First-Order Logic

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Propositional possible worlds

Each possible world is a structure $<\{0,1\}, \boldsymbol{P}, v>$

- $\{0,1\}$ are the *truth values*
- $\boldsymbol{P}$ is the ***signature*** of the formal language: a set of propositional symbols
- v is a *function* $: \boldsymbol{P} \to \{0,1\}$ assigning truth values to the symbols in $\boldsymbol{P}$

Propositional symbols (*signature*)

- Each symbol in $\boldsymbol{P}$ stands for an actual *proposition* (in natural language)
  - In the simple convention, we use the symbols $A, B, C, D, \ldots$
  - Caution: $\boldsymbol{P}$ is not necessarily *finite*

Possible worlds

- The class of structures contains all possible worlds:

  $<\{0,1\}, \boldsymbol{P}, v>$
  $<\{0,1\}, \boldsymbol{P}, v'>$
  $<\{0,1\}, \boldsymbol{P}, v''>$
  $\ldots$

  - Each class of structure shares $\boldsymbol{P}$ and $\{0,1\}$
    - The functions v are different: the assignment of truth values varies, depending on the possible world
    - If $\boldsymbol{P}$ is finite, there are only *finitely* many distinct possible worlds (actually $2^{|\boldsymbol{P}|}$)

First-order possible worlds

Possible worlds made of objects, functions and relations

Each possible world is a structure $<\mathbf{U}, \Sigma, v>$

$\mathbf{U}$ is a set of object, called ***domain*** (also *universe of discourse*)

Σ is a set of symbols, called ***signature***

v is a *function* that gives a *meaning* to the symbol in Σ with respect to $\mathbf{U}$

Signature Σ

- *individual constants*: a, b, c, d, ...
- *function symbols (with arity)*: f /n, g /p, h /q, ...
- *predicate symbols* (with *arity*): P /k, Q /l, R /m, ...

Arity is an integer number that describes the expected number of arguments

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Term

- A single *individual constant* is a **term**
- If f/n is a *functional symbol* (with arity n) and $t_1, \ldots, t_n$ are **terms**, then $f(t_1, \ldots, t_n)$ is a **term**

Atom

- If P/n is a *predicate symbol* (with arity n) and $t_1, \ldots, t_n$ are **terms**, then $P(t_1, \ldots, t_n)$ is an **atom** (i.e a first-order well-formed formula – wff)

First-order possible worlds

Possible worlds made of objects, functions and relations

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Function v (***interpretation***)

- v assigns each *individual constant* to an *object* in $\mathbf{U}$

  $v(a) \in \mathbf{U}$ (a individual constant)

- v assigns each *functional symbol* a *function* defined on $\mathbf{U}$

  $v(f/n) : \mathbf{U}^n \rightarrow \mathbf{U}$ (f/n functional symbol)

- v assigns each *predicate symbol* a *relation* defined on $\mathbf{U}$

  $v(P/m) \subseteq \mathbf{U}^m$ (P/n predicate symbol)

Say it with atoms

- Example structure $<\mathbf{U}, \Sigma, v>$

  Domain **U**

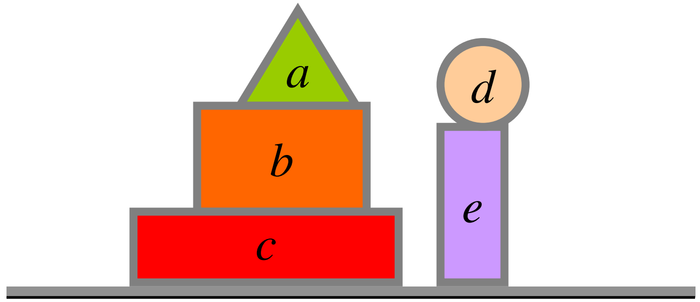
  Objects: { <u>*a*</u>, <u>*b*</u>, <u>*c*</u>, <u>*d*</u>, <u>*e*</u>, <u>*green*</u>, <u>*orange*</u>, <u>*red*</u>, <u>*rose*</u>, <u>*violet* </u>}

  Signature Σ

  Individual constants: *a, b, c, d, e, green, orange, red, rose, violet*

  Function symbols: *colorOf*/1

  Predicate symbols: *Pyramid*/1, *Parallelepiped*/1, *Sphere*/1, *Ontable*/1, *Clear*/1, *Above*/2, =/2



<u>*Objects*</u> *in* **U** *are underlined*
constant symbols in Σ *are not*

How does $<\mathbf{U}, \Sigma, v>$ *satisfy* a set of atoms:

$<\mathbf{U}, \Sigma, v> \models$ *Pyramid*(*a*)
Parallelepiped(*b*), *Parallelepiped*(*c*), *Parallelepiped*(*e*)
Sphere(*d*)
Ontable(*c*), *Ontable*(*e*)
Clear(*a*), *Clear*(*d*)

$<\mathbf{U}, \Sigma, v> \models$ (*colorOf*(*a*) = *green*), (*colorOf*(*b*) = *orange*), (*colorOf*(*c*) = *red*) , (*colorOf*(*d*) = *rose*)

$<\mathbf{U}, \Sigma, v> \models$ *Above*(*a*,*b*), *Above*(*b*,*c*), *Above*(*a*,*c*), *Above*(*d*,*e*)

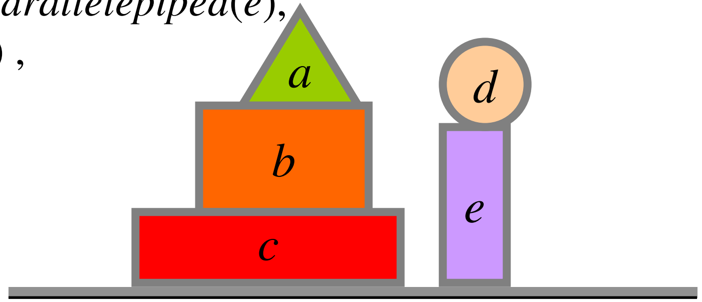
Say it with atoms

Different possible worlds (only interpretation functions v change, in this example)

$\langle \mathbf{U}, \Sigma, v_1 \rangle \models$ *Pyramid*(*a*), *Parallelepiped*(*b*), *Parallelepiped*(*c*), *Sphere*(*d*), *Parallelepiped*(*e*),
(*colorOf*(*a*) = *green*), (*colorOf*(*b*) = *orange*), (*colorOf*(*c*) = *red*) ,
(*colorOf*(*d*) = *rose*) , (*colorOf*(*e*) = *violet*)

Ontable(*c*), *Ontable*(*e*), *Clear*(*a*), *Clear*(*d*)

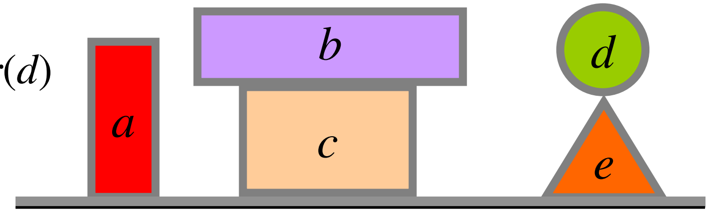
Above(*a*,*b*), *Above*(*b*,*c*), *Above*(*a*,*c*), *Above*(*d*,*e*)



$\langle \mathbf{U}, \Sigma, v_2 \rangle \models$ *Parallelepiped*(*a*), *Parallelepiped*(*b*), *Parallelepiped*(*c*), *Sphere*(*d*), *Pyramid*(*e*),
(*colorOf*(*a*) = *red*), (*colorOf*(*b*) = *violet*), (*colorOf*(*c*) = *pink*) ,
(*colorOf*(*d*) = *green*) , (*colorOf*(*e*) = *orange*)

Ontable(*a*), *Ontable*(*c*), *Ontable*(*e*), *Clear*(*a*), , *Clear*(*b*), *Clear*(*d*)

Above(*b*,*c*), *Above*(*d*,*e*)

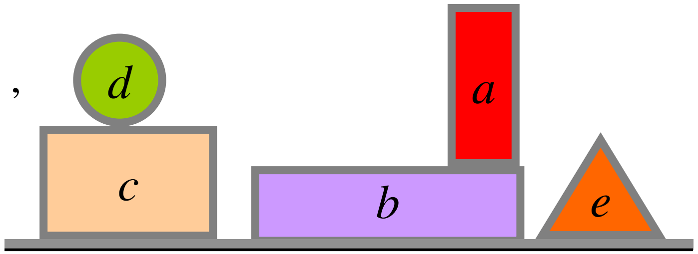


$\langle \mathbf{U}, \Sigma, v_3 \rangle \models$ *Pyramid*(*a*), *Parallelepiped*(*b*), *Parallelepiped*(*c*), *Parallelepiped*(*e*)
Sphere(*d*)

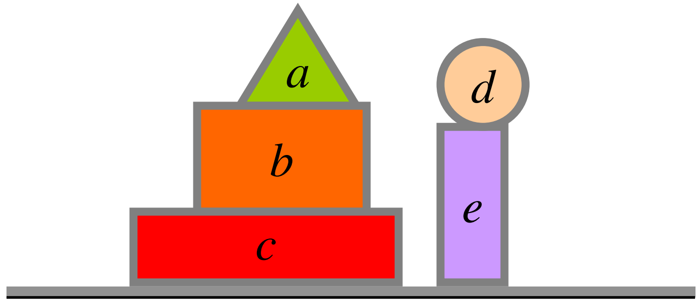
(*colorOf*(*a*) = *green*), (*colorOf*(*b*) = *orange*), (*colorOf*(*c*) = *red*) ,
(*colorOf*(*d*) = *rose*) , (*colorOf*(*e*) = *violet*)

Ontable(*c*), *Ontable*(*e*), *Clear*(*a*), *Clear*(*d*)

Above(*a*,*b*), *Above*(*b*,*c*), *Above*(*a*,*c*), *Above*(*d*,*e*)



Abstraction: variables and quantifiers



(*just* *<u>intuitive</u>* *semantics, for now*)

More general properties

$\neg\forall x\, \exists y\, (Above(x,y))$

$\neg\forall y\, \exists x\, (Above(x,y))$

Defining *new* predicates

$\forall x \forall y\, (On(x,y) \leftrightarrow (Above(x,y)\ \wedge\ \neg\exists z\, (Above(x,z)\ \wedge\ Above(z,y))))$

$\forall x\, (Ontable(x) \leftrightarrow \neg\exists z\, Above(x,z))$

$\forall x\, (Clear(x) \leftrightarrow \neg\exists z\, Above(z,x))$

Abstraction: variables and quantifiers

- "Being brothers means being relatives"

  $\forall x \forall y\ (Brother(x, y) \rightarrow Relative(x, y))$

- "Being relative is a symmetric relation"

  $\forall x \forall y\ (Relative(x, y) \leftrightarrow Relative(y, x))$

- "By definition, being mother is being parent and female"

  $\forall x\ (Mother(x) \leftrightarrow (\exists y\ Parent(x, y) \wedge Female(x)))$

- "A cousin is a son of either a brother or a sister of either parents"

  $$\forall x \forall y\ (Cousin(x,y) \leftrightarrow \exists z \exists w\ (Parent(z, x) \wedge Parent(w, y) \wedge (Brother(z, w) \vee Sister(z, w))))$$

- "Everyone has a mother"

  $\forall x \exists y\ Mother(y, x)$

  BE CAREFUL about the order of quantifiers, in fact:

  $\exists y \forall x\ Mother(y, x)$

  "There is one (common) mother to everyone"

First-order language

- Well-formed formulae (wff)

Starting from a *signature* Σ, add *variables* $x, y, z \ldots$ and allow variables in *terms:*

A single *individual constant* or a *variable* is a **term**

If f/n is a *functional symbol* (with arity n) and $t_1, \ldots, t_n$ are **terms**, then $f(t_1, \ldots, t_n)$ is a **term**

The definition of *atoms* remains unchanged

Every *atom* composed from Σ and the variables is a $\mathrm{wff}(L_{PO})$

$$\varphi \in \mathrm{wff}(L_{PO}) \Rightarrow (\neg\varphi) \in \mathrm{wff}(L_{PO})$$
$$\varphi, \psi \in \mathrm{wff}(L_{PO}) \Rightarrow (\varphi \to \psi) \in \mathrm{wff}(L_{PO})$$
$$\varphi \in \mathrm{wff}(L_{PO}) \Rightarrow (\forall x\, \varphi) \in \mathrm{wff}(L_{PO})$$

$$\varphi, \psi \in \mathrm{wff}(L_{PO}), \quad (\varphi \vee \psi) \Leftrightarrow ((\neg\varphi) \to \psi)$$
$$\varphi, \psi \in \mathrm{wff}(L_{PO}), \quad (\varphi \wedge \psi) \Leftrightarrow (\neg(\varphi \to (\neg\psi)))$$
$$\varphi, \psi \in \mathrm{wff}(L_{PO}), \quad (\varphi \leftrightarrow \psi) \Leftrightarrow ((\varphi \to \psi) \wedge (\psi \to \varphi))$$
$$\varphi \in \mathrm{wff}(L_{PO}) \quad (\exists x\, \varphi) \Leftrightarrow (\neg\forall x\, \neg\varphi)$$

Open formulae, sentences

- ***Bound*** and ***free*** variables

  The occurrence of a *variable* in a wff is ***bound*** if it lies in the scope of a quantifier (for that *variable*)

  The occurrence of a *variable* in a wff is ***free*** if it is not *bound*

  Examples of bound variables:
  $\forall x\, P(x)$
  $\exists x\, (P(x) \rightarrow (A(x) \wedge B(x))$

  Examples of free variables:
  $P(x)$
  $\exists y\, (P(y) \rightarrow (A(x,y) \wedge B(y)))$

- ***Open*** and ***closed*** formulae

  A wff is ***open*** if there is at least one free occurrence of a variable

  Otherwise, the wff is ***closed*** (also called ***sentence***)

  Only *closed* wffs, i.e. *sentences*, have a truth value (see after)

Possible worlds, interpretations, valuation

- Possible world: a structure $<\mathbf{U}, \Sigma, v>$

  $\mathbf{U}$ is a set of object, called ***domain*** (also *universe of discourse*)

  Σ is a set of symbols, called ***signature***

  v is a *function* that:

  - v assigns each *individual constant* to an *object* in $\mathbf{U}$

    $v(a) \in \mathbf{U}$ (a individual constant)

  - v assigns each *functional symbol* a *function* defined on $\mathbf{U}$

    $v(f /n)$: $\mathbf{U}^n \rightarrow \mathbf{U}$ (f /n functional symbol)

  - v assigns each *predicate symbol* a *relation* defined on $\mathbf{U}$

    $v(P /m) \subseteq \mathbf{U}^m$ (P /n predicate symbol)

  Function v does not assign a value to <u>variables</u>

- Valuation (of *<u>variables</u>*): a function s

  Given a possible world $<\mathbf{U}, \Sigma, v>$, a ***valuation*** s

  is a *function* that assigns at each ***variable*** x an ***object*** in $\mathbf{U}$

  $s(x) \in \mathbf{U}$

Satisfaction

- Given a possible world $<\mathbf{U}, \Sigma, v>$ and a valuation s

If φ is an *atom* (i.e. φ has the form $P(t_1, \dots, t_n)$)

$<\mathbf{U}, \Sigma, v> [s] \models \varphi$ iff $<v(t_1)\,[s], \dots, v(t_n)\,[s]> \in v(P)\,[s]$

If φ e ψ are wffs

$<\mathbf{U}, \Sigma, v> [s] \models (\neg\varphi)$ iff $<\mathbf{U}, \Sigma, v> [s] \not\models \varphi$

$<\mathbf{U}, \Sigma, v> [s] \models (\varphi \wedge \psi)$ iff $<\mathbf{U}, \Sigma, v> [s] \models \varphi$ AND $<\mathbf{U}, \Sigma, v> [s] \models \psi$

$<\mathbf{U}, \Sigma, v> [s] \models (\varphi \vee \psi)$ iff $<\mathbf{U}, \Sigma, v> [s] \models \varphi$ OR $<\mathbf{U}, \Sigma, v> [s] \models \psi$

$<\mathbf{U}, \Sigma, v> [s] \models (\varphi \rightarrow \psi)$ iff NOT $<\mathbf{U}, \Sigma, v> [s] \models \varphi$ OR $<\mathbf{U}, \Sigma, v> [s] \models \psi$

Quantified formulae

$<\mathbf{U}, \Sigma, v> [s] \models \forall x\, \varphi$ iff FORALL $\underline{d} \in \mathbf{U}$ we have $<\mathbf{U}, \Sigma, v> [s](x{:}\underline{d}) \models \varphi$

$<\mathbf{U}, \Sigma, v> [s] \models \exists x\, \varphi$ iff it EXISTS $\underline{d} \in \mathbf{U}$ such that $<\mathbf{U}, \Sigma, v> [s](x{:}\underline{d}) \models \varphi$

Where $[s](x{:}\underline{d})$ is the *variant* of function s that assigns $\underline{d}$ to x and remains unaltered for any other variables.

Models

- **Validity** in a possible world, **model**

  A wff φ such that $\langle \mathbf{U}, \Sigma, v \rangle\, [s] \models \varphi$ for any *valuation* s
  is **valid** in $\langle \mathbf{U}, \Sigma, v \rangle$

  $\langle \mathbf{U}, \Sigma, v \rangle$ is also a **model** of φ
  and we write $\langle \mathbf{U}, \Sigma, v \rangle \models \varphi$ (i.e. the reference to s can be omitted)

  A possible world $\langle \mathbf{U}, \Sigma, v \rangle$ is a **model** of a *set* of wff Γ iff
  it is a model for all the wffs in Γ
  and we write $\langle \mathbf{U}, \Sigma, v \rangle \models \Gamma$

- **Truth**

  A **sentence** ψ is **true** in $\langle \mathbf{U}, \Sigma, v \rangle$ if it is **valid** in $\langle \mathbf{U}, \Sigma, v \rangle$

Validity in general

- Validity and logical truth

  A wff (either open or closed) is **valid** (also **logically valid**)
  if it is **valid** in any possible world <$\mathbf{U}$, Σ, v>

  Example:

  $(P(x) \vee \neg P(x))$

  A sentence ψ is a **logical truth**
  if it is **true** in any possible world <$\mathbf{U}$, Σ, v>

  we write then $\models \psi$ (i.e. no reference to <$\mathbf{U}$, Σ, v>)

  Examples:

  $\forall x\ (P(x) \vee \neg P(x))$

  $\forall x \forall y\ (G(x,y) \rightarrow (H(x,y) \rightarrow G(x,y)))$

- Inconsistence

  A wff (either open or closed) is **inconsistent** if its not *satisfiable*

  Example:

  $\forall x\ (P(x) \wedge \neg P(x))$

Entailment

- Definition

Given a set of wffs Γ and one wff φ , we have

$$\Gamma \models \varphi$$

iff all the combinations $<\mathbf{U}, \Sigma, v>$ $[s]$ satisfying Γ also satisfy φ

This definition embraces all possible combinations $<\mathbf{U}, \Sigma, v>$ $[s]$

The only thing that does not vary is the language Σ

In general, a direct calculus of entailment is impossible…

*Say it with function or predicates?

Semantically, functions and predicates are very similar to each other:
can we get rid of functions at all?

- Functions are *relations*

  Hence they can be *represented* via predicates
  For instance, the two sentences:

  $$\forall x \forall y \forall z\ ((\varphi(x,y) \land \varphi(x,z)) \to (y = z))$$

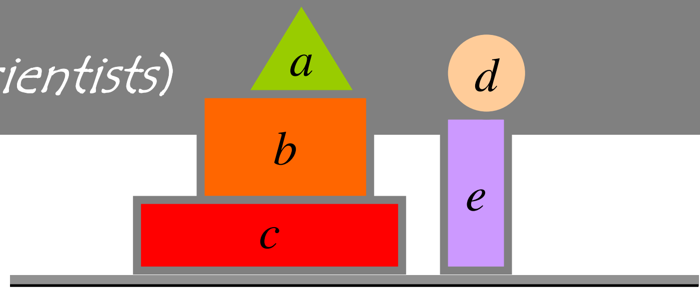
  $$\forall x \exists y\ \varphi(x,y)$$

  say altogether that the meaning of $\varphi(..)$ (i.e. a relation $v(\varphi) \subseteq \mathbf{U}^2$)
  is also a *function* $\mathbf{U} \to \mathbf{U}$

- But only functions can be nested in terms

  Therefore, functions allow for a much greater expressive power
  (*which will reflect into a much greater difficulty in calculus* ...)

*Many-sorted or nil? (just for computer scientists)



green, *colorOf*(*green*), *colorOf*(*colorOf*(*green*)), *colorOf*(*colorOf*(*colorOf*(*green*)))

All these terms are syntactically correct, although they do not make that much sense…

For practical applications, functions should be restricted to given *types* (i.e. *sort*)

The type should be made explicit for each function and predicate symbols (besides *arity*)

Convenience of *nil*

A particular constant: *nil*
which has the conventional interpretation of a *non-object*

This gives an alternative for otherwise meaningless definitions:

$(colorOf(a) = green) \wedge (colorOf(green) = nil)$

And for particular cases as well:

$Above(a,b) \wedge Above(b,c) \wedge Above(c,nil)$