Logic Programs and Minimal Models

Marco Piastra

Logic Program

- An example of logic program:

$$\Pi \equiv \{\{Human(x), \neg Philosopher(x)\}, \{Mortal(y), \neg Human(y)\}, \{Philosopher(socratess)\}, \{Philosopher(plato)\}, \{Philosopher(aristotle)\}\}$$

$$\phi \equiv \exists x\, Mortal(x)$$

$$\neg\phi \equiv \neg\exists x\, Mortal(x) \equiv \forall x\, \neg Mortal(x) \equiv \{\neg Mortal(x)\}$$ (a goal, i.e. a Horn clause)

By applying resolution in an exhaustive way, we obtain:

$$\Sigma \equiv \{[x/socratess], [x/plato], [x/aristotle]\}$$

Looks like a query on an *implicit* database ...

- **Answer Set**

It includes all complete substitutions of the variables in the *goal* corresponding to the closed branches (i.e. with an empty clause) in the SLD tree

Herbrand Universe, Herbrand Base

- Herbrand **terms** and **atoms**

  Given a signature Σ

  A Herbrand **term** is a *ground term* (i.e. a term that contains no variables)

  Examples:

  $f(a)$, $g(a,b)$, $g(f(a),b)$, $g(f(a),g(b,c))$, $g(f(a),g(f(b),c))$, …

  A Herbrand **atom** is a *ground atom* (i.e. an atom that contains no variables)

  Examples:

  $P(f(a))$, $P(g(a,b))$, $Q(g(f(a),b), g(f(a),g(b,c)))$, …

- Herbrand **universe**

  The set of all Herbrand *terms* from Σ

  Example:

  $\mathbf{U}_H \equiv \{f(a), g(a,b), g(f(a),b), g(f(a),g(b,c)), g(f(a),g(f(b),c)), \ldots \}$

- Herbrand **base**

  The set of all Herbrand *atoms* from Σ

  Example:

  $\mathrm{B}_H \equiv \{P(f(a)), P(g(a,b)), Q(g(f(a),b), g(f(a),g(b,c))), \ldots\}$

Herbrand models

- Herbrand **structure**

  A semantic structure $<\mathbf{U}_\mathrm{H}, \Sigma, v_\mathrm{H}>$ such that

- Herbrand **interpretation** v_H

  For constants, $v_\mathrm{H}(c) = c$

  For ground terms, $v_\mathrm{H}(t) = t$

  For predicate symbols, $v_\mathrm{H} \subseteq \mathrm{B}_\mathrm{H}$

  i.e. a **subset** of the Herbrand base B_H

  Example: $v_\mathrm{H} \equiv \{P(a), P(f(b)), P(c), Q(a,g(b,c)), Q(b,c) \ldots \}$

- Herbrand **model**

  $\varphi \in \mathrm{Atom}(L_{PO})$, $<\mathbf{U}_\mathrm{H}, \Sigma, v_\mathrm{H}> [s] \models \varphi$ iff $\varphi \in v_\mathrm{H}$

  $\varphi \in \mathrm{Atom}(L_{PO})$, $<\mathbf{U}_\mathrm{H}, \Sigma, v_\mathrm{H}> [s] \models \neg\varphi$ iff $\varphi \notin v_\mathrm{H}$

  $<\mathbf{U}_\mathrm{H}, \Sigma, v_\mathrm{H}> [s] \models \neg\varphi$ iff $<\mathbf{U}_\mathrm{H}, \Sigma, v_\mathrm{H}> [s] \not\models \varphi$

  $<\mathbf{U}_\mathrm{H}, \Sigma, v_\mathrm{H}> [s] \models \varphi \to \psi$ iff ($<\mathbf{U}_\mathrm{H}, \Sigma, v_\mathrm{H}> [s] \not\models \varphi$ or $<\mathbf{U}_\mathrm{H}, \Sigma, v_\mathrm{H}> [s] \models \psi$)

  $<\mathbf{U}_\mathrm{H}, \Sigma, v_\mathrm{H}> [s] \models \forall x\, \varphi$ iff for all $c \in \mathrm{Cost}(L_{PO})$, $<\mathbf{U}_\mathrm{H}, \Sigma, v_\mathrm{H}> [s](x{:}c) \models \varphi$

Horn clauses and Herbrand models

- **Herbrand Theorem**

  Given a theory of universal sentences Φ,

  $H(\Phi)$ has a model iff Φ has a model

- Corollary (for *Horn clauses*)

  Given a set Φ of Horn clauses, the two following statements are equivalent:

  - Φ is satisfiable
  - Φ has an Herbrand model

  This is not true in general: only if Φ is a set of Horn clauses

Clausole di Horn e modelli di Herbrand

- Corollary to Herbrand theorem (for *Horn clauses*)

  Given a set Φ of Horn clauses, the two following statements are equivalent:
  - Φ is satisfiable
  - Φ has an Herbrand model

  This is not true in general: only if Φ is a set of Horn clauses

- Herbrand **minimal model**

  The minimal model M_Φ for a set of Horn clauses Φ is:

  $M_\Phi \equiv \cap_{\forall i} M_i$ where M_i is a Herbrand model of Φ

- Theorem (van Emden e Kowalski, 1976)

  Let Φ be a set of Horn clauses and φ a ground atom.
  These three statements are equivalent:
  - $\Phi \models \varphi$
  - $\varphi \in M_\Phi$
  - φ is derivable from Φ via resolution with refutation

Logic programming system and minimal model

- Theorem (Apt e van Emden, 1982)

  Let Π be a ***logical program*** (i.e. *a set of definite clauses*).
  The (finite) success set of Π with SLD-resolution (*fair*) coincides with M_Π

- A logic programming system (i.e. Prolog) can generate the *subset* of M_Π corresponding to a specific ***goal***

  A goal $\{\neg\alpha_1, \neg\alpha_2, \ldots, \neg\alpha_m\}$ where the variables $x_1, x_2, \ldots, x_m$ occur
  is equivalent to the sentence $\forall x_1 \forall x_2 \ldots \forall x_n (\neg\alpha_1 \vee \neg\alpha_2 \vee \ldots \vee \neg\alpha_m)$
  which is equivalent to $\neg\exists x_1 \exists x_2 \ldots \exists x_n (\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_m)$

  A logic programming system can generate all possible ***substitutions*** $[x_1/t_1, x_2/t_2, \ldots, x_n/t_n]$
  such that $\Pi \cup \{\neg(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_m)[x_1/t_1, x_2/t_2, \ldots, x_n/t_n]\}$ is *unsatisfiable*

  (that implies $\Pi \models (\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_m)[x_1/t_1, x_2/t_2, \ldots, x_n/t_n]$)

  (that implies $(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_m)[x_1/t_1, x_2/t_2, \ldots, x_n/t_n] \in M_\Pi$)

  Each goal act like a *filter*, i.e. defining the subset of M_Π

*NOTE: a logic programming system with a **fair** strategy can do so…*