Deductive Systems

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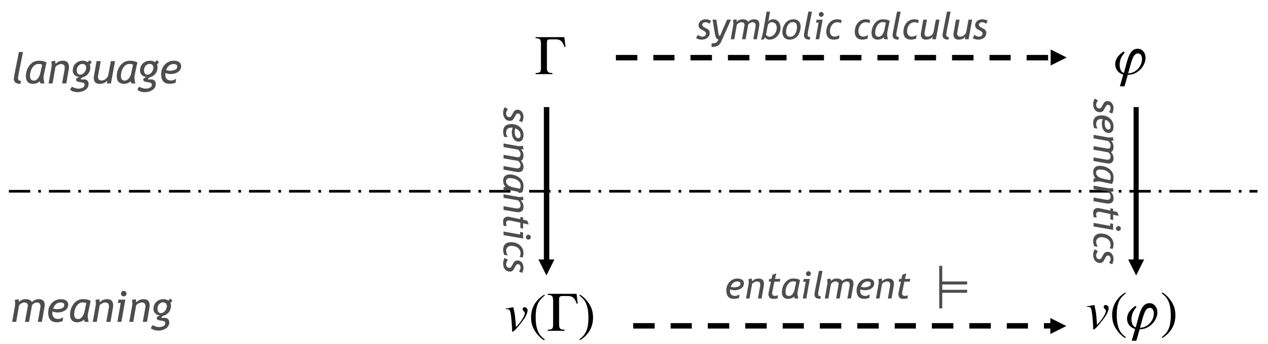
Symbolic calculus?

- A wff φ is **entailed** by a set of wff Γ iff every *model* of Γ is also *model* of φ

  Formally:

  $\Gamma \models \varphi$



Note that, in the definition above, the set of ***all*** possible models is considered

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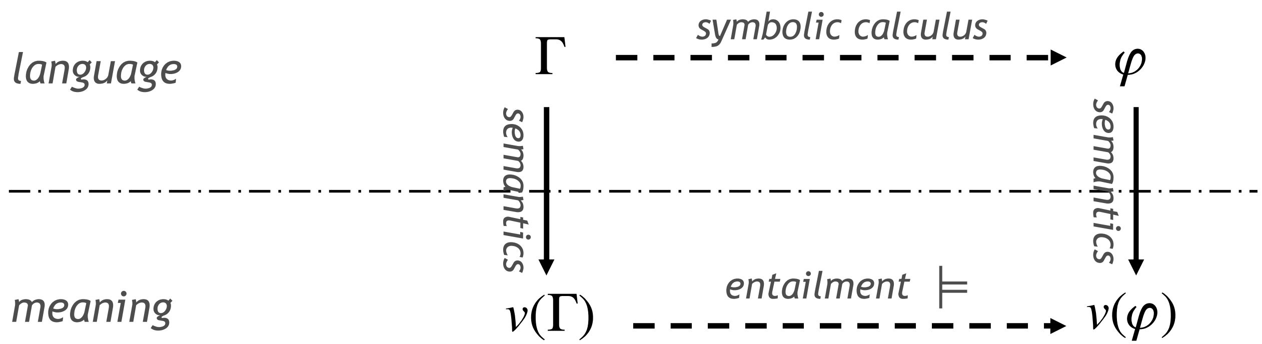

  Note that, in the definition above, the set of *all* possible models is considered

- Can we detect entailment by working on wffs only?

  For instance, by applying reasoning *schemas*…

Axiomatic method (i.e. Hilbert System, 1899)

- Language, *axioms* and *rules of inference*

  $< L_P , \boldsymbol{Ax} , \boldsymbol{Inf} >$

  L_P is a propositional language whose signature is P

  $\boldsymbol{Ax}$ a set of wffs, i.e. the *axioms*

  $\boldsymbol{Inf}$ is a set of inference rules

Axioms

The axioms (*of a logic*) describe the <u>*reasoning schemas*</u> (*of that logic*)

- Axiom <u>*schemas*</u> for propositional logic (Lukasiewicz, 1917)

$$\text{Ax1} \quad \vdash \varphi \to (\psi \to \varphi)$$

$$\text{Ax2} \quad \vdash (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

$$\text{Ax3} \quad \vdash (\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

Each wff obtained by substituting the *meta-variabes* φ, ψ e χ with a wff is an axiom

The wffs thus obtained are also called *instances* of axiom

Examples:

$\vdash A \to (\neg A \to A)$ [Ax1: φ/A, $\psi/\neg A$]

$\vdash (\neg(B \vee C) \to \neg D) \to (D \to (B \vee C))$ [Ax3: $\varphi/(B \vee C)$, ψ/D]

All axiom instances are *tautologies*

(But do not rely on this for the <u>definition</u> of deductive systems)

Inference Rules

Recall that $\{\varphi \to \psi, \varphi\} \models \psi$ is valid

- ***Inference rules*** are fundamental in any *symbolic calculus* (also known as *derivation* or *deduction rules*)

  They work on the structure of wffs

- For *propositional logic*, just one *inference rule* is sufficient

  Modus Ponens (MP):

$$\frac{\begin{array}{l}\varphi \to \psi \\ \varphi\end{array}}{\psi}$$

  It can be written also in this way:

  $\{\varphi \to \psi, \varphi\} \vdash \psi$ (i.e. from $\{\varphi \to \psi, \varphi\}$, ψ is *derivable*)

Proofs (also *derivations*)

- A *proof* (or *derivation*) of a wff φ from a set of wffs Γ

  Is a *finite* sequence of steps: $\Gamma \vdash \phi_1, \Gamma \vdash \phi_2, \ldots, \Gamma \vdash \phi_n$

  Admissible steps, at stage i :

  1) ϕ_i is an instance of an axiom schema Ax_n
  2) ϕ_i is in Γ
  3) ϕ_i has been obtained from two previous steps, via *Modus Ponens*

  In the final step, the wff to be proved is obtained: $\phi_n = \varphi$

  The notation is $\Gamma \vdash \varphi$ "φ is *derivable* Γ"

  There must be many different ways to show that $\Gamma \vdash \varphi$, i.e. many different *proofs*

  Note that:

$\Gamma \vdash \mathrm{Ax}n$	(an axiom or axiom instance can be derived from any Γ)
$\vdash \mathrm{Ax}n$	(an axiom or axiom instance can be derived from an empty Γ)
$\{\varphi, ..\} \vdash \varphi$	(any φ can be derived form a Γ that contains it)

Derivations, an *incomplete* example

- Same problem ("Harry is happy")

$$B \lor D \lor \neg(A \land C),\ B \lor C,\ A \lor D,\ \neg B \vdash D$$

Rewrite the problem in equivalent terms using only $\neg$ *and* $\rightarrow$

$$C \rightarrow (\neg B \rightarrow (A \rightarrow D)),\ \neg B \rightarrow C, \neg A \rightarrow D,\ \neg B \vdash D$$

1:	$\Gamma \vdash \neg B \rightarrow C$	
2:	$\Gamma \vdash \neg B$	
3:	$\Gamma \vdash C$	(*MP* 1,2)
4:	$\Gamma \vdash C \rightarrow (\neg B \rightarrow (A \rightarrow D))$	
5:	$\Gamma \vdash \neg B \rightarrow (A \rightarrow D)$	(*MP* 3,4)
6:	$\Gamma \vdash A \rightarrow D$	(*MP* 2,5)
7:	$\Gamma \vdash \neg A \rightarrow D$	
8*:	$\Gamma \vdash (\neg \varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi)$	(***OK if this was an axiom – it is not***)
9:	$\Gamma \vdash (\neg A \rightarrow D) \rightarrow ((A \rightarrow D) \rightarrow D)$	(*Sost*)
10:	$\Gamma \vdash (A \rightarrow D) \rightarrow D$	(*MP* 7,9)
11:	$\Gamma \vdash D$	(*MP* 6,10)

Derivations: example 0

- Any wff implies itself

$$\vdash \varphi \to \varphi$$

$$
\begin{array}{ll}
1: \vdash (\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)) & \text{(Ax2)} \\
2: \vdash (\varphi \to ((\varphi \to \varphi) \to \varphi)) & \text{(Ax1)} \\
3: \vdash (\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi) & (\mathit{MP}\ 1,2) \\
4: \vdash (\varphi \to (\varphi \to \varphi)) & \text{(Ax1)} \\
5: \vdash \varphi \to \varphi & (\mathit{MP}\ 3,4)
\end{array}
$$

Deduction (meta)-theorem

- Deduction theorem

$$\Gamma \cup \{\varphi\} \vdash \psi \quad \Leftrightarrow \quad \Gamma \vdash \varphi \rightarrow \psi$$

- (semantic dual of Deduction Theorem)

$$\Gamma \cup \{\varphi\} \models \psi \quad \Leftrightarrow \quad \Gamma \models \varphi \rightarrow \psi$$

Derivations: example 1

- The order of hypotheses is irrelevant

$$\vdash (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$$

1:	$(\varphi \to (\psi \to \chi)), \psi, \varphi \vdash (\varphi \to (\psi \to \chi))$	
2:	$(\varphi \to (\psi \to \chi)), \psi, \varphi \vdash \varphi$	
3:	$(\varphi \to (\psi \to \chi)), \psi, \varphi \vdash \psi \to \chi$	$(MP\ 1,2)$
4:	$(\varphi \to (\psi \to \chi)), \psi, \varphi \vdash \psi$	
5:	$(\varphi \to (\psi \to \chi)), \psi, \varphi \vdash \chi$	$(MP\ 3,4)$
6:	$(\varphi \to (\psi \to \chi)), \psi \vdash \varphi \to \chi$	(Ded)
7:	$(\varphi \to (\psi \to \chi)) \vdash \psi \to (\varphi \to \chi)$	(Ded)
8:	$\vdash (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	(Ded)

Derivations: example 2

- Double negation implies affirmation

$\vdash \neg\neg\varphi \rightarrow \varphi$

1:	$\vdash \neg\neg\varphi \rightarrow (\neg\neg\neg\neg\varphi \rightarrow \neg\neg\varphi)$	(Ax1)
2:	$\neg\neg\varphi \vdash \neg\neg\neg\neg\varphi \rightarrow \neg\neg\varphi$	(*Ded*)
3:	$\neg\neg\varphi \vdash (\neg\neg\neg\neg\varphi \rightarrow \neg\neg\varphi) \rightarrow (\neg\varphi \rightarrow \neg\neg\neg\varphi)$	(Ax3)
4:	$\neg\neg\varphi \vdash \neg\varphi \rightarrow \neg\neg\neg\varphi$	(*MP* 3,2)
5:	$\neg\neg\varphi \vdash (\neg\varphi \rightarrow \neg\neg\neg\varphi) \rightarrow (\neg\neg\varphi \rightarrow \varphi)$	(Ax3)
6:	$\neg\neg\varphi \vdash \neg\neg\varphi \rightarrow \varphi$	(*MP* 5,4)
7:	$\neg\neg\varphi \vdash \neg\neg\varphi$	
8:	$\neg\neg\varphi \vdash \varphi$	(*MP* 6,7)
9:	$\vdash \neg\neg\varphi \rightarrow \varphi$	(*Ded*)

Derivations: example 3

- A rule is false if the LHS is true and the RHS is false

$$\vdash \varphi \to (\neg\psi \to \neg(\varphi \to \psi))$$

1:	$\varphi, (\varphi \to \psi) \vdash \psi$	(MP)
2:	$\varphi \vdash (\varphi \to \psi) \to \psi$	(Ded)
3:	$\varphi \vdash ((\varphi \to \psi) \to \psi) \to (\neg\psi \to \neg(\varphi \to \psi))$	(Ax3)
4:	$\varphi \vdash \neg\psi \to \neg(\varphi \to \psi)$	$(MP$ 3,2$)$
5:	$\vdash \varphi \to (\neg\psi \to \neg(\varphi \to \psi))$	(Ded)

Derivations: example 4

- From absurd, anything can be derived ("*Ex absurdo sequitur quodlibet*"):

  $\vdash \varphi \to (\neg\varphi \to \psi)$ (vale a dire $\varphi, \neg\varphi \vdash \psi$)

1:	$\varphi, \neg\varphi \vdash \neg\varphi \to (\neg\psi \to \neg\varphi)$	(Ax1)
2:	$\varphi, \neg\varphi \vdash \neg\varphi$	
3:	$\varphi, \neg\varphi \vdash \neg\psi \to \neg\varphi$	(*MP* 1,2)
4:	$\varphi, \neg\varphi \vdash (\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$	(Ax3)
5:	$\varphi, \neg\varphi \vdash \varphi \to \psi$	(*MP* 4,3)
6:	$\varphi, \neg\varphi \vdash \varphi$	
7:	$\varphi, \neg\varphi \vdash \psi$	(*MP* 5,6)
8:	$\varphi \vdash \neg\varphi \to \psi$	(*Ded*)
9:	$\vdash \varphi \to (\neg\varphi \to \psi)$	(*Ded*)

A set of wffs that contains a contradiction is called *incoherent* (or *inconsistent*)

From an incoherent set, anything can be derived,
including a contradiction like: $\psi \wedge \neg\psi$

Derivations: example 5 (using *theorems*)

- When falsity implies contradiction, then it must be true:

$\vdash (\neg\varphi \to \varphi) \to \varphi$

1:	$\neg\varphi \to \varphi, \neg\varphi \vdash \neg\varphi$	
2:	$\neg\varphi \to \varphi, \neg\varphi \vdash \neg\varphi \to \varphi$	
3:	$\neg\varphi \to \varphi, \neg\varphi \vdash \varphi$	(*MP* 1,2)
4:	$\neg\varphi \to \varphi, \neg\varphi \vdash \varphi \to (\neg\varphi \to \neg(\neg\varphi \to \varphi))$	(Th. 4)
5:	$\neg\varphi \to \varphi, \neg\varphi \vdash \neg\varphi \to \neg(\neg\varphi \to \varphi)$	(*MP* 3,4)
6:	$\neg\varphi \to \varphi, \neg\varphi \vdash \neg(\neg\varphi \to \varphi)$	(*MP* 1,5)
7:	$\neg\varphi \to \varphi \vdash \neg\varphi \to \neg(\neg\varphi \to \varphi)$	(*Ded*)
8:	$\neg\varphi \to \varphi \vdash (\neg\varphi \to \neg(\neg\varphi \to \varphi)) \to ((\neg\varphi \to \varphi) \to \varphi)$	(Ax3)
9:	$\neg\varphi \to \varphi \vdash (\neg\varphi \to \varphi) \to \varphi$	(*MP* 7,8)
10:	$\neg\varphi \to \varphi \vdash \neg\varphi \to \varphi$	
11:	$\neg\varphi \to \varphi \vdash \varphi$	(*MP* 9,10)
12:	$\vdash (\neg\varphi \to \varphi) \to \varphi$	(*Ded*)

Derivations: Theorem "X"

- Resolution rule (see the first *incomplete* example):

$$\vdash (\neg\varphi \to \psi) \to ((\varphi \to \psi) \to \psi)$$

1:	$(\neg\varphi \to \psi), (\varphi \to \psi) \vdash (\neg\varphi \to \psi)$	
2:	$(\neg\varphi \to \psi), (\varphi \to \psi) \vdash (\neg\varphi \to \psi) \to (\neg\psi \to \varphi)$	(Ax3)
3:	$(\neg\varphi \to \psi), (\varphi \to \psi) \vdash (\neg\psi \to \varphi)$	(*MP* 1,2)
4:	$(\neg\varphi \to \psi), (\varphi \to \psi), \neg\psi \vdash \varphi$	(*Ded*)
5:	$(\neg\varphi \to \psi), (\varphi \to \psi), \neg\psi \vdash \varphi \to \psi$	
6:	$(\neg\varphi \to \psi), (\varphi \to \psi), \neg\psi \vdash \psi$	(*MP* 4,5)
7:	$(\neg\varphi \to \psi), (\varphi \to \psi) \vdash (\neg\psi \to \psi)$	(*Ded*)
8:	$(\neg\varphi \to \psi), (\varphi \to \psi) \vdash ((\neg\psi \to \psi) \to \psi)$	(Th. 5)
9:	$(\neg\varphi \to \psi), (\varphi \to \psi) \vdash \psi$	(*MP* 7,8)
10:	$(\neg\varphi \to \psi) \vdash ((\varphi \to \psi) \to \psi)$	(*Ded*)
11:	$\vdash (\neg\varphi \to \psi) \to ((\varphi \to \psi) \to \psi)$	(*Ded*)

Rewritten in an equivalent way:

$$\vdash (\varphi \vee \psi) \to ((\neg\varphi \vee \psi) \to \psi)$$

Correctness

- *Correctness*

  All wffs that are *derivable* from axioms Axn are *tautologies* (valid wffs)

  $$\vdash \varphi \Rightarrow \models \varphi$$

  It can be verified directly that the axioms schemas *Ax1*, *Ax2* e *Ax3* are schemas of tautologies as well

  The inference rule of *Modus Ponens* is correct, as it preserves entailment

  Any wff entailed by a set of tautologies is a tautology

Completeness

- *Completeness*

  All *tautologies* (i.e. *valid* wffs) are *derivable* from axiom schemas Axn

  $$\models \varphi \;\Rightarrow\; \vdash \varphi$$

  Why:

  See textbook

Properties of derivations

- *Coherence* (definition)

  A set of wffs Γ is *coherent* if it exists at least a wff φ which is not *derivable* from Γ

  (see theorem 3)

- *Refutation*

  $\Gamma \cup \{\neg\varphi\}$ is *incoherent* $\Leftrightarrow$ $\Gamma \vdash \varphi$

  $\Gamma \cup \{\neg\varphi\}$ is incoherent implies that for any ψ, $\Gamma \cup \{\neg\varphi\} \vdash \psi$

In particular	$\Gamma \cup \{\neg\varphi\} \vdash \varphi$
From deduction theorem	$\Gamma \vdash \neg\varphi \to \varphi$
From Theorem 5	$\Gamma \vdash (\neg\varphi \to \varphi) \to \varphi$
MP	$\Gamma \vdash \varphi$

Properties of derivations

- *Coherence* *is equivalent to* *satisfiability*

  A set of wffs Γ is *satisfiable* if it is *coherent*

  If Γ was incoherent, then it would be possible to derive a contradiction (Theorem 3)

  But, given that derivability implies entailment, then Γ should be *unsatisfiable*

  A set of wffs Γ which is *coherent* is also *satisfiable*

  (see textbook)

- *Syntactic compactness*

  Consider a set of wffs Γ (not necessarily finite)

  $\Gamma \vdash \varphi \quad \Rightarrow$ There exist a *finite* subset $\Sigma \subseteq \Gamma$ such that $\Sigma \vdash \varphi$

  (See textbook for a proof)

Properties of derivations

- *Syntactic monotony*

  For any Γ and Δ, if $\Gamma \vdash \varphi$ then $\Gamma \cup \Delta \vdash \varphi$

  In fact, any derivation of φ from Γ remains valid even if Γ grows larger

- *Transitivity*

  If for any $\varphi \in \Sigma$ we have $\Gamma \vdash \varphi$, then if $\Sigma \vdash \psi$ then $\Gamma \vdash \psi$

  One can apply the deduction theorem and *MP* repeatedly

Theorems

Given a set of wffs Γ, the **theorems** of Γ is the set of all wff φ that can be derived from Γ ← (Γ can be empty)

The set of theorems of Γ is also written as $Th(\Gamma)$

Due to the definition of *derivability*, this means that any such φ descends from $\boldsymbol{Ax} \cup \Gamma$

This definition has general validity: it applies to any axiomatized logic

Any *theorem* of Γ is also an *entailment* of Γ

$\varphi \in Th(\Gamma) \;\Rightarrow \Gamma \models \varphi$

Why? (a simple exercise for the reader ...)

Independence of the axiom schemas Axn

- Minimality

  The proof of completeness requires using them all (see textbook)

- Independence

  The three schemas are **logically independent**:

  It is not possible to derive any of them from the other two

- There exist other axiomatizations of propositional logic

  In one of them, just a single schema is used

  Using axiom schemas, however, is unavoidable

  In other words, using an *infinite* set of axioms is unavoidable

Theorems, theories, axiomatizations

- Theory = set of wffs (yes, just that)

  Any set Σ (however defined) is a ***theory***

- Theorem = a wff derivable from a set of wffs

  Given a set of wffs Γ, the set of **theorems** of Γ
  is the set of all wffs that can be *derived* from Γ

  $$Th(\Gamma) = \{\varphi : \Gamma \vdash \varphi\}$$

- Axiomatizations = a set of wff that describes a *theory*

  A set of wffs Γ is an ***axiomatization*** of a theory Σ iff

  $$\Sigma \equiv Th(\Gamma)$$

  Axiom schemas $\mathrm{Ax}n$ describe the *theory* of *valid* wffs
  in (***classical***) **propositional logic**