$\varphi, \psi \in \mathrm{wff}(L_{FO}) \Rightarrow (\varphi \rightarrow \psi) \in \mathrm{wff}(L_{FO})$

$\varphi, \psi \in \mathrm{wff}(L_{FO}) \Rightarrow (\varphi \vee \psi) \in \mathrm{wff}(L_{FO}), \quad (\varphi \vee \psi) \Leftrightarrow ((\neg\varphi) \rightarrow \psi)$

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$\varphi, \psi \in \mathrm{wff}(L_{FO}) \Rightarrow (\varphi \leftrightarrow \psi) \in \mathrm{wff}(L_{FO}), \quad (\varphi \leftrightarrow \psi) \Leftrightarrow ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$

$\varphi \in \mathrm{wff}(L_{FO}) \Rightarrow (\forall x\, \varphi) \in \mathrm{wff}(L_{FO})$ ← $x$ *can be any variable*

$\varphi \in \mathrm{wff}(L_{FO}) \Rightarrow (\exists x\, \varphi) \in \mathrm{wff}(L_{FO})$

# Satisfaction

- Given a possible world $<\mathbf{U}, \Sigma, v>$ and a valuation $s$ (on that world)

  A valuation is a function $s$ : *Variables* $\rightarrow \mathbf{U}$ ⟵ *A valuation s transforms all variables into constants*

  If $\varphi$ is an *atom* (i.e. $\varphi$ has the form $P(t_1, \ldots, t_n)$)
  $<\mathbf{U}, \Sigma, v> [s] \models \varphi$ iff $<v(t_1)\,[s], \ldots, v(t_n)\,[s]> \in v(P)\,[s]$

  If $\varphi$ e $\psi$ are wffs

  $<\mathbf{U}, \Sigma, v> [s] \models (\neg\varphi)$ iff $<\mathbf{U}, \Sigma, v> [s] \not\models \varphi$

  $<\mathbf{U}, \Sigma, v> [s] \models (\varphi \wedge \psi)$ iff $<\mathbf{U}, \Sigma, v> [s] \models \varphi$ AND $<\mathbf{U}, \Sigma, v> [s] \models \psi$

  $<\mathbf{U}, \Sigma, v> [s] \models (\varphi \vee \psi)$ iff $<\mathbf{U}, \Sigma, v> [s] \models \varphi$ OR $<\mathbf{U}, \Sigma, v> [s] \models \psi$

  $<\mathbf{U}, \Sigma, v> [s] \models (\varphi \rightarrow \psi)$ iff NOT $<\mathbf{U}, \Sigma, v> [s] \models \varphi$ OR $<\mathbf{U}, \Sigma, v> [s] \models \psi$

  Quantified formulae

  $<\mathbf{U}, \Sigma, v> [s] \models \forall x\, \varphi$ iff FORALL $\underline{d} \in \mathbf{U}$ we have $<\mathbf{U}, \Sigma, v> [s](x{:}\underline{d}) \models \varphi$

  $<\mathbf{U}, \Sigma, v> [s] \models \exists x\, \varphi$ iff it EXISTS $\underline{d} \in \mathbf{U}$ such that $<\mathbf{U}, \Sigma, v> [s](x{:}\underline{d}) \models \varphi$

  Where $[s](x{:}\underline{d})$ is the *variant* of function $s$ that assigns $\underline{d}$ to $x$ and remains unaltered for any other variables.

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***translates into*** $<\mathbf{U}, \Sigma, v>$

- ***Sentences***

$<\mathbf{U}, \Sigma, v>[s] \models (\forall x\ (\exists y\ Likes(x, y)))$ because

FORALL $\underline{cat1} \in \mathbf{U}$, $<\mathbf{U}, \Sigma, v>[s](x{:}\underline{cat1}) \models (\exists y\ Likes(x, y))$ because

it EXISTS $\underline{cat2} \in \mathbf{U}$, $<\mathbf{U}, \Sigma, v>([s](x{:}\underline{cat1}))(y{:}\underline{cat2}) \models Likes(x, y)$

# What is *true*?

- **A world of *cats***

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| tom | x | | | |
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***translates into*** $<\mathbf{U}, \Sigma, v>$

- ***Sentences***

$<\mathbf{U}, \Sigma, v>[s] \not\models (\exists x\, (\forall y\, Likes(x, y)))$ because

FORALL $\underline{cat1} \in \mathbf{U}$, $<\mathbf{U}, \Sigma, v>[s](x{:}\underline{cat1}) \not\models (\forall y\, Likes(x, y))$ because

it EXISTS $\underline{cat2} \in \mathbf{U}$, $<\mathbf{U}, \Sigma, v>([s](x{:}\underline{cat1}))(y{:}\underline{cat2}) \not\models Likes(x, y)$

# Variables and quantifiers: further examples

- "Being brothers means being relatives"

  $\forall x \forall y\ (Brother(x, y) \rightarrow Relative(x, y))$

- "Being relative is a symmetric relation"

  $\forall x \forall y\ (Relative(x\ ,y) \leftrightarrow Relative(y, x))$

- "By definition, being mother is being parent and female"

  $\forall x\ (Mother(x) \leftrightarrow (\exists y\ Parent(x, y)\ \wedge\ Female(x)))$

- "A cousin is a son of either a brother or a sister of either parents"

  $\forall x \forall y\ (Cousin(x,y)$

  $\leftrightarrow \exists z \exists w\ (Parent(z, x)\ \wedge\ Parent(w, y)\ \wedge\ (Brother(z, w)\ \vee\ Sister(z, w))))$

- "Everyone has a mother"

  $\forall x \exists y\ Mother(y, x)$

  BE CAREFUL about the order of quantifiers, in fact:

  $\exists y \forall x\ Mother(y, x)$

  "There is one (common) mother to everyone"

# Open formulae, sentences

- ***Bound*** and ***free*** variables

  The occurrence of a *variable* in a wff is ***bound*** if it lies in the scope of a quantifier (for that *variable*)

  The occurrence of a *variable* in a wff is ***free*** if it is not *bound*

  Examples of bound variables:
  $\forall x\ P(x)$
  $\exists x\ (P(x) \to (A(x) \land B(x))$

  Examples of free variables:
  $P(x)$
  $\exists y\ (P(y) \to (A(x,y) \land B(y)))$

- ***Open*** and ***closed*** formulae: ***sentences***

  A wff is ***open*** if there is at least one free occurrence of a variable

  Otherwise, the wff is ***closed*** (also called ***sentence***)

  Only *closed* wffs, i.e. *sentences*, have a truth value (see after)

# Models

- **Validity** in a possible world, **model**

  A wff $\varphi$ such that $<\mathbf{U}, \Sigma, v> [s] \models \varphi$ for any *valuation* $s$ is **valid** in $<\mathbf{U}, \Sigma, v>$

  $<\mathbf{U}, \Sigma, v>$ is also a **model** of $\varphi$

  and we write $<\mathbf{U}, \Sigma, v> \models \varphi$ (i.e. the reference to $s$ can be omitted)

  A possible world $<\mathbf{U}, \Sigma, v>$ is a **model** of a *set* of wff $\Gamma$ iff it is a model for all the wffs in $\Gamma$

  and we write $<\mathbf{U}, \Sigma, v> \models \Gamma$

- **Truth**

  A **sentence** $\psi$ such that $<\mathbf{U}, \Sigma, v> [s] \models \psi$ for one valuation $s$ is **valid** in $<\mathbf{U}, \Sigma, v>$

  *If the sentence is true for one valuation $s$, then is true for all valuations*

  A **sentence** $\psi$ is **true** in $<\mathbf{U}, \Sigma, v>$ if it is **valid** in $<\mathbf{U}, \Sigma, v>$

# Validity in general

- Validity and logical truth

  A wff (either open or closed) is **valid** (also **logically valid**)
  if it is **valid** in any possible world $<\mathbf{U}, \Sigma, v>$

  Example:

  $(P(x) \lor \neg P(x))$

  A sentence $\psi$ is a **logical truth**
  if it is **true** in any possible world $<\mathbf{U}, \Sigma, v>$

  we write then $\models \psi$ (i.e. no reference to $<\mathbf{U}, \Sigma, v>$)

  Examples:

  $\forall x\, (P(x) \lor \neg P(x))$

  $\forall x \forall y\, (G(x,y) \to (H(x,y) \to G(x,y)))$

- Inconsistence

  A wff (either open or closed) is **inconsistent** if its not *satisfiable*

  Example:

  $\forall x\, (P(x) \land \neg P(x))$

# Entailment

- Definition

  Given a set of wffs $\Gamma$ and one wff $\varphi$ , we have

  $$\Gamma \models \varphi$$

  iff all possible worlds $<\mathbf{U}, \Sigma, v>$ $[s]$ satisfying $\Gamma$ also satisfy $\varphi$

  This definition embraces all possible combinations $<\mathbf{U}, \Sigma, v>$ $[s]$
  The only thing that does not vary is the language $\Sigma$

  Is this problem decidable?

  *In general, a direct calculus of entailment is impossible…*

# *Say it with functions or predicates?

Semantically, functions and predicates are very similar to each other:
can we get rid of functions at all?

- Functions are *relations*

  Hence they can be *represented* via predicates
  For instance, the two sentences:

$$\forall x \forall y \forall z \; ((\varphi(x,y) \land \varphi(x,z)) \rightarrow (y = z))$$

$$\forall x \exists y \; \varphi(x,y)$$

  say altogether that the meaning of $\varphi(..)$ (i.e. a relation $v(\varphi) \subseteq \mathbf{U}^2$)
  is also a *function* $\mathbf{U} \rightarrow \mathbf{U}$

- But only functions can be nested in terms

  Therefore, functions allow for a much greater expressive power
  (*which will reflect into a much greater difficulty in calculus* ...)

# *The discreet charme of functions

- Representing data structures: *lists of items* $[a, b, c, \ldots]$

  Symbols in $\Sigma$

  $cons/2$
  *it's a function* that associates items (e.g. $a$) to a list (e.g. $[b, c]$)
  $cons(a, cons(b, cons(c, nil)))$ represents the list $[a, b, c]$

  $Append/3$
  *it's a predicate*: each pair of lists $x$ and $y$ is associated to their *concatenation* $z$

  $nil$
  *it's a constant*, represents the *empty list*.

  Axioms ($\mathbf{AL}$)

  $$\forall x\ Append(nil, x, x)$$
  $$\forall x\ \forall y\ \forall z\ (Append(x, y, z) \rightarrow \forall s\ Append(cons(s, x), y, cons(s, z)))$$

  Examples of entailment

  $$\{\mathbf{AL} + \exists z\ Append(cons(a, nil), cons(b, cons(c, nil), z)\ \}$$
  $$\models Append(cons(a, nil), cons(b, cons(c, nil)), cons(a, cons(b, cons(\mathrm{c}, \mathrm{nil}))))$$
  $$\{\mathbf{AL} + \exists x\ \exists y\ Append(x, y, cons(a, cons(b, nil)))\}$$
  $$\models Append(cons(a, nil), cons(b, nil), cons(a, cons(b, nil)))$$
  $$\models Append(nil, cons(a, cons(b, nil)), cons(a, cons(b, nil)))$$
  $$\models Append(cons(a, cons(b, nil)), nil, cons(a, cons(b, nil)))$$